On the consequences of twisted Poincaré symmetry upon QFT on Moyal NC spaces

G. Fiore, Universitá "Federico II" and INFN, Napoli.

Ringberg Symposium, February 3-6 2008

in honor of W. Zimmermann's 80th birthday

based on joint work with J. Wess, Phys. Rev. D75: 105022, 2007

Introduction

The idea of spacetime NC is rather old: goes back to Heisenberg.

Simplest NC: constant commutators

Moyal space: $$[\hat{x}^\mu, \hat{x}^\nu] = i\mathbf{1}\theta^{\mu\nu} \qquad (1)$$

Algebra $\widehat{\mathcal{A}}$ of functions on Moyal space: generated by $\mathbf{1}, \hat{x}^\mu$ fulfilling (1). With $\mu = 0, 1, 2, 3$ and $\eta_{\mu\nu}$: deformed Minkowski space. $\theta^{\mu\nu} = 0$: $\mathcal{A}$ generated by commuting x^μ.

(1) are translation invariant, not Lorentz-covariant.

Contributions to the construction of QFT on it start in 1994-95. I would divide them into 3 groups, according to the used approaches. By no means are they equivalent!

1. DFR Approach

(Doplicher, Fredenhagen, Roberts 1994-95; Bahns, Piacitelli,...): field quantization in (rigorous) operator formalism on deformed Minkowski space. (1) motivated by the interplay of QM and GR in what they call the **Principle of gravitational stability against localization of events:**

The gravitational field generated by the concentration of energy required by the Heisenberg Uncertainty Principle to localise an event in spacetime should not be so strong to hide [a] *the event itself to any distant observer - distant compared to the Planck scale.*

(Goes back to Wheeler?)

---

[a] By black hole formation

In the first, simplest version $\theta^{\mu\nu}$ are not fixed constants, but central operators (obeying additional conditions) which on each irrep become fixed constants $\sigma^{\mu\nu}$, the joint spectrum of $\theta^{\mu\nu}$.

In more recent versions $\theta^{\mu\nu}$ is no more central, but commutation relations remain of Lie-algebra type.

It seems that the wished Lorentz covariance is sooner or later lost.

Speculations (heard from Doplicher): $\theta^{\mu\nu}$ should be finally related to v.e.v. of $R^{\mu\nu}$, which in turn should be influenced by the presence of matter quantum fields in spacetime (through quantum equations of motions).

2. Path-integral quantization

Initiated by Filk 1996. A lot of physicists: N. Seiberg, E. Witten, M. R. Douglas, A.S. Schwarz, S. Minwalla, M. Van Raamsdonk, J. Gomis, T. Mehen, L. Alvarez-Gaume, M.A. Vazquez-Mozo, N. A. Nekrasov, R.J. Szabo,...., H. Grosse, R. Wulkenhaar,...

String people motivation: low-energy effective theory from string theory in a constant background B-field.

(Wick-rotated) Lorentz covariance is lost, but this is expected in effective string theory because of the B-field.

Many pathologies: violation of causality, non-unitarity (for $\theta^{0i} \neq 0$), UV-IR mixing of divergences, subsequent non-renormalizability, claimed changes of statistics, etc.

UV-IR MIXING:

Planar Feynman diagrams remain as the undeformed, apart from a phase factor, in particular have the same UV divergences. Nonplanar Feynman diagrams which were UV divergent become finite for generic non-zero external momentum, but diverge as the latter go to zero, even with massive fields: IR divergences! Necessary ∞-ely many counterterms $\Rightarrow$ nonrenormalizability.

H. Grosse & R. Wulkenhaar's cure (for scalar theories): add x-dependent harmonic potential term $\Omega^2 x^2 \varphi \star \varphi$ to the lagrangian in Euclidean path-integral formulation of QFT. Then renormalizable theory. Actually $\Omega^2 x^2 \varphi \star \varphi$ is the only other marginal/relevant operator in the renormalization group flow.

These two "brave riders" are chasing "Landau's ghost" out of the castle.

3. Twisted Poincaré covariant approaches

This is the framework of our work, subject of this talk.
It recovers Poincaré covariance in a deformed version.
Field quantization either in an operator or in a path-integral approach (on the Euclidean).

Chaichian *et al*, Wess, Koch *et al*, Oeckl:
(1) are twisted Poincaré group covariant.

How to implement twisted Poincaré covariance in QFT?

Different proposals, [Chaichian *et al* 04,05,06], [Tureanu06], [Balachandran *et al* 05,06] [Lizzi *et al* 06], [Bu *et al* 06], [Zahn 06], [Abe 06]...:
a) do coordinates x, y of different spacetime points commute?
b) deform the CCR of $a_p, a_p^\dagger$ for free fields?

