Zimmermann's subtraction scheme and the perturbative solution to R.G. evolution equations

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Abstract

In the framework of Euclidean field theory we show that an infrared safe slightly modified version of Zimmermann's subtraction scheme generates the perturbative solutions to the Wilson-Polchinski renormalization group equations. [1] [2]

[1] More details in: http://www.ge.infn.it/~becchi/prague-2007.pdf

[2] The 1-P.I. R.G. equations: M. Bonini et al., Nucl.Phys.B409 (1993) 441

We consider an Euclidean scalar field theory in 4 dimensions. Introducing the UV-IR cut-off Fourier transformed propagator:

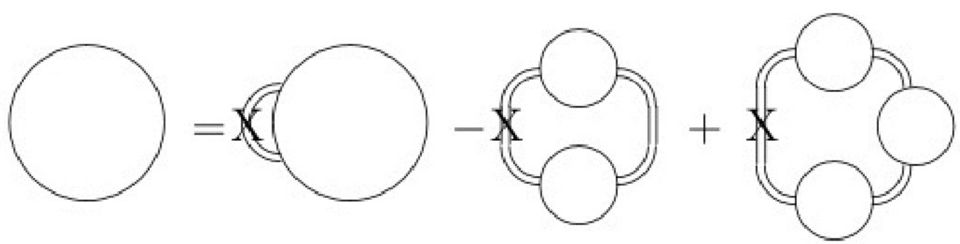
$$\tilde{\hat{S}}(p) = \frac{e^{-\frac{p^2}{\Lambda_0^2}} - e^{-\frac{p^2}{\Lambda^2}}}{p^2}$$

and

$$\Lambda^2 \frac{\partial}{\partial \Lambda^2} \tilde{\hat{S}}(p) \equiv \dot{\tilde{\hat{S}}}(p) = -\frac{e^{-\frac{p^2}{\Lambda^2}}}{\Lambda^2} \ .$$

one defines the 1-P.I Effective Action V_{Λ,Λ_0} whose evolution equation is represented in the figure:

$$\Lambda\partial_\Lambda V_{\Lambda,\Lambda_0} \equiv \Lambda\partial_\Lambda$$



$$+ \cdots \equiv R_{\Lambda,\Lambda_0}$$

where double lines correspond to the propagator $\hat{S}$ and the crossed double one to $\dot{\hat{S}}$ while circles correspond to the 1-PI parts generated by V_{Λ,Λ_0}.

Expanding:

$$V_{\Lambda,\Lambda_0}[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} (dp_i \tilde{\phi}(p_i)) \delta(\sum_{j=1}^{n} p_j) V_n(p_1, \cdot\cdot, p_n, \Lambda, \Lambda_0)$$

and introducing an analogous expansion for $R_{\Lambda,\Lambda_0}[\phi]$ one translates the evolution equation into an infinite system of integral equations for the coefficients:

$$\begin{aligned}
V_2(0,0,\Lambda,\Lambda_0) &= \mu^2 + \int_{\Lambda_R}^{\Lambda} \frac{d\lambda}{\lambda} R_2(0,0,\lambda,\Lambda_0) \\
\partial_{p^2} V_2(p,-p,\Lambda,\Lambda_0)|_{p=0} &= \zeta^2 + \int_{\Lambda_R}^{\Lambda} \frac{d\lambda}{\lambda} \partial_{p^2} R_2(p,-p,\lambda,\Lambda_0)|_{p=0} \\
V_4(0,\cdot\cdot,0,\Lambda,\Lambda_0) &= g + \int_{\Lambda_R}^{\Lambda} \frac{d\lambda}{\lambda} R_4(0,\cdot\cdot,0,\lambda,\Lambda_0) \ ,
\end{aligned}$$

and, for $n+k>4$,

$$\partial_p^k \, V_n(p_1, \cdot\cdot, p_n, \Lambda, \Lambda_0) = \int_{\Lambda_0}^{\Lambda} \frac{d\lambda}{\lambda} \, \partial_p^k \, R_n(p_1, \cdot\cdot, p_n, \lambda, \Lambda_0)$$

If $R_n \sim \Lambda^{4-n} r_n(p/\Lambda)$, up to logs and Λ_0^{-1} corrections, the choice of boundary conditions is unique if V_{Λ,Λ_0} is required to be regular in the $\Lambda_0 \to \infty$ limit.

On the other hand one can show that if

$$\sup |\partial_q^k \, V_n(p_1, \cdot\cdot, p_n, \Lambda, \Lambda_0)| \leq \Lambda^{4-n-k} P_{n,k} \left(p_1, \cdot\cdot, p_n, \log\left(\frac{\Lambda}{\Lambda_R}\right)\right)$$

where $P_{n,k}$ is a polynomial, an analogous bound holds true for every single contribution to $\partial_q^k \, R_n(p_1, \cdot\cdot, p_n, \Lambda, \Lambda_0)$ uniformly in Λ_0.

