Scattering Amplitudes of Massive $\mathcal{N} = 2$ Gauge Theories in Three Dimensions

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Outline

I. Invitation to amplitudeology
   - Twistors
   - BDS
   - BCFW recursion relations

II. Mass-deformed $\mathcal{N} = 2$ amplitudes in $d = 3$
   - Mass-deformed Chern-Simons theory (CSM)
   - Yang-Mills-Chern-Simons theory (YMCS)
   - Massive spinor-helicity in $d = 3$
   - Trouble with YMCS external gauge fields
   - On-shell SUSY algebras
   - Four-point amplitudes: superamplitude for CSM
   - Massive BCFW in $d = 3$

III. Conclusions and looking forward
Part I: Amplitudeology
Amplitudeology

- Parke-Taylor formula: massive simplification of amplitudes in “spinor-helicity” variables; Witten’s twistor string theory

- BCFW recursion: $n$-point amplitudes from $n-1$-point amplitudes

- Unitarity methods to construct loop-level amplitudes

- BCJ relations: duality between colour and kinematics

- KLT relations: gauge-theory amplitudes $^2 = gravity$ amplitudes

- Grassmannian formulation

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Penrose’s twistors

- Penrose’s concept of twistors turns out to be an immensely powerful technique for describing massless amplitudes.

- Idea is to coordinatize space by the bundle of light-rays passing through a given point: i.e. by the local celestial sphere.

- Imagine two observers at different places in the galaxy. Knowledge of their celestial spheres is enough to determine their locations:
Twistors

Homogeneous coordinates of $CP^3$:

$$Z_I = (Z_1, Z_2, Z_3, Z_4), \quad Z_I \sim \lambda Z_I, \quad \lambda \in \mathbb{C}.$$ 

For a given twistor $Z_I$, the incidence relation ($\implies$ null condition)

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \sigma^\mu x_\mu \begin{pmatrix} Z_3 \\ Z_4 \end{pmatrix} \implies \text{Im} \left( Z_1 Z_3^* + Z_2 Z_4^* \right) = 0,$$

fixes $x^\mu = (0, \vec{x}_0) + k^\mu \tau$ with $k^2 = 0$, i.e. specifies a single light ray, going through a specific point in space.

Two (or more) twistors $Z_I$ and $Z'_I$ incident to the same point $\vec{x}_0$ specify two (or more) different light rays through that point, i.e. $(0, \vec{x}_0) + k^\mu \tau$ and $(0, \vec{x}_0) + k'^\mu \tau$.

For fixed $\vec{x}_0$, the incidence relation takes $CP^3 \to CP^1 \sim S^2$ which is nothing but the celestial sphere at $\vec{x}_0$. 
Spinor-helicity variables

On-shell massless particle representations

\[ p^{a\dot{a}} = p_\mu (\sigma^\mu)^{a\dot{a}} = \lambda^a \bar{\lambda}^{\dot{a}}, \quad \langle i,j \rangle = \epsilon_{ab} \lambda_i^a \lambda_j^b, \quad [i,j] = \epsilon_{\dot{a} \dot{b}} \bar{\lambda}_i^{\dot{a}} \bar{\lambda}_j^{\dot{b}}, \]

with which the Parke-Taylor formula for MHV tree-level gluon scattering amplitudes is expressed:

\[ \langle -_1, \ldots, -, +_i, -, \ldots, -, +_j, -, \ldots, -, -_n \rangle \propto \delta^4 \left( \sum_{i=1}^{n} \lambda_i^a \bar{\lambda}_i^{\dot{a}} \right) \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle}. \]

Notice that expression is “holomorphic” i.e. does not depend on \( \bar{\lambda} \). Fourier transform w.r.t. \( \bar{\lambda} \) \cite{Witten2003}

\[ \int d^4x \prod_i \frac{d^2 \bar{\lambda}_i}{(2\pi)^2} \exp \left( i \sum_i \mu_{i\dot{a}} \bar{\lambda}_i^{\dot{a}} \right) \exp \left( ix_{a\dot{a}} \sum_i \lambda_i^a \bar{\lambda}_i^{\dot{a}} \right) f(\{\lambda\}) \]

\[ = \int d^4x \prod_i \left[ \delta^2 (\mu_{i\dot{a}} + x_{a\dot{a}} \lambda_i^a) \right] f(\{\lambda\}) \rightarrow \text{define twistor } Z_I = (\lambda^a, \mu_{\dot{a}}). \]