Summary of our work

We note that a proper enforcement of the “twisted Poincaré” covariance of [Chaichian *et al*], [Wess], [Koch *et al*], [Oeckl] requires (nontrivial) "braided" commutation relations between any pair of coordinates x, y generating two different copies of the Grönewold-Moyal space, or equivalently a $\star$-tensor product $f(x) \star g(y)$ (in the parlance of [Aschieri *et al*]).

Then all $(x-y)^\mu$ behave like undeformed coordinates.

Consequently, one can formulate QFT in a way physically equivalent to the undeformed counterpart, as observables involve only coordinate differences. (Similarly for n-particle QM)

Plan

1. Introduction
2. Twisted Poincaré Hopf algebra, several spacetime variables, $\star$-products
3. Revisiting Wightman axioms for QFT and their consequences
4. Free fields
5. Interacting fields
6. (Some) Conclusions?

2. The Hopf $*$-algebra $H \equiv U_\theta \mathcal{P}$

This is $U\mathcal{P}$ ($\mathcal{P} =$ Poincaré Lie algebra) "twisted" [Drinfel'd 83] with $\mathcal{F}$: $U\mathcal{P}, H$ have

1. same $*$-algebra and counit ε($\equiv$ trivial representation)
2. coproducts $\Delta, \hat{\Delta}$ related by

$$\Delta(g) \equiv \sum_I g^I_{(1)} \otimes g^I_{(2)} \longrightarrow \hat{\Delta}(g) = \mathcal{F}\Delta(g)\mathcal{F}^{-1} \equiv \sum_I g^I_{(\hat{1})} \otimes g$$

The *twist* $\mathcal{F}$ is not uniquely determined. The simplest choice is

$$\mathcal{F} \equiv \sum_I \mathcal{F}_I^{(1)} \otimes \mathcal{F}_I^{(2)} := \exp\left(\tfrac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu\right).$$

$$\hat{\Delta}(P_\mu) = P_\mu \otimes \mathbf{1} + \mathbf{1} \otimes P_\mu = \Delta(P_\mu),$$
$$\hat{\Delta}(M_\omega) = M_\omega \otimes \mathbf{1} + \mathbf{1} \otimes M_\omega + P[\omega, \theta] \otimes P \neq \Delta(M_\omega).$$

where $M_\omega = \omega^{\mu\nu} M_{\mu\nu}$. Translations undeformed!

If not familiar with Hopf algebras: the coproduct says how to construct the tensor product of any two representations.

Cocommutative Hopf algebra $U\mathbf{g}$:

$$g \in \mathbf{g} \to \Delta(g) = (g \otimes \mathbf{1} + \mathbf{1} \otimes g) \equiv g_1 + g_2 \in U\mathbf{g}\,.$$

Clearly $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$. One can extend Δ as a $*$-algebra map

$$\Delta : U\mathbf{g} \to U\mathbf{g} \otimes U\mathbf{g}\,, \qquad \Delta(ab) := \Delta(a)\Delta(b) \tag{2}$$

unambiguously (in fact: $\Delta\big([g,g']\big) = \big[\Delta(g), \Delta(g')\big]$ if $g, g' \in \mathbf{g}$). $\hat{\Delta}$ also fulfills (2), $\hat{\Delta}(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$, compatibility with ϵ and $*$ ($\mathcal{F}$ is unitary). Then $\hat{\Delta}$ can replace Δ in constructing the tensor product of two representations of $U\mathbf{g}$: deformed coalgebra.

3. With the above $\mathcal{F}$ same antipode, $\hat{S} = S$.
   $S(g) = -g$ if $g \in \mathbf{g}$, $S(\mathbf{1}) = \mathbf{1}$, $S(ab) = S(b)S(a)$.

(1) are H-covariant, i.e. $\widehat{\mathcal{A}}$ is a H-module algebra, as $\mathcal{A}$ is a $U\mathcal{P}$ -module algebra: $\exists$ (left) actions $\triangleright, \hat{\triangleright}$ of $U\mathcal{P}, H$ on $\mathcal{A}, \widehat{\mathcal{A}}$ ($g \in U\mathcal{P}$ acts on $\mathcal{A}$ as corresponding differential operator, e.g. $P_\mu \sim i\partial_\mu$):

- $\triangleright, \hat{\triangleright}$ act in the same way on 1st degree polynomials in $x^\nu, \hat{x}^\nu$

$$P_\mu \triangleright x^\rho = i\delta_\mu^\rho = P_\mu \hat{\triangleright} \hat{x}^\rho, \qquad M_\omega \triangleright x^\rho = 2i(x\omega)^\rho, \qquad M_\omega \hat{\triangleright} \hat{x}^\rho = 2i(\hat{x}\omega$$

and more generally on irreps (irreducible representations); $\Rightarrow$
Same classification of elementary particles as unitary irreps of $\mathcal{P}$!

