Tilted Convolution: A Novel Tool in the Construction of QFT's

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- Algebraic approach to QFT
- Wedge algebras
- Spacetime symmetries and locality
- Scheme for constructing QFT's
- Tilted convolution
- New models
- Conclusions

1. Algebraic approach to QFT

Algebra of all quantum observables: $\mathcal{B}(\mathcal{H})$

Observables localized in (un)bounded regions $\mathcal{R} \subset \mathbb{R}^d$ correspond to sub–(von Neumann)–algebras

$$\mathcal{A}(\mathcal{R}) \subset \mathcal{B}(\mathcal{H}),$$

subject to the conditions of isotony

$$\mathcal{A}(\mathcal{R}_1) \subset \mathcal{A}(\mathcal{R}_2) \quad \text{if} \quad \mathcal{R}_1 \subset \mathcal{R}_2\,.$$

In particular,

$$\mathcal{A}(\mathbb{R}^d) = \mathcal{B}(\mathcal{H}).$$

Insight gained in 50 years of AQFT:
A specific QFT is completely fixed by specifying an assignment (net) $\mathcal{A}$ on $\mathbb{R}^d$

$$\mathcal{A} : \mathcal{R} \mapsto \mathcal{A}(\mathcal{R})$$

and a distinguished vector $\Omega \in \mathcal{H}$ (vacuum).

Remark: It is sufficient to specify this assignment for a subbase $\boldsymbol{\mathcal{W}} = \{\mathcal{W} \subset \mathbb{R}^d\}$ of the topology of $\mathbb{R}^d$, $e.g.$ "double cones" or "wedge regions".

Reminder: Net and vacuum state $(\mathcal{A}, \Omega)$ contain all physical information. One can extract from it

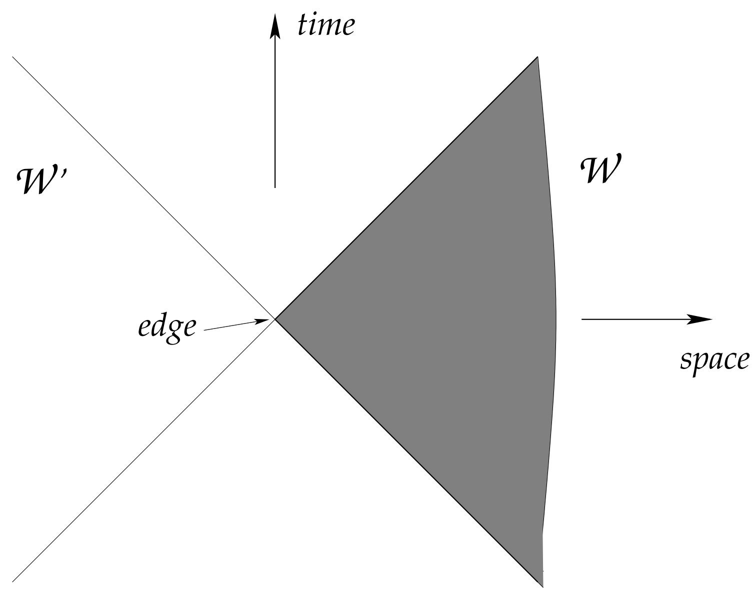
- spacetime symmetries, dynamics
- particle content, collision cross sections
- superselection structure, charges, statistics
- properties of thermal states
- quantum fields, operator product expansions
- short distance structure ...

Task: Construction of nets

- constructive approaches:
  perturbation theory, lattice approximation, functional integration ...
- abstract approach to proving existence:
  operator algebras ...

AQFT may not be the most convenient setting for doing actual computations. Yet: rigorous proofs for the existence of QFT's may simplify.

2. Wedge algebras



Set of all wedges: $\boldsymbol{\mathcal{W}} = \{\mathcal{W} \subset \mathbb{R}^d\}$.