The particles (light rays) interact at a common point in space-time.
In large-N gauge theories we have fields $\phi = \phi^a T^a$, where $T^a$ is (for example) a $SU(N)$ generator. Colour ordering refers to (e.g. for 4-particle scattering)

$$\left\langle \phi^a_1 (p_1) \phi^a_2 (p_2) \phi^a_3 (p_3) \phi^a_4 (p_4) \right\rangle = \mathcal{M} (p_1, p_2, p_3, p_4) \text{Tr}[T^a_1 T^a_2 T^a_3 T^a_4] + \ldots$$

this restricts to the $(p_1 + p_2)^2$ and $(p_1 + p_4)^2$, i.e. adjacent, channels.

In $\mathcal{N} = 4$, $d = 4$ SYM, the MHV amplitudes have a conjectured all-orders form [Bern, Dixon, Smirnov, 2005]

$$\log \frac{\mathcal{M}_{\text{MHV}}}{\mathcal{M}_{\text{tree, MHV}}} = - \sum^n_{i=1} \left[ \frac{1}{8 \epsilon^2} f^{(-2)} \left( \frac{\lambda \mu_{IR}^{2\epsilon}}{(-s_{i,i+1})^\epsilon} \right) + \frac{1}{4 \epsilon} g^{(-1)} \left( \frac{\lambda \mu_{IR}^{2\epsilon}}{(-s_{i,i+1})^\epsilon} \right) \right] + f(\lambda) R^4 \frac{4}{4} + \text{finite.}$$

where $f^{(-n)}(\lambda)$ in the $n$-th logarithmic integral of the cusp anomalous dimension $f(\lambda)$.

IR divergences have been regulated by going above four dimensions, i.e. $d = 4 - 2\epsilon$ with $\epsilon < 0$. 

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Alday & Maldacena taught us that at strong coupling, the dual of the amplitude is the dual of a null-polygonal Wilson loop: i.e. a string worldsheet:

\[
\mathcal{M}_{\text{MHV}}^{\text{tree}} = \left\langle \frac{1}{N} \text{Tr} P \exp \int_C d\tau \, i\dot{x}^\mu A_\mu \right\rangle = \exp \left( -\frac{\sqrt{\lambda}}{2\pi} \text{(Area of Min. Surf.)} \right)
\]

Moreover, the duality holds also at weak coupling [Brandhuber, Heslop, Travaglini, 2007]. Reason: under T-duality \( p_i \leftrightarrow x_{i+1} - x_i \), and AdS is mapped to itself. Amplitude is dual to high energy scattering on an IR brane à la Gross & Mende, T-duality maps it to the null-polygon in the UV, i.e. on the boundary.

The picture which has emerged is that there is a full dual \( PSU(2,2|4) \) symmetry and a Yangian symmetry relating the two [Drummond, Henn, Plefka, 2009].
Recursion relations

\[ A(z) = \]

\[ p_i \rightarrow \tilde{p}_i = p_i + zq, \]
\[ p_j \rightarrow \tilde{p}_j = p_j - zq, \]
\[ \tilde{p}_i^2 = 0 = \tilde{p}_j^2, \]
\[ q^2 = q \cdot p_i = q \cdot p_j = 0. \]

[Britto, Cachazo, Feng, Witten, 2005]