- $\triangleright, \hat{\triangleright}$ differ on higher degree polynomials in $x, \hat{x}$,

$$g \triangleright (ab) = \sum_I \big(g_{(1)} \triangleright a\big)\big(g_{(2)} \triangleright b\big)$$
$$g \hat{\triangleright} (\hat{a}\hat{b}) = \sum_I \big(g^I_{(\hat{1})} \hat{\triangleright} \hat{a}\big)\big(g^I_{(\hat{2})} \hat{\triangleright} \hat{b}\big) \quad \Leftrightarrow \quad g \hat{\triangleright} (a \star b) = \sum_I \big(g^I_{(\hat{1})} \hat{\triangleright} a\big) \star \big(g^I_{(\hat{2}}$$

(these resp. reduce to usual or *deformed* Leibniz rule if $g \in \mathcal{P}$),

and more generally on tensor products of representations.

Several spacetime variables

Let $\mathcal{A}^n$ be the n-fold tensor product algebra of $\mathcal{A}$,
$x_1^\mu \equiv x^\mu \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1}$, $\quad x_2^\mu \equiv \mathbf{1} \otimes x^\mu \otimes \ldots \otimes \mathbf{1}$,...
$\mathcal{A}^n$ is $U\mathcal{P}$-covariant, i.e. $[x_i^\mu, x_j^\nu] = 0$ are compatible with $\triangleright$.

$[\hat{x}_i^\mu, \hat{x}_j^\nu] = 0$ is not compatible with $\hat{\triangleright}$ (apply e.g. $M_\omega \hat{\triangleright}$) .

The H**-covariant** NC generalization of $\mathcal{A}^n$ is the unital $*$-algebra $\widehat{\mathcal{A}}^n$ generated by real variables $\hat{x}_i^\mu$ fulfilling

$$[\hat{x}_i^\mu, \hat{x}_j^\nu] = \mathbf{1} i\theta^{\mu\nu}, \tag{3}$$

dictated by the braiding associated to the quasitriangular structure $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$ of H.

Note: some authors erroneously impose (3) only for $i = j$, and rhs(3)=0 for $i \neq j$.

$\star$-Products

Equivalent formulation of $\widehat{\mathcal{A}}^n$: For $n \geq 1$ let $\mathcal{A}_\theta^n$ be the algebra coinciding with $\mathcal{A}^n$ as a vector space, but with the new product

$$a \star b := \sum_I (\overline{\mathcal{F}}_I^{(1)} \triangleright a)(\overline{\mathcal{F}}_I^{(2)} \triangleright b), \qquad (4)$$

with $\overline{\mathcal{F}} \equiv \mathcal{F}^{-1}$. This encodes both the $\star$-product within each copy of $\mathcal{A}$, and the "$\star$−tensor product" algebra [Aschieri *et al*]. $\mathcal{A}_\theta^n$ has $\star$-commutation relations isomorphic to (3), $\Rightarrow \widehat{\mathcal{A}}^n, \mathcal{A}_\theta^n$ are isomorphic H-module $*$-algebras: the above $\mathcal{F}$ gives

$$x_i^\mu \star x_j^\nu = x_i^\mu x_j^\nu + i\theta^{\mu\nu}/2 \qquad \Rightarrow \qquad [x_i^\mu \overset{\star}{,} x_j^\nu] = \mathbf{1} i\theta^{\mu\nu},$$

$$a(x_i) \star b(x_j) = \exp[\frac{i}{2}\partial_{x_i}\theta\partial_{x_j}]a(x_i)b(x_j), \qquad (4')$$

after which we must set $x_i = x_j$ if $i = j$.

Strictly speaking, (4') makes sense if a, b belong to some subalgebra $\mathcal{A}^{n\prime} \subset \mathcal{A}^n$ such that the θ-power series is termwise well-defined and convergent. (Not enough for field theory.)
If $a, b \in \mathcal{A}^{n\prime}$ admit Fourier transform then

$$a(x_i) \star b(x_j) = \int d^4k \int d^4q \check{a}(k) \check{b}(q) \exp[i(k \cdot x_i + q \cdot x_j - k\theta q/2)]$$

(with $k\theta q := k_\mu \theta^{\mu\nu} q_\nu$). This can be used as a *definition* of $\star$-product $a, b \in L^1(\mathbb{R}^4) \cap \widehat{L^1(\mathbb{R}^4)}$, or even if a, b are distributions.

In the sequel we express NC only by $\star$-products.

