Spin Coherent State through Path Integral & Semi-Classical Physics

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Abstract

In the first part, basic properties of the **Coherent State** of a linear Harmonic Oscillator are described using Schroedinger Wave Mechanics. Then coherent states being a "over-complete" set have been used as a tool for the evaluation of the path integral. After that, spin states just analogous to the coherent state of a Harmonic oscillator are defined which are called **Spin-Coherent State**. Their properties will be described followed by some specific applications. Second part consists of a short introduction to the semi-classical quantization.

Contents

1 Introduction

Classical as well as Quantum mechanics are well developed & well understood, nevertheless the intermediate stage of these two are still not explored satisfactorily. In spite of having a huge difference between the classical philosophy and quantum philosophy, physicists expect there should be a way of making transition from quantum world to classical one and indeed peoples have explored a lot in support of this. **Our main aim of this project is to try to focus on this intermediate area through the study of Coherent state.**

We know coherent states are the quantum states which are mimic to the classical one. Therefore these states are among the best candidates which can help us to reveal our query. Therefore we take coherent states as our basic thing to explore this physics.Apart from this there are many other applications of these coherent states, we will see some of the them.

To understand the connection between the classical mechanics and quantum mechanics it would be better to consider Hamilton-Jacobi formalism since this formalism contains the essential hints of the wave nature of particles which is one of the most fundamental ingredients of quantum mechanics (wave-particle duality). So, to describe the semi-classical physics we will follow the Hamilton-Jacobi formalism.

In the higher spin limit spin states can be identified as coherent states by an appropriate transformation. This is known as spin-coherent states.

2 Coherent States of the Linear Harmonic Oscillator

From the properties of the stationary states of linear harmonic oscillator we know, the expectation values $\langle \mathbf{X} \rangle$ & $\langle \mathbf{P} \rangle$ of the position and momentum of the oscillator are zero in these states. Now, in Classical Mechanics, it is well known that the position x & momentum p are oscillating functions of time, which always remain zero **only if** the energy of the motion is also zero. Also, Correspondence Principle says that Quantum Mechanics(QM) must yield the same results as Classical Mechanics(CM) in the limiting case where the harmonic oscillator has an energy much greater than the quantum energy unit $\hbar\omega$ i.e. in the limit of large quantum number.

Thus we ask the following question : **is it possible to construct quantum mechanical states leading to physical predictions which are almost identical to the classical ones, at least for a macroscopic oscillator?** We shall see these states indeed exist: they are coherent linear superposition of the stationary states $|\phi_n\rangle$. These are called "quasi-classical states" or "coherent states" of the linear harmonic oscillator. So, **a coherent state in QM is a specific kind of quantum state whose dynamics most closely resemble the behaviour of the corresponding classical system**.

The position, the momentum, and the energy of a harmonic oscillator are described in QM by operators which do not commute; they are incompatible physical observables. It is not possible to construct a state in which they are all perfectly well defined, instead we will look for a state such that the mean values $\langle \mathbf{X} \rangle$, $\langle \mathbf{P} \rangle$ and $\langle H \rangle$ are as close as possible to the corresponding classical values for all time. This will lead us to a compromise in which none of these three observables is perfectly known, nevertheless the root-mean-square deviations $\Delta\mathbf{X}, \Delta\mathbf{P}, \Delta H$ are completely negligible in the macroscopic limit.