$\boxed{\text{Working hypothesis: Wedge algebras } \{\mathcal{A}(\mathcal{W})\}_{\mathcal{W} \in \boldsymbol{\mathcal{W}}} \text{ are the building blocks of a quantum field theory!}}$

Remark on QFT on non–commutative Minkowski space:

$$[X_\mu, X_\nu] = i\,\Theta_{\mu\,\nu} \cdot 1\,.$$

If $d > 2$ there exist lightlike coordinates $X_\pm$ with $[X_+, X_-] = 0$, *i.e.* observables on such spaces can be "localized" in wedges $\mathcal{W}$.

3. Spacetime symmetries and locality

Symmetry transformations and causal commutation relations can be established for $(\mathcal{A}, \Omega)$ if "condition of geometric modular action" is fulfilled.

CGMA:
Ω is cyclic and separating for $\mathcal{A}(\mathcal{W})$, $\mathcal{W} \in \boldsymbol{\mathcal{W}}$, and the corresponding modular conjugations $J_{\mathcal{W}}$ induce (unspecified) transformations of $\{\mathcal{A}(\mathcal{W})\}_{\mathcal{W} \in \boldsymbol{\mathcal{W}}}$.

<u>Theorem:</u> $(d = 4)$ CGMA implies

- Uniqueness of geometric action:
  there is a continuous (anti)unitary representation U of the Poincaré group $\mathcal{P}_+$ such that
  $$J_{\mathcal{W}} = U(\lambda_{\mathcal{W}}) = J_{\mathcal{W}'}\,, \ \mathcal{W} \in \boldsymbol{\mathcal{W}}\,,$$
  where $\lambda_{\mathcal{W}}$ is the reflection about the edge of $\mathcal{W}$.
  "Condition of modular stability" implies sp $U(\mathbb{R}^d) \subset \overline{V}_+$.

- Covariance:
  $$U(\lambda)\, \mathcal{A}(\mathcal{W}) U(\lambda)^{-1} = \mathcal{A}(\lambda \mathcal{W})\,, \ \lambda \in \mathcal{P}_+\,.$$

- Wedge duality:
  $$\mathcal{A}(\mathcal{W})' = J_{\mathcal{W}} \mathcal{A}(\mathcal{W}) J_{\mathcal{W}} = \mathcal{A}(\mathcal{W}')\,.$$

4. Scheme for constructing QFT's

Detailed information about the general structure of nets sheds new light on "constructive problems".

Strategy:

- Construct (reducible) representation U of $\mathcal{P}_+$ on $\mathcal{H}$ with $\mathrm{sp}\, U(\mathbb{R}^d) \subset \overline{V}_+$. Trivial sub-representation corresponds to Ω; irreducible sub-representations correspond to particles; input on multiplicities taken from collision theory.
- Fix a wedge $\mathcal{W}_0$ and pick some von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ subject to the conditions
  - (a) $U(\lambda)\mathcal{M}U(\lambda)^{-1} \subset \mathcal{M}$ if $\lambda\mathcal{W}_0 \subset \mathcal{W}_0$
  - (b) $U(\lambda')\mathcal{M}U(\lambda')^{-1} \subset \mathcal{M}'$ if $\lambda'\mathcal{W}_0 \subset \mathcal{W}_0'$
  - (c) Ω is cyclic and separating for $\mathcal{M}$

Consistent definition of wedge algebras:

$$\mathcal{A}(\lambda\mathcal{W}_0) \doteq U(\lambda)\mathcal{M}U(\lambda)^{-1}, \quad \lambda \in \mathcal{P}_+ .$$

Corresponding local, Poincaré covariant net:

$$\mathcal{R} \longmapsto \mathcal{A}(\mathcal{R}) \doteq \bigcap_{\mathcal{W} \supset \mathcal{R}} \mathcal{A}(\mathcal{W}).$$

5. Tilted convolution

Crucial step: identify suitable algebra $\mathcal{M} \subset \mathcal{B}$
Novel tool [Grosse, Lechner]: deformations of $\mathcal{M}_0$
Generalization [D.B., Summers]: "tilted convolution"

$$U(x) = e^{iPx} = \int e^{ipx} \, dE(p) \,, \quad x \in \mathbb{R}^d$$

$$\alpha_x \doteq \mathsf{Ad}\, U(x) \quad \text{on } \mathcal{B}(\mathcal{H})$$

Definition of left/right integrals:

$$\int \alpha_{Qp}(A) dE(p) \,, \quad \int dE(p) \alpha_{Qp}(A)$$

where Q suitable skew–symmetric $d \times d$–matrix.