Alternative generators of $\mathcal{A}_\theta^n$

$$\xi_i^\mu = x_{i+1}^\mu - x_i^\mu, \qquad X^\mu = \sum_{j=1}^n a_j x_j^\mu \qquad (i<n, \quad \sum_{j=1}^n a_j = 1$$

1. $[X^\mu \stackrel{\star}{,} X^\nu] = \mathbf{1} i\theta^{\mu\nu}$, so X^μ generate a $\mathcal{A}_{\theta,X}$, whereas $\forall b \in \mathcal{A}_\theta^n$

$$\xi_i^\mu \star b = \xi_i^\mu b = b \star \xi_i^\mu \qquad \Rightarrow \qquad [\xi_i^\mu \stackrel{\star}{,} b] = 0, \qquad (5)$$

ξ_i^μ generate a $\star$-central subalgebra $\mathcal{A}_\xi^{n-1}$, and $\mathcal{A}_\theta^n \sim \mathcal{A}_\xi^{n-1} \otimes \mathcal{A}_{\theta,X}$.

2. $\mathcal{A}_\xi^{n-1}, \mathcal{A}_{\theta,X}$ are actually H-module subalgebras, with

$$\begin{aligned} g\hat{\triangleright} a &= g \triangleright a & a \in \mathcal{A}_\xi^{n-1}, \quad g \in H \\ g\hat{\triangleright}(a \star b) &= \big(g_{(1)} \triangleright a\big) \star \big(g_{(2)} \hat{\triangleright} b\big), & b \in \mathcal{A}_\theta^n, \end{aligned} \qquad (6)$$

i.e. *on $\mathcal{A}_\xi^{n-1}$ the H-action is undeformed*, including the related part of the Leibniz rule. [By (10) $\star$ can be also dropped.] All ξ_i^μ are translation invariant.

Remark 1. $(x-y)^\mu\star = (x-y)^\mu\cdot$, same spectral decomposition on all $\mathbb{R}$ (including 0). On each irrep of $\mathcal{A}_\theta^n$ this amounts to multiplication by either a space-like, or a null, or a time-like 4-vector, in the usual sense.

Summing up, coordinate differences can be treated as classical variables; any x_i^μ is a combination of $\star$-commutative ξ_i^μ and the $\star$-noncommutative X^μ, e.g. if $X := x_1$

$$x_i = \sum_{j=1}^{i-1} \xi_j + X.$$

X =Global "noncommutative translation".

1.,2. can be reformulated in terms of $\hat{x}_i, \widehat{\mathcal{A}}^n$, etc. $\hat{X}$ is like the "quantum shift operator" of [Chaichian *et al*].

The **differential calculus is not deformed**, as $P_\mu \triangleright \partial_{x_i^\nu} = 0$ implies $\partial_{x_i^\nu} \star = \partial_{x_i^\nu} = \star \partial_{x_i^\nu}$:

$$\partial_{x_i^\mu} \star x_j^\nu = \delta_\mu^\nu \delta_j^i + x_j^\nu \star \partial_{x_i^\mu} \qquad\qquad \left[\partial_{x_i^\mu} \stackrel{\star}{,} \partial_{x_j^\nu}\right] = 0$$

($\hat{\partial}_{x_i^\mu}$ on $\widehat{\mathcal{A}}^n$ is isomorphic). In the sequel we shall drop the symbol $\star$ beside a derivative.

Formally, also **integration over the space is not deformed** :

$$\int d^4x \; a \star b = \int d^4x \; ab \tag{7}$$

Stoke's theorem still applies.

Consequences for QFT

Wightman axioms grouped into subsets **QM**, **R** ([Strocchi]):

QM1. The states are described by vectors of a (separable) Hilbert space $\mathcal{H}$.

QM2. The group of space-time translations $\mathbb{R}^4$ is represented on $\mathcal{H}$ by strongly continuous unitary operators $U(a)$. The spectrum of the generators P_μ is contained in $\bar{V}_+ = \{p_\mu : p^2 \geq 0,\, p_0 \geq 0\}$. There is a unique Poincaré invariant state Ψ_0, the *vacuum state*.

QM3. The fields (in the Heisenberg representation) $\varphi^\alpha(x)$ [α enumerates field species and/or $SL(2,\mathbb{C})$-tensor components] are operator (on $\mathcal{H}$) valued tempered distributions on Minkowski space, with Ψ_0 a *cyclic* vector for the fields, i.e. polynomials of the (smeared) fields applied to Ψ_0 give a set $\mathcal{D}_0$ dense in $\mathcal{H}$.

Taking v.e.v.'s we define Wightman functions (distributions):

$$\mathcal{W}^{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n) = \left(\Psi_0, \varphi^{\alpha_1}(x_1) \star \ldots \star \varphi^{\alpha_n}(x_n)\Psi_0\right), \qquad (8)$$

or (their combinations) Green's functions

$$G^{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n) = \left(\Psi_0, T[\varphi^{\alpha_1}(x_1)\star \ldots \star\varphi^{\alpha_n}(x_n)]\Psi_0\right); \qquad (9)$$

no problem in defining time-ordering T as on commutative Minkowski space, even if $\theta^{0i} \neq 0$,

$$T[\varphi^{\alpha_1}(x)\star\varphi^{\alpha_2}(y)] = \varphi^{\alpha_1}(x)\star\varphi^{\alpha_2}(y)\vartheta(x^0-y^0)+\varphi^{\alpha_2}(y)\star\varphi^{\alpha_1}(x)\vartheta(y^0-x^0)$$

as $\vartheta(x^0-y^0)$ are $\star$-central ($\vartheta \equiv$Heavyside function). [The $\star$'s preceding all ϑ can be and have been dropped, by (10).]