2.1 Classical Linear Harmonic Oscillator

The classical equations of motion (1D oscillator) :

$$\frac{d}{dt}x(t) = \frac{1}{m}p(t), \qquad \frac{d}{dt}p(t) = -m\omega^2 x(t) \tag{1}$$

Let's introduce the operators :

$$\hat{x(t)} \equiv \beta x(t), \qquad \hat{p(t)} \equiv \frac{1}{\beta\hbar}p(t) \tag{2}$$

where, $\beta = \sqrt{\frac{m\omega}{\hbar}}$.
So, (1) can be written

$$\frac{d}{dt}\hat{x(t)} = \omega p(t), \qquad \frac{d}{dt}\hat{p(t)} = -\omega \hat{x(t)} \tag{3}$$

The classical state of the oscillator is described by $x(t), p(t)$ at t. Now, we will combine and introduce another quantity $\alpha(t)$ defined by :

$$\alpha(t) \equiv \frac{1}{\sqrt{2}}[\hat{x}(t) + \hat{p}(t)] \tag{4}$$

So, Eq(3) can be rewritten as :

$$\frac{d}{dt}\alpha(t) = -i\omega\alpha(t) \tag{5}$$

whose solution is :

$$\alpha(t) = \alpha_0 \exp(-i\omega t) \tag{6}$$

where,

$$\alpha_0 = \alpha(0) = \frac{1}{\sqrt{2}}[\hat{x}(0) + i\hat{p}(0)] \tag{7}$$

Hence,

$$\hat{x}(t) = \frac{1}{\sqrt{2}}[\alpha_0 \exp(-i\omega t) + \alpha_0^* \exp(i\omega t)] \tag{8a}$$

$$\hat{p}(t) = -\frac{i}{\sqrt{2}}[\alpha_0 \exp(-i\omega t) - \alpha_0^* \exp(i\omega t)] \tag{8b}$$

Energy which is constant over time,

$$H = \frac{1}{2m}p(0)^2 + \frac{m\omega^2}{2}x(0)^2 = \hbar\omega|\alpha_0|^2 \tag{9}$$

For a macroscopic oscillator, the energy must be much greater than the quantum $\hbar\omega$, so : $|\alpha_0| \rangle\rangle 1$.

2.2 Conditions Defining Quasi-Classical States

We are looking for a quantum mechanical state for which at every instant the mean values $\langle X \rangle$, $\langle P \rangle$, $\langle H \rangle$ are practically equal to the values x, p & H which correspond to a given classical motion.
QM $\Rightarrow$

$$\hat{X} = \beta X = \frac{1}{\sqrt{2}}(a + a^\dagger) \qquad \hat{P} = \frac{1}{\hbar\beta}P = -\frac{i}{\sqrt{2}}(a - a^\dagger) \tag{10}$$

and,

$$H = \hbar\omega(aa^\dagger + \frac{1}{2}) \tag{11}$$

For an arbitrary state $|\psi(t)\rangle$ the time evolution of the matrix element $\langle a \rangle(t) = \langle \psi(t)|a|\psi(t)\rangle$ is given by:

$$i\hbar\frac{d}{dt}\langle a\rangle(t) = \langle[a,H]\rangle(t) = \omega\langle a\rangle(t) \tag{12}$$

Hence,

$$\langle a\rangle(t) = \langle a\rangle(0)\exp(-i\omega t) \tag{13a}$$

$$\langle a^\dagger\rangle(t) = \langle a^\dagger\rangle(0)\exp(i\omega t) = \langle a\rangle^*(0)\exp(i\omega t) \tag{13b}$$

See, Eq(13) are analogous to the classical Eq(6)!
Substituting (13) in (10) :

$$\langle\hat{X}\rangle(t) = \frac{1}{\sqrt{2}}[\langle a\rangle(0)\exp(-i\omega t) + \langle a\rangle^*(0)\exp(i\omega t)] \tag{14a}$$

$$\langle\hat{P}\rangle(t) = -\frac{i}{\sqrt{2}}[\langle a\rangle(0)\exp(-i\omega t) - \langle a\rangle^*(0)\exp(i\omega t)] \tag{14b}$$