Thus, at least in a perturbative construction ($\hbar$ ordered) in which at every order $\partial_q^k \, R_n(p_1, \cdot\cdot, p_n, \Lambda, \Lambda_0)$ receives a finite number of contributions one has:

- an analogous bound holds true for $\partial_q^k \, R_n(p_1, \cdot\cdot, p_n, \Lambda, \Lambda_0)$ uniformly in Λ_0.
- the iterative the solutions of the integral equations satisfy the same bound
- the iterative the solutions of the integral equations have a regular U.V. ($\Lambda_0 \to \infty$) limit which coincides with the iterative solution to the integral equations in the U.V. limit.

We want to show that in the $\Lambda_0 \to \infty$ limit an alternative construction to the iterative, loop expanded, solutions of the R.G. integral equations is given by an Euclidean variant of Zimmermann's (Lowenstein-Zimmermann) subtraction method.

The unsubtracted, and hence possibly divergent, Feynman integral corresponding to the diagram Γ contributing to a $2n$ external leg, m loop, Schwinger function $S_{2n}^{(m)}$ has the form:

$$S_\Gamma(p) = \int \frac{d^{4m}k}{(2\pi)^{4m}} I_\Gamma(p,k) \ ,$$

where $k \equiv k_1, ..., k_m$ is a basis of internal momenta of the diagram and $p \equiv p_1, ..., p_{2n-1}$ a basis of external momenta.

$I_\Gamma(p,k)$ is built with the propagator:

$$\tilde{\hat{S}}(p) = \frac{1 - e^{-\frac{p^2}{\Lambda^2}}}{p^2}$$

and vertices

$$(\mu^2\phi^2 + \zeta^2(\partial\phi)^2)/2 \quad , \quad g\phi^4/4!$$

The subtraction procedure consists in replacing $I_\Gamma(p,k)$ with the renowned *forest formula*:

$$R_\Gamma(p,k) \equiv \mathcal{S}_\Gamma \sum_{F\in\mathcal{F}_\Gamma} \prod_{\gamma\in F} (-t^d_\gamma \mathcal{S}_\gamma) I_\Gamma(p.k) \ .$$

where:

- $\mathcal{F}_\Gamma$ is the set of all forests of Γ
- $\mathcal{S}_\gamma$ defines the *momentum routing* of the sub-diagram γ
- t^d_γ takes the $\hat{p}^{(\gamma)}$ Taylor expansion of $I_\gamma(p,k)$ up to degree d_γ, the superficial divergence of γ,
- t^d_γ **replaces** Λ **with** Λ_R **in the propagators**

Notice the analogy with Lowenstein-Zimmermann's scheme.

Let us call $\mathcal{V}_\Lambda[\phi]$ the functional generator of the subtracted 1-P.I. Feynman amplitudes.

We have to show that its coefficient functions $\mathcal{V}_n(p,\Lambda)$ satisfy the above system of integral evolution equations in the limit $\Lambda_0 \to \infty$.

We consider the Λ-derivative of a generic subtracted Feynman integral corresponding to a 1-PI diagram and hence contributing to $\mathcal{V}_\Lambda$.

- Due to the absolute convergence of the momentum integral we are allowed to commute the Λ-derivative with the momentum integration.

- Un-subtracted Feynman integrands depends on Λ only through the propagators $\hat{S}$

- Sub-diagram subtraction terms generated by the Taylor operators $t_\gamma^{d_\gamma}$ are Λ-independent since they are computed at $\Lambda = \Lambda_R$.

For a generic 1-PI diagram Γ one has:

$$R_\Gamma(p,k) = (1 - t_\Gamma^{d_\Gamma})\hat{R}_\Gamma(p,k)$$

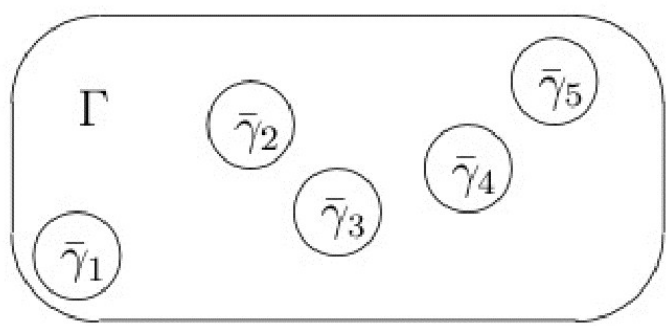
where

$$\hat{R}_\Gamma(p,k) = \mathcal{S}_\Gamma \sum_{F\in\mathcal{F}'_\Gamma} \prod_{\gamma\in F} (-t_\gamma^d \mathcal{S}_\gamma) I_\Gamma(p.k)$$

and $\mathcal{F}'_\Gamma$ is the set of forests not containing Γ as an element.

$$\partial_\Lambda R_\Gamma(p,k) = \partial_\Lambda \hat{R}_\Gamma(p,k) \ ,$$

Let $\bar{F} \in \bar{\mathcal{F}}'_\Gamma$ be a forest with disjoint elements $\bar{\gamma}_i \in \bar{F}$,