Argue as for ordinary QFT [Streater & Wightman 1964].

QM2 $\Rightarrow$ **Wightman and Green's functions are translation invariant** and therefore may depend only on the ξ_i^μ.

$$
\begin{aligned}
\mathcal{W}^{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n) &= W^{\alpha_1,\ldots,\alpha_n}(\xi_1,\ldots,\xi_{n-1}), \\
\mathcal{G}^{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n) &= G^{\alpha_1,\ldots,\alpha_n}(\xi_1,\ldots,\xi_{n-1}).
\end{aligned}
\tag{10}
$$

From QM3, QM2, QM1 it follows

W1. $\mathcal{W}^{\{\alpha\}}(x_1,\ldots,x_n) = W^{\{\alpha\}}(\xi_1,\ldots,\xi_{n-1})$ are tempered distributions.

W2. (**Spectral condition**) The support of the Fourier transform $\tilde{W}$ of W is contained in the product of forward cones, i.e.

$$
\tilde{W}^{\{\alpha\}}(q_1,\ldots q_{n-1}) = 0, \qquad \text{if } \exists j: \quad q_j \notin \overline{V}_+. \tag{11}
$$

W3. $\mathcal{W}^{\{\alpha\}}$ fulfill the same **Hermiticity and Positivity** properties following from those of the scalar product in $\mathcal{H}$.

Ordinary relativistic conditions on QFT:

R1. (**Lorentz Covariance**) $SL(2,\mathbb{C})$ is represented on $\mathcal{H}$ by strongly continuous unitary operators $U(A)$, and under the Poincaré transformations $U(a,\ A)=U(a)\,U(A)$

$$U(a,A)\,\varphi^\alpha(x)\,U(a,A)^{-1}=S^\alpha_\beta(A^{-1})\,\varphi^\beta\big(\Lambda(A)x+a\big), \qquad (12)$$

with S a finite-dimensional representation of $SL(2,\mathbb{C})$.

R2. (**Microcausality or locality**) The fields either commute or anticommute at spacelike separated points

$$[\,\varphi^\alpha(x),\varphi^\beta(y)\,]_\mp=0, \qquad \text{for}\ \ (x-y)^2<0. \qquad (13)$$

As a consequence of QM2,R1 in ordinary QFT one finds

W4. (Lorentz Covariance of Wightman functions)

$$\mathcal{W}^{\alpha_1..\alpha_n}(\Lambda(A)x_1,\ldots,\Lambda(A)x_n) = S^{\alpha_1}_{\beta_1}(A)\ldots S^{\alpha_n}_{\beta_n}(A)\mathcal{W}^{\beta_1..\beta_n}(x_1,\ldots,x_n). \tag{14}$$

R1 needs a "twisted" reformulation R1$_\star$, which we defer. R1$_\star$ should imply that $W^{\{\alpha\}}$ are $SL_\theta(2,\mathbb{C})$ tensors, anyway. But, as the $W^{\{\alpha\}}$ should be built only in terms of ξ_i^μ and other $SL(2,\mathbb{C})$ tensors (like $\partial_{x_i^\mu}$, $\eta_{\mu\nu}$, γ^μ, polarization vectors, spinors, etc.), which are all annihilated by $P_\mu\triangleright$, $\mathcal{F}$ should act as id and $W^{\{\alpha\}}$ should transform under $M^{\rho\sigma}$ as for $\theta=0$. Therefore **we shall require W4 also if** $\theta\neq 0$ as a temporary substitute of R1.

R2$_\star$? Simplest: with a $\star$-commutator; makes sense, as space-like separation is well-defined. Alternatively, $\exists$ some reasonable weakening? In fact, an open question also on commutative space; the same restrictions should apply.

R2$_\star$. $$[\,\varphi^\alpha(x) \stackrel{\star}{,} \varphi^\beta(y)\,]_\mp = 0, \qquad \text{for } (x-y)^2 < 0.$$

Argue as [S. & W. 1964] to prove QM1-3, W4, R2$_\star$ are (independent and) compatible: they can be fulfilled by free fields (see below)! So in particular **the noncommutativity structure of a Moyal space is compatible with R2$_\star$!**

As consequences of R2$_\star$ one again finds

W5. (Locality) if $(x_j - x_{j+1})^2 < 0$

$$\mathcal{W}^{\{\alpha\}}(x_1,...x_j,x_{j+1},...x_n) = \pm\mathcal{W}^{\{\alpha\}}(x_1,...x_{j+1},x_j,...x_n). \tag{15}$$

W6. (Cluster property) For any spacelike a and for $\lambda \to \infty$

$$\mathcal{W}^{\{\alpha\alpha'\}}(x_1,...x_j,x_{j+1}+\lambda a,..., x_n+\lambda a) \to \mathcal{W}^{\{\alpha\}}(x_1,...,x_j)\,\mathcal{W}^{\{\alpha'\}}(x_{j+1},...,x_n) \tag{16}$$

(convergence as distributions); true also with permuted x_i's.