Comparing (14) with (8), we see that, in order to have at all times t :

$$\langle\hat{X}\rangle(t) = \hat{x}(t), \qquad \langle\hat{P}\rangle(t) = \hat{p}(t) \tag{15}$$

it is necessary and sufficient to set, at the instant $t = 0$, the condition :

$$\langle a\rangle(0) = \alpha_0 \tag{16a}$$

The normalized state vector $|\psi(t)\rangle$ of the oscillator must therefore satisfy the condition :

$$\langle\psi(0)|a|\psi(0)\rangle = \alpha_0 \tag{16b}$$

Now, we must require the mean value of the Hamiltonian equal classical energy i.e.,

$$\langle H\rangle = \hbar\omega\langle a^\dagger a\rangle(0) + \frac{\hbar\omega}{2} = \hbar\omega|\alpha_0|^2 \tag{16c}$$

Since for a classical oscillator, $|\alpha_0|$ is much greater than 1, we can neglect $\frac{\hbar\omega}{2}$ with respect to the other term. Then this gives us the 3rd condition on the state vector: $\langle a^\dagger a\rangle(0) = |\alpha_0|^2$ i.e.,

$$\langle\psi(0)|\alpha^\dagger a|\psi(0)\rangle = |\alpha_0|^2 \tag{16d}$$

We shall see that conditions (16b) and (16d) are sufficient to determine the normalized state vector $|\psi(0)\rangle$.

Hence, we require

$$\boxed{\langle\psi(0)|a|\psi(0)\rangle = \alpha_0, \qquad \langle\psi(0)|\alpha^\dagger a|\psi(0)\rangle = |\alpha_0|^2} \tag{17}$$

$$\Updownarrow \tag{18}$$

$$\boxed{\langle\hat{X}\rangle(t) = \hat{x}(t), \qquad \langle\hat{P}\rangle(t) = \hat{p}(t), \qquad \langle H\rangle = E} \tag{19}$$

2.3 Quasi-Classical States are Eigenvectors of the operator a

Let's define an operator

$$b = a - \alpha_0 I \tag{20}$$

Then,

$$b^\dagger b = (a^\dagger - \alpha_0^* I)(a - \alpha_0 I) = a^\dagger a - \alpha_0^* a - a^\dagger \alpha_0 + \alpha_0^* \alpha \tag{21}$$

and the square of the norm of the ket $b|\psi(0)\rangle$ is

$$\langle\psi(0)|b^\dagger b|\psi(0)\rangle = |\alpha_0|^2 - \alpha_0^*\alpha - \alpha\alpha_0^* + \alpha_0^*\alpha = 0 \tag{22}$$

Therefore,

$$b|\psi(0)\rangle = 0 \tag{23a}$$

$$\boxed{a|\psi(0)\rangle = \alpha_0|\psi(0)\rangle} \tag{23b}$$

Conversely, if the normalized vector $|\psi_0\rangle$ satisfies the relation (23b), it is obvious that conditions (17) and (19) are satisfied.
We therefore get the result : **the quasi-classical state, associated with a classical motion characterized by the parameter α_0, is such that $|\psi(0)\rangle$ is an eigenvector of the operator a with eigenvalue α_0.**
Now onwards we shall denote the eigenvector of the operator a with eigenvalue α by $|\alpha\rangle$ i.e.,

$$\boxed{a|\alpha\rangle = \alpha|\alpha\rangle} \tag{24}$$

where, $|\alpha\rangle$ is called the **"Coherent State"** of the oscillator.

2.4 Expansion of $|\alpha\rangle$ in stationary states $|\phi_n\rangle$ of the Hamiltonian

Since $|\phi_n\rangle$ forms a complete state, so

$$|\alpha\rangle = \sum_n c_n(\alpha)|\phi_n\rangle \tag{25}$$

Then,

$$a|\alpha\rangle = \sum_n c_n(\alpha)\sqrt{n}|\phi_{n-1}\rangle \tag{26}$$

Now,

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{27a}$$

$$\Rightarrow a\sum_n c_n(\alpha)|\phi_n\rangle = \alpha\sum_n c_n(\alpha)|\phi_n\rangle \tag{27b}$$