Not definable on all of $\mathcal{B}(\mathcal{H})$; integrals exist as strong limits of Bochner integrals for elements of

$$\mathcal{C}(\mathcal{H}) \doteq \{A \in \mathcal{B}(\mathcal{H}) : x \mapsto \alpha_x(A) \text{ smooth}\}$$

on domain

$$\mathcal{D} \doteq \{\Psi \in \mathcal{H} : x \mapsto U(x)\Psi \text{ smooth}\} \,.$$

Remark: $\mathcal{D}$ stable under action of left/right integrals.

(i) Left/right integrals are equal

Computations require proper treatment of expressions such as $dE(p)AdE(q)$ (not a product measure).

Argument based on $dE(p)f(P) = dE(p)f(p)$:

$$
\begin{aligned}
&\int dE(p)\alpha_{Qp}(A) \\
&= \int dE(p)U(Qp)AU(Qp)^{-1}\int dE(q) \\
&= \iint dE(p)e^{\cancel{ipQp}}A\, dE(q)e^{-iqQp} \\
&= \iint dE(p)\, e^{-iqQp}Ae^{-\cancel{iqQq}}\, dE(q) \\
&= \iint dE(p)\, e^{ipQq}Ae^{-iqQq}\, dE(q) \\
&= \int dE(p)\int U(Qq)AU(Qq)^{-1}dE(q) \\
&= \int \alpha_{Qq}(A)dE(q)\,.
\end{aligned}
$$

In particular:

$$\Big(\int dE(p)\alpha_{Qp}(A)\Big)^* \supset \int dE(p)\alpha_{Qp}(A^*)$$

(ii) Integrals preserve commutation relations

Let $A, B \in \mathcal{C}(\mathcal{H})$ such that for all $p, q \in \mathrm{supp}E(\cdot)$

$$\alpha_{Qp}(A)\alpha_{-Qq}(B) = \alpha_{-Qq}(B)\alpha_{Qp}(A)\,.$$

Then

$$\begin{aligned}
&\int dE(p)\alpha_{Qp}(A) \int dE(q)\alpha_{-Qq}(B) \\
&= \int dE(p)\alpha_{Qp}(A) \int \alpha_{-Qq}(B)dE(q) \\
&= \iint dE(p)\alpha_{-Qq}(B)\alpha_{Qp}(A)dE(q) \\
&= \iint dE(p)e^{-ip\cancel{Q}q}BU(Qq+Qp)Ae^{-iq\cancel{Q}p}dE(q) \\
&= \iint dE(p)U(-\cancel{Q}p)BU(Qp+Qq)AU(-\cancel{Q}q)dE(q) \\
&= \iint dE(p)\alpha_{-Qp}(B)\alpha_{Qq}(A)dE(q) \\
&= \int \alpha_{-Qp}(B)dE(p) \int dE(q)\alpha_{Qq}(A)
\end{aligned}$$

i.e. integrated operators still commute!

(iii) Integrals preserve covariance

Isometry group $\mathcal{P}_+^\uparrow$ of $\mathbb{R}^d$: $x \mapsto \lambda x \doteq \Lambda x + a$

$$U(\lambda)U(x) = U(\Lambda x)U(\lambda)$$
$$U(\lambda)dE(p) = dE(\Lambda p)U(\lambda)$$

Thus

$$U(\lambda)\Big(\int \alpha_{Qp}(A)dE(p)\Big)U(\lambda)^{-1}$$
$$= \int \alpha_{\Lambda Q \Lambda^{-1}p}(U(\lambda)AU(\lambda)^{-1})dE(p)\,.$$

Observation [Grosse, Lechner]: Fix standard wedge

$$\mathcal{W}_0 \doteq \{x \in \mathbb{R}^d : x_1 > |x_0|\}$$

and associated $d \times d$–matrix

$$Q \doteq \begin{pmatrix} 0 & -\kappa & \\ \kappa & 0 & \\ & & 0 \end{pmatrix}, \qquad \kappa > 0\,.$$

(a) Let $\lambda \in \mathcal{P}_+^\uparrow$ s.t. $\lambda\mathcal{W}_0 \subset \mathcal{W}_0$; then $\Lambda Q \Lambda^{-1} = Q$.