Summarizing: QFT framework with **QM1-3, W4, R2**$_\star$ or alternatively with constraints **W1-6** on $\mathcal{W}^{\{\alpha\}}$ exactly as in QFT on Minkowski space.

We stress that these results should hold for all $\theta^{\mu\nu}$, and not only if $\theta^{0i} = 0$, as in other approaches.

Free fields

Free field e.o.m. remain undeformed (as $\Box$, $\not{\partial}$, etc), hence also their constraints on $\mathcal{W}^{\{\alpha\}}, \mathcal{G}^{\{\alpha\}}$ and on the field comm. relations. For simplicity Hermitean scalar field $\varphi_0(x)$ of mass m. One finds

$$
\begin{aligned}
&(\Box_x + m^2)\varphi_0 = 0, \qquad \Rightarrow \\
&\varphi_0(x) = \varphi_0^+(x) + \varphi_0^-(x) = \int d\mu(p)\, [e^{-ip\cdot x} a^p + a_p^\dagger e^{ip\cdot x}],
\end{aligned} \tag{17}
$$

where $d\mu(p) := \delta(p^2 - m^2)\vartheta(p^0)d^4p$, and

$$
\begin{aligned}
W(x-y) &= \int \frac{d\mu(p)}{(2\pi)^3} e^{-ip\cdot(x-y)} = -iF^+(x-y) \\
G(x-y) &= \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon},
\end{aligned} \tag{18}
$$

(18) *are independent of* $\mathbf{R2}_\star$ *or any other assumption about field commutation relations*, which are not used in the proof.

Adding **R2**$_\star$ and reasoning as in the proof of the Jost-Schroer Thm. (4-15 in [S. & W. 1964]) one proves (up to a factor > 0) the **free field commutation relation**

$$[\varphi_0(x) \stackrel{\star}{,} \varphi_0(y)] = iF(x-y), \qquad F(\xi) := F^+(-\xi) - F^+(\xi) \quad (19)$$

(F undeformed!). Applying ∂_{y^0} and then setting $y^0 = x^0$ [this is compatible with (7)] one even finds **the c.c.r.**

$$[\varphi_0(x^0, \mathbf{x}) \stackrel{\star}{,} \dot{\varphi}_0(x^0, \mathbf{y})] = i\,\delta^3(\mathbf{x} - \mathbf{y}). \quad (20)$$

As a consequence of (24), also the n-point Wightman functions coincide with the undeformed ones, i.e. vanish if n is odd and are sum of products of two point functions if n is even (factorization). This agrees with the cluster property, as expected.

Two ways to get φ_0 fulfilling (19) from (17). The first is by plugging $a^p, a_p^\dagger$ satisfying the commutation relations

$$a^\dagger_p a^\dagger_q = e^{ip\theta' q}\, a^\dagger_q a^\dagger_p, \qquad a^p a^q = e^{ip\theta' q}\, a^q a^p,$$

$$a^p a^\dagger_q = e^{-ip\theta' q}\, a^\dagger_q a^p + 2\omega_{\mathbf{p}} \delta^3(\mathbf{p}-\mathbf{q}), \qquad \theta' = \theta \qquad (21)$$

$$[a^p, f(x)] = [a^\dagger_p, f(x)] = 0,$$

($p\theta q := p_\mu \theta^{\mu\nu} q_\nu$), as adopted in [Balachandran *et al* 05,06] first, then [Lizzi *et al* 06, Abe 06]. The choice $\theta' = 0$ gives the CCR, assumed in most of the literature, explicitly in [Doplicher *et al* 95], apparently in [Chaichian *et al* 04,05,06], [Tureanu06], or implicitly in path-integral approach to quantization. Correspondingly, one finds non-local $\star$-commutation relations

$$\varphi_0(x) \star \varphi_0(y) = e^{i\partial_x(\theta-\theta')\partial_y} \varphi_0(x) \star \varphi_0(y) + i\, F(x-y), \quad (22)$$

unless $\theta' = \theta$. [But taking $\theta' = \theta$ and using $\varphi_0(x)\varphi_0(y)$ instead of $\varphi_0(x) \star \varphi_0(y)$ one also finds non-local relations.]