It is possible to reorganize the above sum over forests getting:

$$\hat{R}_\Gamma(p,k) = \mathcal{S}_\Gamma \sum_{\bar{F}\in\bar{\mathcal{F}}'_\Gamma} \left[\prod_{\bar{\gamma}\in\bar{F}} ((-t_{\bar{\gamma}}^{d_{\bar{\gamma}}} \mathcal{S}_{\bar{\gamma}}) \hat{R}_{\bar{\gamma}}(p,k)) \right] I_{\Gamma/(\prod_{\bar{\gamma}\in\bar{F}}\gamma)}(p,k)$$

In this equation the *reduced diagram* $\Gamma/(\prod_{\gamma\in\bar{F}}\gamma)$ is built with the lines and vertices of Γ not belonging to any element of $\bar{F}$ and of a further set of vertices corresponding to the elements $\bar{\gamma}$ of $\bar{F}$ shrunk to point vertices.
Then:

$$\begin{aligned}\Lambda^2\partial_{\Lambda^2} R_\Gamma(p,k) \quad &= \mathcal{S}_\Gamma \sum_{\bar{F}\in\bar{\mathcal{F}}'_\Gamma} \left[\prod_{\bar{\gamma}\in\bar{F}} ((-t_{\bar{\gamma}}^{d_{\bar{\gamma}}} \mathcal{S}_{\bar{\gamma}}) \hat{R}_{\bar{\gamma}}(p,k)) \right] \\ &\sum_{l\in L(\Gamma/(\prod_{\bar{\gamma}\in\bar{F}}\bar{\gamma}))} \dot{\hat{S}}(p_l+k_l) I_{\Gamma/(\prod_{\bar{\gamma}\in\bar{F}}\bar{\gamma}\cup l)}(p,k)\end{aligned}$$

where $\Gamma/(\prod_{\gamma\in\bar{F}}\gamma\cup l) = \Gamma/(\prod_{\gamma\in\bar{F}}\gamma)/l$.

Now we interchange the sum over the forests with that over the line l getting:

$$\Lambda^2 \partial_{\Lambda^2} R_\Gamma(p,k) = \sum_{l \in L(\Gamma)} \dot{\hat{S}}(p_l + k_l) \mathcal{S}_\Gamma \sum_{F \in \mathcal{F}_{\Gamma/l}} \prod_{\gamma \in F} (-t^d_\gamma \mathcal{S}_\gamma) I_{\Gamma/l}(p,k) \ .$$

The following remarks are in order:

- If Γ is 1-PI, Γ/l is a chain 1-PI sub-diagrams pairwise connected by lines.
- Thus $I_{\Gamma/l}(p,k)$ factorizes into a chain of line and 1-PI factors γ_i , $i = 0, \cdots n$ closed by the line l.
- A forest F in Γ/l appears as the union of, possibly trivial, forests in the 1-PI factors.

Therefore the sum over the forests in Γ/l decomposes into the product of the sums over the forests in the γ_i's and one has:

$$\Lambda^2 \partial_{\Lambda^2} R_\Gamma(p,k) = \mathcal{S}_\Gamma \sum_{l \in L(\Gamma)} \dot{\hat{S}}\,(p_l + k_l) R_{\gamma_0}(p,k) \prod_{i=1}^{n} \left[\hat{S}(p_i + k_i)) R_{\gamma_i}(p,k) \right] \ .$$

Summing over all the possible diagrams it clearly appears that the structure of the right-hand side of this equation coincides with the chain structure of the right-hand side of the evolution equation of the effective proper generator $V_{\Lambda,\infty}[\phi]$. Furthermore:

- One should verify that the combinatorial factors, starting from $1/2$ in the evolution equation, combine correctly. This is however fairly obvious.
- The forest formula guarantees that the integral equations for the coefficients $\mathcal{V}_2(p,\Lambda)$ and $\mathcal{V}_4(p,\Lambda)$ contain the correct boundary values at Λ_R.
- And a standard analysis shows that $\mathcal{V}_n(p,\Lambda)$ satisfies the bound given above for $|\partial_q^k \ V_n(p_1, \cdot\cdot, p_n, \Lambda, \infty)|$.

In conclusion, comparing the R-G and subtraction approach one has :

- In both cases one is dealing with an infinity of quantities and hence the chosen ordering is crucial.
- The subtraction approach deals with diagrams and hence the resulting amplitudes depend on the ordering of diagrams (loop ordering, ..)
- The R-G integral equations are not strictly related to diagrams, hence a wider class of recursive construction is *in principle* open
- However the right-hand side of the evolution equation is the sum of a series, and such appear the integral equations for the coefficients due to the two point insertions.
- Therefore, either one refers to a perturbative framework, in which the right-hand side is a finite sum,
- Or one has to use, for Λ big enough, precise bounds for the full propagator and for the amplitudes constructed iteratively. This is excluded e.g in 4-d scalar field theories due e.g. to the mass problem.