$$\Rightarrow \sum_{n=1}^{\infty} c_n(\alpha)\sqrt{n}|\phi_{n-1}\rangle = \alpha\sum_{n=0}^{\infty} c_n(\alpha)|\phi_n\rangle \tag{27c}$$

$$\Rightarrow \sum_{n=0}^{\infty} c_{n+1}(\alpha)\sqrt{n+1}|\phi_n\rangle = \alpha\sum_{n=0}^{\infty} c_n(\alpha)|\phi_n\rangle \tag{27d}$$

$$\Rightarrow c_{n+1}(\alpha)\sqrt{n+1} = \alpha c_n(\alpha) \tag{27e}$$

$$\Rightarrow \boxed{c_n(\alpha) = \frac{\alpha^n}{\sqrt{n!}}c_0(\alpha)} \tag{27f}$$

It shows that if $c_0(\alpha)$ is fixed, then $c_n(\alpha)$ are also fixed.Let's choose $c_0(\alpha)$ as real, positive and then let's put the normalization condition $\langle\alpha\rangle = 1$ to fix the $c_0(\alpha)$.Hence,

$$\sum_n |c_n(\alpha)|^2 = 1 \tag{28a}$$

$$\Rightarrow |c_0(\alpha)|^2\sum_n \frac{|\alpha|^{2n}}{n!} = 1 \tag{28b}$$

$$\Rightarrow |c_0(\alpha)|^2\exp(|\alpha|^2) = 1 \tag{28c}$$

$$\Rightarrow c_0(\alpha) = \exp(-\frac{|\alpha|^2}{2}) \tag{28d}$$

Therefore,

$$\boxed{|\alpha\rangle = \exp(-\frac{|\alpha|^2}{2})\sum_n \frac{\alpha^n}{\sqrt{n!}}|\phi_n\rangle} \tag{28e}$$

The relation (27f) says that **the probability distribution follows Poisson distribution** i.e.

$$P_n(\alpha) = |c_n(\alpha)|^2 = \frac{|\alpha|^{2n}}{n!}\exp(-|\alpha|^2) \tag{29}$$

2.5 Minimum Uncertainty

Let's calculate $(\Delta X)_\alpha$, $(\Delta P)_\alpha$ and, $(\Delta H)_\alpha$:

$$\langle X\rangle_\alpha = \langle\alpha|X|\alpha\rangle \tag{30a}$$

$$= \langle\alpha\sqrt{\frac{\hbar}{2m\omega}}|(a + a^\dagger)|\alpha\rangle \tag{30b}$$

$$\Rightarrow \langle X\rangle_\alpha = \sqrt{\frac{2\hbar}{m\omega}}Re(\alpha) \tag{30c}$$

and, exactly by similar calculations,

$$\langle P\rangle_\alpha = \sqrt{2m\hbar\omega}Im(\alpha) \tag{31}$$

$$\langle X^2\rangle_\alpha = \frac{\hbar}{2m\omega}[(\alpha + \alpha^*)^2 + 1] \tag{32}$$

$$\langle P^2\rangle_\alpha = \frac{m\hbar\omega}{2}[1 - (\alpha - \alpha)^2] \tag{33}$$

Therefore,

$$(\Delta X)_\alpha = \sqrt{\frac{\hbar}{2m\omega}} \tag{34}$$

$$(\Delta P)_\alpha = \sqrt{\frac{m\hbar\omega}{2}} \tag{35}$$

and consequently,

$$\boxed{(\Delta X)_\alpha(\Delta P)_\alpha = \frac{\hbar}{2}} \tag{36}$$

So, **the coherent state satisfies the minimum uncertainty relation**.