$$\mathcal{W}(x_1,x_2,x_3,x_4) = W(x_1-x_2)W(x_3-x_4) + e^{i\partial_{x_2}(\theta-\theta')\partial_{x_3}}W(x_1-x_3)W(x_2-x_4) + \ldots \tag{23}$$

The first term at the rhs comes from the v.e.v.'s of $\varphi_0(x_1)\star\varphi_0(x_2)$ and $\varphi_0(x_3)\star\varphi_0(x_4)$; it is Lorentz invariant and factorized. The second, nonlocal term comes from the v.e.v.'s of $\varphi_0(x_1)\star\varphi_0(x_3)$ and $\varphi_0(x_2)\star\varphi_0(x_4)$, after commuting $\varphi_0(x_2),\varphi_0(x_3)$. Only if $\theta'=\theta$ is Lorentz invariant and factorizes to $W(x_1-x_3)W(x_2-x_4)$. As it depends only on x_1-x_3, x_2-x_4, it is invariant under $(x_1,x_2,x_3,x_4) \to (x_1,x_2+\lambda a,x_3,x_4+\lambda a)$. By taking a space-like and $\lambda \to \infty$, we conclude that if $\theta' \neq \theta$ $\mathcal{W}$ violates W4 and W6, as expected.

The second, "exotic" way to realize the free com. rel. (19) is: Assume $P_\mu \triangleright a^\dagger_p = p_\mu a^\dagger_p$, $P_\mu \triangleright a^p = -p_\mu a^p$ and extend the $\star$-product law also to $a^p, a^\dagger_p$. It amounts to $\theta' = -\theta$ (inserting $\star$'s) and nontrivial com. rel. between the $a^p, a^\dagger_p$ and functions:

$$a^\dagger_p \star a^\dagger_q = e^{-ip\theta q}\, a^\dagger_q \star a^\dagger_p, \qquad a^p \star a^q = e^{-ip\theta q}\, a^q \star a^p,$$

$$a^p \star a^\dagger_q = e^{ip\theta q}\, a^\dagger_q \star a^p + 2\omega_{\mathbf{p}} \delta^3(\mathbf{p} - \mathbf{q}), \tag{24}$$

$$a^p \star e^{iq\cdot x} = e^{-ip\theta q}\, e^{iq\cdot x} \star a^p, \qquad a^\dagger_p \star e^{iq\cdot x} = e^{ip\theta q}\, e^{iq\cdot x} \star a^\dagger_p.$$

Whence $[\varphi_0(x) \stackrel{\star}{,} f(y)] = 0$. The first three relations define an example of a general deformed Heisenberg algebra [G. F. 95]

$$\begin{aligned} a^q \star a^p &= R^{qp}_{rs}\, a^s \star a^r \qquad & a^\dagger_p \star a^\dagger_q &= R^{sr}_{pq}\, a^\dagger_r \star a^\dagger_s \\ a^p \star a^\dagger_q &= \delta^p_q + R^{rp}_{qs}\, a^\dagger_r \star a^s & & \end{aligned} \tag{25}$$

covariant under a triangular Hopf algebra H.

Here $R^{pq}_{rs} := \langle\mathbf{p}|\langle\mathbf{q}|\mathcal{R}\,|\mathbf{r}\rangle|\mathbf{s}\rangle$, $\mathcal{R} := \mathcal{F}_{21}\mathcal{F}^{-1}$ is the triangular structure of H, $\{|p\rangle\}$ is the generalized basis of the 1-particle Hilbert space consisting of (on-shell) eigenvectors of P_μ; $\delta^p_q = 2\omega_{\mathbf{p}}\delta^3(\mathbf{p}-\mathbf{q})$ is Dirac's delta (up to normalization).

Up to normalization of R, and with $p, q, r, s \in \{1, ..., N\}$, relations (25) are also identical to the ones defining the older q-deformed Heisenberg algebras of [Pusz & Woronowicz], [Wess & Zumino], based on a quasitriangular $\mathcal{R}$ in (only) the *fundamental* representation of $H = U_q su(N)$.

Interacting QFT (T-ordered perturb. th.)

Def. Normal ordering: $\mathcal{A}_\theta^n$-bilinear map of field algebra into itself such that $(\Psi_0, :M: \Psi_0) = 0$, in particular $:\mathbf{1}: = 0$. Applying it to (21) we find that it is consistent to define

$$:a^p a^q: = a^p a^q, \quad :a^\dagger_p a^q: = a^\dagger_p a^q, \quad :a^\dagger_p a^\dagger_q: = a^\dagger_p a^\dagger_q, \quad :a^p a^\dagger_q: = a^\dagger_q a^p e^{-ip\theta' q}$$