- \( A(z) = \frac{F(z)}{G(z)} \) i.e. amplitude is rational.
- Poles in \( z \) are simple.
- \( \lim_{z \to \infty} A(z) = 0. \)

\[ \Rightarrow A(z) = \sum_p \text{Res}[A(z), z_p]/(z - z_p) \]
Recursion relations cont’d

\[ P = \sum_L p = - \sum_R p = \text{no } z \text{ dep.} \]

\[ P = \sum_L p = - \sum_R p = \text{depends on } z \]

\[ A(z) = \sum_{\text{splittings}} \frac{A_L(z_p)A_R(z_p)}{P^2(z)} \implies A = A(0) = \sum_{\text{splittings}} \frac{A_L(z_p)A_R(z_p)}{P^2}, \]

\[ A_L(z_p) = A(\ldots, p_i(z_p), \ldots, P(z_p)), \quad A_R(z_p) = A(\ldots, p_j(z_p), \ldots, -P(z_p)), \]

\[ P^2(z_p) = 0. \]
A few slides motivating amplitudes in three-dimensions
Three-dimensional theories: ABJM

• Propagating degrees of freedom are scalars and fermions. Results have not been interpreted in terms of “helicity”.

\[ A^\text{tree}_4 = \delta^3(P)\delta^6(Q) / \sqrt{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}, \]

six-particile result also known [Agarwal, Beisert, McLoughlin, 2008], [Bargheer, Loebbert, Meneghelli, 2010].

• BCFW and dual super-conformal invariance [Gang, Huang, Koh, Lee, Lipstein, 2011].

• Extensions to loop-level performed [Chen, Huang, 2011] [Bianchi, Leoni\(^{(2)}\), Mauri, Penati, Santambrogio, 2011] [Caron-Huot, Huang, 2013].

• Yangian constructed [Bargheer, Loebbert, Meneghelli, 2010].

• Grassmanian proposed [Lee, 2010].

• Light-like Wilson loop seems to match amplitudes [Henn, Plefka, Wiegandt, 2010], [Bianchi, Leoni, Mauri, Penati, Santambrogio, 2011].
Three-dimensional theories: $\mathcal{N} = 8$ SYM and ABJM

- Strong coupling IR fixed point of $\mathcal{N} = 8$ SYM is believed to be ABJM.

- Can be seen using M2-to-D2 Higgsing of ABJM.

- On-shell supersymmetry algebras of the two theories (and analogues with less SUSY) may be mapped to each other [Agarwal, DY (2012)].

- One-loop MHV vanish, one-loop non-MHV are finite [Lipstein, Mason (2012)]. – ABJM 1-loop amps either vanish or are finite.

- $\mathcal{N} = 8$ amps have dual conformal covariance [Lipstein, Mason (2012)], ABJM amps have dual conformal invariance.

- 4-pt. 2-loop amplitudes agree in the Regge limit between the two theories [Bianchi, Leoni (2013)].
Mass-deformed three-dimensional theories: $\mathcal{N} \geq 4$

Chern-Simons-Matter amplitudes

[Agarwal, Beisert, McLoughlin (2008)]

- Amplitudes computed at the tree and one-loop level.

- Exploited $SU(2|2)$ algebra to relate amplitudes to one another – same constraints at play in $\mathcal{N} = 4$ SYM spin chains!

Now we will look at massive Chern-Simons-Matter theory with $\mathcal{N} = 2$, and also at another way of introducing mass: Yang-Mills-Chern-Simons theory.
Part II:
Mass-deformed $\mathcal{N} = 2$ amplitudes in $d = 3$
\( \mathcal{N} = 2 \) massive Chern-Simons-matter theory

\[
S_{CSM} = \kappa \int \epsilon^{\mu\nu\rho} \text{Tr}(A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho) \\
- 2 \int \text{Tr} |D_\mu \Phi|^2 + 2i \int \text{Tr} \bar{\Psi}(D_\mu \gamma^\mu \Psi + m \Psi) \\
- \frac{2}{\kappa^2} \int \text{Tr} \left( |[\Phi, [\Phi^\dagger, \Phi]] + e^2 |\Phi|^2 \right) + \frac{2i}{\kappa} \int \text{Tr}([\Phi^\dagger, \Phi][\bar{\Psi}, \Psi] + 2[\bar{\Psi}, \Phi][\Phi^\dagger, \Psi])
\]

- Gauge field is non-dynamical: external states are \( \Phi \)'s and \( \Psi \)'s.