(b) Let $\lambda' \in \mathcal{P}_+^\uparrow$ s.t. $\lambda'\mathcal{W}_0 \subset {\mathcal{W}_0}'$; then $\Lambda Q \Lambda^{-1} = -Q$.

(c) For $p \in \overline{V}_+$ one has $Qp \in \overline{\mathcal{W}}_0$, *i.e.* $\overline{V}_+$ is "tilted" into $\overline{\mathcal{W}}_0$, and $-Qp \in {\overline{\mathcal{W}}_0}'$.

6. New models

Resulting relations for the tilted integrals admit local and covariant deformations of theories.

Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ such that for the given $\mathcal{W}_0$

(a) $U(\lambda)\mathcal{M}U(\lambda)^{-1} \subset \mathcal{M}$ if $\lambda\mathcal{W}_0 \subset \mathcal{W}_0$

(b) $U(\lambda')\mathcal{M}U(\lambda')^{-1} \subset \mathcal{M}'$ if $\lambda'\mathcal{W}_0 \subset \mathcal{W}_0'$

(c) Ω is cyclic and separating for $\mathcal{M}$

Deformed algebras (Q fixed):

$$\widehat{\mathcal{M}} \doteq \{\widehat{A} : A \in \mathcal{M} \cap \mathcal{C}(\mathcal{H})\}''$$

$$\widehat{A} \doteq \int \alpha_{Qp}(A) dE(p)\,.$$

Proposition: The algebra $\widehat{\mathcal{M}}$ has properties (a) to (c).

Net of algebras for arbitrary wedges $\mathcal{W} = \lambda\mathcal{W}_0$:

$$\mathcal{W} \to \widehat{\mathcal{A}}(\mathcal{W}) \doteq \alpha_\lambda(\widehat{\mathcal{M}})\,.$$

Consistent definition. Net again local and covariant.

Starting from any QFT (e.g. a free theory) one arrives at a new theory!

Remarks on collision states:

If original theory describes a massive particle, then two particle scattering states can be constructed in deformed theory (frame–dependent). Fixing $\mathcal{W}_0$:

$$\widehat{|p,q\rangle}_{in} = e^{\mp ipQq}\,|p,q\rangle_{in}\,, \qquad p_1 \gtrless q_1$$
$$\widehat{|p,q\rangle}_{out} = e^{\pm ipQq}\,|p,q\rangle_{out}\,, \quad p_1 \gtrless q_1\,.$$

Scattering kernel: $p_1 > q_1, p'_1 > q'_1$

$$_{out}\widehat{\langle p,q}|\widehat{p',q'\rangle}_{in} = e^{-ipQq-ip'Qq'}\,{}_{out}\langle p,q|p',q'\rangle_{in}\,.$$

Remarks:

• S–matrix is deformed, thus the deformed net is not isomorphic to the original one.

• S–matrix not Lorentz invariant in $d > 2$ in spite of $\mathcal{W}$–locality and covariance.

Can be understood if one re–interprets the deformed net as a theory on non–commutative Minkowski space [Grosse, Lechner]; breakdown of Lorentz invariance also familiar from theories with long range forces.

• collision cross sections do not change under deformations, *i.e.* observation of breakdown of Lorentz invariance only possible in specific experiments.

Conclusions

- AQFT sheds new light on constructive problems of QFT. Task: construction of $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$
- can be transfered to more general spacetimes
- provides mathematical tools which are not available in other approaches to QFT
- deserves continuing attention