Note the phase. More generally, in any monomial reorders all a^p to the right of all $a^\dagger_q$ introducing a $e^{-iq\theta' p}$ for each flip $a^p \leftrightarrow a^\dagger_q$. Assuming (21) or (24), (i.e. free field com. rel.) one finds

$$
\begin{aligned}
:\varphi_0(x): \; &= \; \varphi_0(x)\\
:\varphi_0(x) \star \varphi_0(y): \; &= \; \varphi_0(x) \star \varphi_0(y) - (\Psi_0, \varphi_0(x) \star \varphi_0(y) \Psi_0)\\
:\varphi_0(x) \star \varphi_0(y) \star \varphi_0(z): &= \varphi_0(x) \star \varphi_0(y) \star \varphi_0(z) - (\Psi_0, \varphi_0(x) \star \varphi_0(y) \Psi_0)\, \varphi
\end{aligned}
$$

$$- (\Psi_0, \varphi_0(x) \star \varphi_0(z) \Psi_0)\, \varphi_0(y) - \varphi_0(x)\, (\Psi_0, \varphi_0(y) \star \varphi_0(z) \Psi_0$$

Well-defined operators also if coinciding coordinates (e.g. $y \to x$). Moreover, **the same Wick theorem will hold**:

$$T\left[\varphi_0(x) \star \varphi_0(y)\right] = :\varphi_0(x) \star \varphi_0(y): + \Big(\Psi_0, T\left[\varphi_0(x) \star \varphi_0(y)\right] \Psi$$

$$T\left[\varphi_0(x) \star \varphi_0(y) \star \varphi_0(z)\right] = :\varphi_0(x) \star \varphi_0(y) \star \varphi_0(z): + \Big(\Psi_0, T\left[\varphi_0(x) \star \varphi_0(y)\right] \Psi_0\Big) :$$

$$+ \Big(\Psi_0, T\left[\varphi_0(x) \star \varphi_0(z)\right] \Psi_0\Big) :\varphi_0(y): + \Big(\Psi_0, T\left[\varphi_0(y) \star \varphi_0(z)\right] \Psi_0\Big) :\varphi_0$$

$$\dots$$

Interacting theory. Wish to apply the Gell-Mann–Low formula

$$G(x_1,\dots,x_n) = \frac{\Big(\Psi_0, T\left\{\varphi_0(x_1) \star \dots \star \varphi_0(x_n) \star \exp\left[-i\lambda \int dy^0 \; H_I(y^0)\right]\right\} \Psi}{\Big(\Psi_0, T \exp\left[-i \int dy^0 \; H_I(y^0)\right] \Psi_0\Big)} \tag{27}$$

Here $\varphi_0 \equiv$free "in" field, and $H_I(x^0)$ is the interaction Hamiltonian in the interaction representation.

Same heuristic derivation of (27): only time-differences involved!

Choose

$$H_I(x^0) = \lambda \int d^3x \ :\varphi_0^{\star m}(x):_\star \qquad \varphi_0^{\star m}(x) \equiv \underbrace{\varphi_0(x) \star \ldots \star \varphi_0(x)}_{m \text{ times}}$$

This is a well-defined, Hermitean operator, with zero v.e.v. Expanding the $\exp$ we evaluate the generic $O(\lambda^h)$ term in the numerator or denominator as in the undeformed case: applying Wick Thm to the field monomial and $(\Psi_0, :M: \Psi_0) = 0$ we find exactly the *same* integrals over y-variables of products of free propagators having coordinate differences as arguments. Each term is represented by a Feynman diagram. So the **Green's functions coincide with the undeformed ones** and can be computed by Feynman diagrams with the undef. Feynman rules. So, at least perturbatively, **this QFT is completely equivalent to the undeformed one** (no more pathologies like UV-IR mixing!).

Conclusions. What do we learn?

- Glass: half full or half empty?
- Various approaches to QFT on NC spaces. Operator ones (as by [Doplicher, Fredenhagen, Roberts, Bahns, Piacitelli]) on (deformed) Minkowski spaces look safer starting points, but still no completely satisfactory guiding principle.
- Twisting or not Poincaré group, and doing it properly, makes radical differences.
- A sensible theory with twisted Poincaré seems possible: equivalent to the undeformed one. Avoids all complications (IV-UR, causality/unitarity violation, statistics violation, cluster property violation,...).
- Obtained by matching operator $(a, a^\dagger)$ and spacetime noncommutativities somehow to compensate each other

- No new physics, nor a more satisfactory formulation of old one (e.g. by an inthrinsic UV regularization)...
- ... but can be used as a Lab to:
  1. look for and test equivalent formulations of QFT on NC spaces: Wick rotation into EQFT, path integral quantization, etc.;
  2. clarify notions asymptotic states, spin-statistics, CPT, etc., on NC spaces;
  3. properly formulate covariance properties of fields under twisted symmetries ($R1_\star$), and clarify their connection to the ordinary ones.
  4. properly formulate gauge field theory on NC spaces.