- Mass is set by \( e \): this quantity does not run, \( m = e^2 / \kappa \).

- \( \kappa = k/(4\pi) \), \( k \) is CS level.

- Couplings in potential include \( \Phi^6 \), \( \Phi^4 \), and \( \Phi^2 \Psi^2 \).
$\mathcal{N} = 2$ Yang-Mills-Chern-Simons theory

Chern-Simons theory as a mass-term in $d = 3$ [Deser, Jackiw, Templeton (1982)]:

\[ S_{YM} = \frac{\text{Tr}}{e^2} \int \left[ -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} - D_\mu \Phi D^\mu \Phi + F^2 + i \bar{\Psi}_I \gamma^\mu D_\mu \Psi_I + \epsilon_{AB} \bar{\Psi}_A \Phi, \Psi_B \right], \]
\[ S_{CS} = \frac{m}{e^2} \text{Tr} \int \left[ \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \frac{2i}{3} \epsilon^{\mu\nu\rho} A_\mu A_\nu A_\rho + i \bar{\Psi}_I \Psi_I + 2 F \Phi \right], \]

Magic Arithmetic:

\[
\begin{array}{ccc}
S_{YM} & + & S_{CS} \\
\hline
S_{YMCS} & \Rightarrow & \text{massless} \\
& & + \text{non-dynamical} \\
& & \text{massive}
\end{array}
\]

• Auxilliary field $F$ gives mass term for $\Phi$.

• Fermion mass-term present in $S_{CS}$.

• Gauge field kinetic term is $A^\mu \left( \partial^2 \eta_{\mu\nu} - \partial_\mu \partial_\nu - m \epsilon_{\mu\nu\rho} \partial^\rho \right) A^\nu \Rightarrow$

\[ \Delta_{\mu\nu}(p) = \frac{1}{p^2(p^2 + m^2)} \left( p^2 \eta_{\mu\nu} - p_\mu p_\nu + im \epsilon_{\mu\nu\rho} p^\rho \right). \]
Massive spinor-helicity in $d = 3$

Recall: $p^{\alpha \dot{\alpha}} = \lambda^{\alpha \bar{\lambda} \dot{\alpha}}$ for massless spinors in $d = 4$. The reason for this is that the Lorentz group (up to signature) is $SO(4) \sim SU(2) \times SU(2)$, hence we have $\alpha$ and $\dot{\alpha}$.

- Massive momentum in $d = 3$ also has 3 d.o.f.
- Lorentz group is $SO(3) \sim SU(2)$, thus distinction between $\alpha$ and $\dot{\alpha}$ disappears; extra momentum-component becomes a three-dimensional mass
  \[ p^{\alpha \beta} = \lambda^{\alpha \bar{\lambda} \beta} - im\epsilon^{\alpha \beta}. \]
  \[ p^2 = -m^2, \quad \langle \lambda \bar{\lambda} \rangle = \epsilon_{\beta \alpha} \lambda^{\alpha \bar{\lambda} \beta} = -2im. \]
- Square-bracket from four dimensions is replaced by a barred notation: $\langle ij \rangle$, $\langle i \bar{j} \rangle$, $\langle i j \rangle$, and $\langle i \bar{j} \rangle$. 

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Trouble with external gauge fields in YMCS

In YMCS, the electric field does not commute with itself:

\[ [E^i(x), E^j(x')] \sim \epsilon^{ij} \delta^2(x - x') \]

A usual mode expansion like

\[
A^a_\mu(x) = \int \frac{d^2p}{(2\pi)^2} \frac{1}{\sqrt{2p^0}} \left( \epsilon_\mu(p) a^a_1(p)e^{ip\cdot x} + \epsilon^{*}_\mu(p) a^a_1(p)e^{-ip\cdot x} \right)
\]

does not fit the bill! [Haller, Lim-Lombridas, (1994)].