Similarly if we calculate

$$\langle H\rangle_\alpha = \sum_n P_n(\alpha)E_n = \sum_n P_n(\alpha)(n + \frac{1}{2}\hbar\omega) \tag{37a}$$

or, equivalently

$$\langle H\rangle_\alpha = \hbar\omega\langle\alpha|(a^\dagger a + \frac{1}{2})|\alpha\rangle \tag{37b}$$

and,

$$\langle H^2\rangle_\alpha = \hbar^2\omega^2\langle\alpha|(a^\dagger a + \frac{1}{2})^2|\alpha\rangle \tag{37c}$$

$$\boxed{(\Delta H)_\alpha = \hbar\omega|\alpha| \qquad \Rightarrow \qquad \frac{(\Delta H)_\alpha}{\langle H\rangle_\alpha} \approx \frac{1}{|\alpha|}\langle\langle 1} \tag{37d}$$

2.6 Existence of a Unitary Operator $D(\alpha)$

In this section, our aim is to examine whether there is any unitary transformation which transforms the ground state $|\phi_0\rangle$ of the oscillator to the coherent state $|\alpha\rangle = D(\alpha)|\phi_0\rangle$ with $DD^\dagger = D^\dagger D = 1$.Now,

$$|\alpha\rangle = \exp(-\frac{|\alpha|^2}{2})\sum_n \frac{\alpha^n}{\sqrt{n!}}|\phi_n\rangle \tag{38a}$$

$$\Rightarrow |\alpha\rangle = \exp(-|\alpha|^2/2)\sum_n \frac{(\alpha a^\dagger)^n}{n!}|\phi_0\rangle \tag{38b}$$

$$\Rightarrow |\alpha\rangle = \exp(-|\alpha|^2/2)\exp(\alpha a^\dagger)|\phi_0\rangle \tag{38c}$$

Now, if we define $D(\alpha)$ by

$$D(\alpha) \equiv \exp(-|\alpha|^2/2)\exp(\alpha a^\dagger) \tag{38d}$$

then we will get

$$D^\dagger D = \exp(-|\alpha|^2)\exp(\alpha^* a + \alpha a^\dagger)\exp(|\alpha|^2/2) \neq 1 \tag{38e}$$

So to make $D(\alpha)$ unitary we need to multiply (38e) by $\exp(-\alpha^* a - \alpha a^\dagger)\exp(|\alpha|^2/2)$. Therefore, if we choose

$$D(\alpha) \equiv \exp(-|\alpha|^2/2)\exp(\alpha a^\dagger)\exp(-\alpha^* a) \tag{38f}$$

then that can fulfil the requirements. Further, multiplying (38d) by the extra term $\exp(-\alpha^* a)$ does not have any effect on the coherent state $|\alpha\rangle$ since $\exp(-\alpha^* a)|\phi_0\rangle = |\phi_0\rangle$. Therefore,

$$D(\alpha) \equiv \exp(-|\alpha|^2/2)\exp(\alpha a^\dagger)\exp(-\alpha^* a) \tag{38g}$$

$$\Rightarrow D(\alpha) = \exp(-|\alpha|^2/2)\exp(\alpha a^\dagger - \alpha^* a)\exp(|\alpha|^2/2) \tag{38h}$$

$$\Rightarrow D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) \tag{38i}$$

Hence, we can define the unitary operator by

$$\boxed{D(\alpha) \equiv \exp(-|\alpha|^2/2)\exp(\alpha a^\dagger)\exp(-\alpha^* a) = \exp(\alpha a^\dagger - \alpha^* a)} \tag{38j}$$

2.7 Wave Function $\alpha(x)$

Let's see the form of the coherent state in the position basis.