We learned this the hard way:

- YMCS amplitudes with external gauge fields computed using a standard mode expansion do not respect the SUSY algebra!
Two different theories, two different SUSY algebras

- CSM theory: \( \Phi = \Phi_1 + i\Phi_2, \Psi = \Psi_1 + i\Psi_2, \) \( SO(2) \) R-symmetry

\[
\{Q^\beta J, Q^{\alpha I}\} = \frac{1}{2} (P^{\alpha \beta} \delta^{IJ} + m \epsilon^{\beta \alpha} \epsilon^{JI} R)
\]

- YMCS theory: Real scalar \( \Phi \sim \Phi_2, \) gauge field d.o.f. \( A \sim \Phi_1: \) no R-symmetry although \( \Psi_1 \) and \( \Psi_2 \) do enjoy \( SO(2) \)

\[
\{Q^\beta J, Q^{\alpha I}\} = \frac{1}{2} P^{\alpha \beta} \delta^{IJ}
\]
On-shell SUSY algebras

Solutions to the massive Dirac equation ($\not{p} \bar{\lambda} = i m \bar{\lambda}$, $\not{p} \lambda = -i m \lambda$):

$$\bar{\lambda}(p) = -\frac{1}{\sqrt{p_0 - p_1}} \left( \begin{array}{c} p_2 + i m \\ p_1 - p_0 \end{array} \right), \quad \lambda(p) = \frac{1}{\sqrt{p_0 - p_1}} \left( \begin{array}{c} p_2 - i m \\ p_1 - p_0 \end{array} \right).$$

CSM:

$$Q_I |\Phi_1\rangle = -\frac{1}{2} \bar{\lambda} |\Psi_I\rangle,$$

$$Q_I |\Phi_2\rangle = -\frac{1}{2} \bar{\lambda} \epsilon_{IJ} |\Psi_J\rangle,$$

$$Q_I |\Psi_J\rangle = \frac{1}{2} \delta_{IJ} \lambda |\Phi_1\rangle + \frac{1}{2} \epsilon_{IJ} \lambda |\Phi_2\rangle.$$

YMCS:

$$Q_I |A\rangle = \frac{1}{2} \lambda |\Psi_I\rangle,$$

$$Q_I |\Phi\rangle = -\frac{1}{2} \bar{\lambda} \epsilon_{IJ} |\Psi_J\rangle,$$

$$Q_I |\Psi_J\rangle = -\frac{1}{2} \delta_{IJ} \bar{\lambda} |A\rangle + \frac{1}{2} \epsilon_{IJ} \lambda |\Phi\rangle.$$

CSM theory has $SO(2)$ R-symmetry: $a_{\pm} \equiv (\Phi_1 \pm i \Phi_2)/\sqrt{2}$, $\chi_{\pm} = (\Psi_1 \pm i \Psi_2)/\sqrt{2}$

$$Q_+ |a_+\rangle = -\frac{1}{\sqrt{2}} \bar{\lambda} |\chi_+\rangle, \quad Q_+ |\chi_-\rangle = \frac{1}{\sqrt{2}} \lambda |a_-\rangle,$$

$$Q_- |a_-\rangle = -\frac{1}{\sqrt{2}} \bar{\lambda} |\chi_-\rangle, \quad Q_- |\chi_+\rangle = \frac{1}{\sqrt{2}} \lambda |a_+\rangle,$$

$$Q_- |a_+\rangle = Q_+ |\chi_+\rangle = Q_+ |a_-\rangle = Q_- |\chi_-\rangle = 0.$$
CSM four-point amplitudes

**SUSY algebra is a powerful constraint:**

\[ 0 = Q_- \langle \chi+a+a-a_- \rangle = \lambda_1 \langle a+a+a-a_- \rangle + \bar{\lambda}_3 \langle \chi+a+\chi-a_- \rangle + \bar{\lambda}_4 \langle \chi+a+a-\chi_- \rangle \]
\[ \implies \langle 1\bar{3} \rangle \langle \chi+a+\chi-a_- \rangle = -\langle 1\bar{4} \rangle \langle \chi+a+a-\chi_- \rangle \]

**Including crossing relations, tree-level four-point amplitudes all related to one single amplitude.**

**Can be packaged into two superamplitudes:**

\[ A_{\Phi\Phi\Psi\Psi} = \frac{\langle 24 \rangle}{\langle 32 \rangle} \delta^3(P)\delta^2(Q), \quad A_{\Phi\Psi\Phi\Psi} = \frac{\langle 41 \rangle \langle 4\bar{1} \rangle - \langle 43 \rangle \langle 4\bar{3} \rangle}{\langle 1\bar{2} \rangle \langle 4\bar{1} \rangle} \delta^3(P)\delta^2(Q), \]
\[ P^{\alpha\beta} = \sum_{i=1}^4 \lambda_i^{(\alpha} \bar{\lambda}_i^{\beta)} , \quad Q^\alpha = \sum_{i=1}^4 \lambda_i^\alpha \bar{\eta}_i + \bar{\lambda}_i^\alpha \eta_i , \quad \delta^2(Q) = Q^\alpha Q_\alpha , \]
\[ \Phi = a_+ + \bar{\eta} \chi_+ , \quad \Psi = \chi_- + \eta a_- . \]
YMCS four-point amplitudes

• SUSY algebra less constraining: three-amplitude relations instead of two-amplitude relations:

\[ Q_2 \langle \Psi_1 \Psi_2 A \Psi_1 \rangle = 0 \]
\[ = -\lambda_1 \langle \Phi \Psi_2 A \Psi_1 \rangle - \bar{\lambda}_2 \langle \Psi_1 AA \Psi_1 \rangle + \lambda_3 \langle \Psi_1 \Psi_2 \Psi_2 \Psi_1 \rangle - \lambda_4 \langle \Psi_1 \Psi_2 A \Phi \rangle \]

• Can use the SUSY algebra to obtain all four-point amplitudes with external gauge fields from those without.

• Four-fermion amplitudes:

\[ \langle \chi^{+} \chi^{+} \chi^{-} \chi^{-} \rangle = \langle \chi^{-} \chi^{-} \chi^{+} \chi^{+} \rangle = -\frac{2 \langle 34 \rangle}{u + m^2} \left[ \langle 12 \rangle + im \frac{\langle 42 \rangle}{\langle 41 \rangle} \right] , \]
\[ \langle \chi^{+} \chi^{-} \chi^{-} \chi^{+} \rangle = \langle \chi^{-} \chi^{+} \chi^{+} \chi^{-} \rangle = \frac{2 \langle 41 \rangle}{s + m^2} \left[ \langle 23 \rangle + im \frac{\langle 13 \rangle}{\langle 12 \rangle} \right] . \]
Example of a nastier-looking amplitude:

\[
\langle \chi + AA\chi^- \rangle = -\frac{\langle 41 \rangle\langle 41 \rangle}{\langle 24 \rangle\langle 43 \rangle}\langle \Psi_2 \Psi_2 \Psi_1 \Psi_1 \rangle + \frac{\langle 43 \rangle}{\langle 24 \rangle\langle 43 \rangle}\langle \Psi_1 \Psi_2 \Psi_2 \Psi_1 \rangle - \frac{\langle 41 \rangle\langle 24 \rangle}{\langle 24 \rangle\langle 43 \rangle}\langle \Phi \Phi \chi + \chi^- \rangle
\]

\[
= \frac{1}{\langle 24 \rangle} \left[ -2 \frac{\langle 41 \rangle\langle 23 \rangle}{\langle 31 \rangle} (s + 2m^2) + 2 \frac{\langle 12 \rangle\langle 34 \rangle}{\langle 31 \rangle} (s + 4m^2) 
- im \left( \frac{\langle 32 \rangle\langle 34 \rangle}{\langle 31 \rangle\langle 41 \rangle} (s + 4m^2) \right) \right] \frac{1}{u + m^2}
\]

\[
+ \frac{1}{\langle 43 \rangle} \left[ \frac{\langle 12 \rangle\langle 34 \rangle}{\langle 24 \rangle} (s - u) - 2 \frac{\langle 23 \rangle\langle 41 \rangle}{\langle 24 \rangle} (t + s) - 2 \frac{\langle 23 \rangle\langle 41 \rangle\langle 13 \rangle}{\langle 12 \rangle\langle 34 \rangle} (s + 2m^2)
+ 2im \frac{\langle 13 \rangle\langle 14 \rangle}{\langle 24 \rangle\langle 12 \rangle} (t + s) + im \frac{\langle 23 \rangle}{\langle 12 \rangle} (u - t) \right] \frac{1}{s + m^2}.
\]
**BCFW for massless lines in** \( d = 3 \)