$$\alpha(x) = \langle x|\alpha\rangle = \langle x|D(\alpha)|\phi_0\rangle \tag{39}$$

$$\Rightarrow \alpha(x) = \langle x|\exp(\sqrt{\frac{m\omega}{\hbar}}\frac{\alpha-\alpha^*}{\sqrt{2}}X)\exp(-i\sqrt{\frac{1}{m\omega\hbar}}\frac{\alpha+\alpha^*}{\sqrt{2}}P)\exp(\frac{(\alpha^*)^2-\alpha^2}{4}|\phi_0\rangle \tag{40}$$

$$\Rightarrow \alpha(x) = \exp(\sqrt{\frac{m\omega}{\hbar}}\frac{\alpha-\alpha^*}{\sqrt{2}}x)\exp(\frac{(\alpha^*)^2-\alpha^2}{4})\phi_0(x-\sqrt{\frac{\hbar}{2m\omega}}) \tag{41}$$

$$\Rightarrow \alpha(x) = \exp(\frac{(\alpha^*)^2-\alpha^2}{4})\exp(i\langle P\rangle_\alpha x/\hbar)\phi_0(x-\langle X\rangle_\alpha) \tag{42}$$

where, $\langle X\rangle_\alpha$ & $\langle P\rangle_\alpha$ are given by (30c) and (31) respectively. If we replace the form of the $\phi_0(x)$ then the form of the wave packet associated with the coherent state $|\alpha\rangle$ is given by:

$$\boxed{|\alpha(x)|^2 = \sqrt{\frac{m\omega}{\pi\hbar}}\exp\{-\frac{1}{2}[\frac{x-\langle X\rangle_\alpha}{\Delta X_\alpha}]^2\}} \tag{43}$$

which is a **Gaussian**. This is expected since in the coherent state minimum uncertainty relation is satisfied and, in general it can be shown that **if any system is in a state where this relation $\Delta P\Delta X = \hbar/2$ is satisfied, then the corresponding weve function is Gaussian**.

2.8 Over-Complete Set

Since, $|\alpha\rangle$ are eigenvectors of the non-Hermitian operator a, there is no obvious reason for these states to satisfy orthogonality & closure relations. Let's see whether these state satisfy any closure relation.

Firstly, it is straightforward to show that

$$\boxed{|\langle\alpha|\alpha'\rangle|^2 = \exp(-|\alpha - \alpha'|^2} \tag{44}$$

$$\Downarrow \tag{45}$$

scalar product can never be zero i.e. they do not satisfy orthogonality condition.

However, we will see these satisfy some kind of closure relation. Let's define an operator by

$$\hat{I} \equiv \frac{1}{\pi}\int (dRe(\alpha))(dIm(\alpha))|\alpha\rangle\langle\alpha| \tag{46}$$

Then,

$$\langle\psi_1|\hat{I}|\psi_2\rangle = \sum_{m,n} c_m^* c_n \langle\phi_m|\hat{I}|\phi_n\rangle \tag{47a}$$

$$= \frac{1}{\pi}\sum_{m,n} c_m^* c_n \int \langle\phi_m|(dRe(\alpha))(dIm(\alpha))|\alpha\rangle\langle\alpha|\phi_n\rangle \tag{47b}$$

$$= \frac{1}{2i\pi}\sum_{m,n} c_m^* c_n \int \langle\phi_m|\alpha\rangle\langle\alpha|\phi_n\rangle(d\alpha)(d\bar{\alpha}) \tag{47c}$$

where, $\alpha = Re(\alpha) + iIm(\alpha)$. Using the form of the coherent state

$$\langle\psi_1|\hat{I}|\psi_2\rangle = \frac{1}{2i\pi}\sum_{m,n}\frac{c_m^* c_n}{\sqrt{m!n!}}\int \exp(-|\alpha|^2)\alpha^m\bar{\alpha}^n(d\alpha)(d\bar{\alpha}) \tag{47d}$$

Let, $\alpha = \rho\exp(i\theta)$, or, $d\alpha d\bar{\alpha} = 2i\rho d\rho d\theta$ then (47d) becomes

$$\langle\psi_1|\hat{I}|\psi_2\rangle = \frac{1}{\pi}\sum_{m,n}\frac{c_m^* c_n}{\sqrt{m!n!}}\int_0^\infty \rho d\rho\exp(-\rho^2)\rho^m\rho^n\int_o^{2\pi} d\theta\exp(i(m-n)\theta) \tag{47e}$$