**Recall:** we had a linear shift in \( d = 4 \)

\[
p_i \rightarrow p_i + z q, \quad p_j \rightarrow p_j - z q
\]

this will **not** work in \( d = 3 \)

\[
q = \alpha p_i + \beta p_j + \gamma p_i \land p_j
\]

requiring \( p_i^2 = p_j^2 = 0 \) means requiring \( q \cdot p_i = q \cdot p_j = q^2 = 0 \) but then \( \alpha = \beta = \gamma = 0 \).

**Resolution:** Use a non-linear shift [Gang, Huang, Koh, Lee, Lipstein (2011)]

\[
p_i \rightarrow \frac{1}{2}(p_i + p_j) \pm z^2 q \pm z^{-2} \tilde{q}, \quad q + \tilde{q} = \frac{1}{2}(p_i - p_j)
\]

then \( q^2 = \tilde{q}^2 = q \cdot (p_i + p_j) = \tilde{q} \cdot (p_i + p_j) = 0 \) and \( 2 q \cdot \tilde{q} = -p_i \cdot p_j \) can be solved!

N.B. undeformed case is now \( z = 1 \).
In terms of spinor variables the BCFW shift is expressed as
\[
\begin{pmatrix}
\lambda_i \\
\lambda_j
\end{pmatrix} \rightarrow \begin{pmatrix}
\frac{1}{2} (z + z^{-1}) & i \frac{1}{2} \left( z - z^{-1} \right) \\
-i \frac{1}{2} \left( z - z^{-1} \right) & \frac{1}{2} (z + z^{-1})
\end{pmatrix} \begin{pmatrix}
\lambda_i \\
\lambda_j
\end{pmatrix}.
\]
This can be extended to the massive case just by doing the same to the $\bar{\lambda}$'s:
\[
\begin{pmatrix}
\bar{\lambda}_i \\
\bar{\lambda}_j
\end{pmatrix} \rightarrow \begin{pmatrix}
\frac{1}{2} (z + z^{-1}) & i \frac{1}{2} \left( z - z^{-1} \right) \\
-i \frac{1}{2} \left( z - z^{-1} \right) & \frac{1}{2} (z + z^{-1})
\end{pmatrix} \begin{pmatrix}
\bar{\lambda}_i \\
\bar{\lambda}_j
\end{pmatrix}.
\]
We then can express the recursion relation as
\[
A(z = 1) = -\frac{1}{2\pi i} \sum_{f,j} \oint_{z_{f,j}} \frac{A_L(z)A_R(z)}{\hat{p}_f(z)^2 + m^2} \frac{1}{z - 1},
\]
where $f$ labels splittings, $\hat{p}_f(z)^2 + m^2 = a_f z^{-2} + b_f + c_f z^2$, and $j$ labels its four roots.
Applying BCFW to CSM and YMCS

• The question of applicability has to do with the large-$z$ behaviour of the amplitudes.

• We need $A(z) \to 0$ when $z \to \infty$.

• The YMCS component amplitudes do not have this property.

• The CSM component amplitudes don’t either, but the superamplitude does.

• Thus the CSM theory seems amenable to BCFW recursion.
Future directions

• Compute 6-pt. amplitudes in CSM and see if BCFW gives the same result.

• Explore the theories at loop-level.

• Does there exist a superamplitude expression for YMCS?

• Understand how to compute amplitudes with external gauge fields in YMCS.
Future directions

- Compute 6-pt. amplitudes in CSM and see if BCFW gives the same result.
- Explore the theories at loop-level.
- Does there exist a superamplitude expression for YMCS?
- Understand how to compute amplitudes with external gauge fields in YMCS.

Thanks!