Then simple calculation gives

$$\boxed{\langle\psi_1|\hat{I}|\psi_2\rangle = \sum_n |c_n|^2 = \langle\psi_1|\psi_2\rangle} \tag{47f}$$

Hence,

$$\Downarrow \tag{47g}$$

$$\boxed{\hat{I} \equiv \frac{1}{\pi}\int (dRe(\alpha))(dIm(\alpha))|\alpha\rangle\langle\alpha| = \mathbf{I}} \tag{47h}$$

Therefore, coherent states satisfy closure relation but not orthonormalization condition. These type of set is called an overcomplete set.

2.9 Time Evolution of the Coherent State

Time-independent Hamiltonian $\Rightarrow$
$|\alpha(t)\rangle = \exp(-\frac{|\alpha_0|^2}{2})\sum_n \frac{\alpha_0^n}{\sqrt{n!}}\exp(-i(n+\frac{1}{2})\omega t)|\phi_n\rangle$.
Comparing this result with (28e), we see, to get $|\alpha(t)\rangle$ from $|\alpha_0\rangle$ we just have to change α_0 to $\alpha_0\exp(-i\omega t)$ and multiply the obtained ket by $\exp(-i\omega t)$

Hence,

$$\boxed{|\alpha(t)\rangle = \exp(-i\omega t/2)|\alpha = \alpha_0\exp(-i\omega t)\rangle} \tag{48}$$

Therefore, **a coherent state remains an eigenvector of the annihilation operator a for ALL TIME, with an eigenvalue** $\alpha_0\exp(-i\omega t)$.

Following the above results we obtain:

$$\langle X\rangle(t) = \sqrt{\frac{2\hbar}{m\omega}}Re[\alpha_0\exp(-i\omega t)] \tag{49a}$$

$$\langle P\rangle(t) = \sqrt{2m\hbar\omega}Im[\alpha_0\exp(-i\omega t)] \tag{49b}$$

and, as expected

$$\langle H\rangle = \hbar\omega[|\alpha_0|^2 + \frac{1}{2}] \tag{49c}$$

average energy is time-independent.
Also,simple calculations show that the rms deviations become

$$\boxed{\Delta X = \sqrt{\frac{\hbar}{2m\omega}}, \qquad \Delta P = \sqrt{\frac{m\hbar\omega}{2}}} \tag{50}$$

Hence, ΔX & ΔP **are time-independent i.e. the coherent state wave packet remains a minimum wave packet for all time**. We know, a **free** Gaussian wave packet spreads out with time as it propagates but here we are seeing this Gaussian wave packet, in the parabolic potential(harmonic oscillator potential) it remains a Gaussian without any distortion. Physically, **this happens because the tendency of the wave-packet to spread out is compensated by the parabolic potential, which tries to push the wave packet towards the origin.**

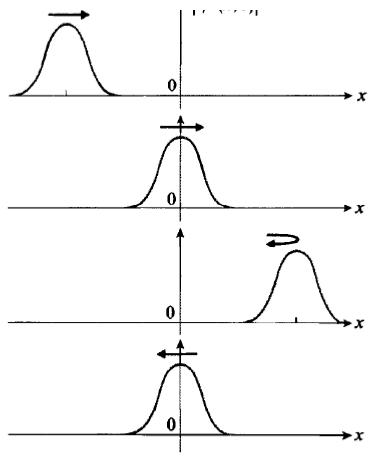

Figure 1: Propagation of the Gaussian Wave Packet in SHO potential

Therefore,

- Coherent States describe a maximal kind of coherence & classical kind of behaviour.
- When $|\alpha|$ is very large then the rms deviations of position, momentum as well as Hamiltonian becomes much smaller than that of the mean values. So, one can obtain a quantum mechanical state where position and momentum can be adjusted with very high accuracy simultaneously which resembles classical motion. Therefore, coherent state describes the motion of a macroscopic harmonic oscillator, for which the position, momentum and the energy can be considered to be classical quantities.
- Vacuum state of the oscillator is also a coherent state.

2.10 A Different way of looking the coherent state

Here, I will try to show an alternative way of getting the properties of the coherent states.

- **General Properties of the two observables whose commutator is a constant**: Let's take the example of position & momentum which satisfy $[Q, P] = i\hbar$. Let's define an operator by

$$S(\lambda) \equiv \exp(\lambda P/i\hbar) \tag{51}$$

with, real λ. Clearly this is unitary operator which is the **consequence of the anti-unitarity of** $\lambda P/i\hbar$. Also,

$$S^{\dagger}(\lambda) = S^{-1}(\lambda) = S(-\lambda) \tag{52a}$$

$$S(\lambda_1)S(\lambda_2) = S(\lambda_1 + \lambda_2) \tag{52b}$$

$$[Q, S(\lambda)] = \lambda S(\lambda) \tag{52c}$$

Now, let's assume $Q|p\rangle = q|q\rangle$, then

$$Q(S(\lambda)|q\rangle) = (q + \lambda)(\lambda)|q\rangle) \tag{52d}$$

So, **starting with an eigenvector of Q, one can construct another eigenvector of Q by applying** $S(\lambda)$**.Also,since** λ **can take any real value, so the spectrum of Q is continuous & composed of all possible values on the real axis**.

Many properties can be obtained, like :

All the eigenvalues of Q have the same degree of degeneracy.

- Now, for the SHO $[a, a^{\dagger}] = 1 = \mathit{constant}$. Following the previous procedure if we define:

$$S(\lambda) = \exp(\lambda a^{\dagger}) \tag{53a}$$

with complex λ, then clearly this is not a unitary operator. This is expected because λa is NOT anti-Hermitian. Following the similar argument as we saw in the section 2.6 it can be proved easily that we will end up with

$$\boxed{S(\lambda) \equiv \exp(\lambda a^\dagger - \lambda^* a)} \tag{53b}$$

Similar to the previous one, if we assume

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{53c}$$

then,

$$a(S(\lambda)|\alpha\rangle) = (\alpha + \lambda)((S(\lambda)|\alpha\rangle) \tag{53d}$$

This shows, **spectrum of a is the whole complex plane. Importantly, all the properties of the Coherent state can be derived following this procedure**, though I won't derive again all the properties following this procedure.

3 Spin Coherent States

Recall the case for the Harmonic Oscillator: though, in general, we can't determine position and momentum simultaneously with arbitrary accuracy, nevertheless in the case of parabolic potential(SHO potential) we can construct a state having dynamics very much analogous to the classical one. For spin angular momentum also we can find such a state,called Spin Coherent State, where average of the spin-angular momentum follow classical like dynamics though three components of the spin angular momentum do not commute.

3.1 Spin States

Let's consider a single particle with spin S.

(a) Defining Ground State:

$$\boxed{|0\rangle \equiv |s, s\rangle} \tag{54a}$$

& so,

$$\Downarrow \tag{54b}$$

$$\boxed{S_z|0\rangle = S_z|s, m_s = s\rangle = m_s|s, s\rangle = s|s, s\rangle} \tag{54c}$$

Let's define:

$$\boxed{S_-|l\rangle \equiv c|l+1\rangle} \tag{55}$$

where, $S_- \equiv S_x - iS_y$, $S_+ \equiv S_x + iS_y$ and $|l\rangle \equiv |s, l\rangle$.

Note: Here we have defined

$$\begin{aligned}
|0\rangle &\equiv |s, s\rangle \equiv |s, m_s = s\rangle \\
S_-|0\rangle &\propto |s, s-1\rangle \equiv |1\rangle \\
(S_-)^2|0\rangle &\propto |s, s-2\rangle \equiv |2\rangle \\
&. \\
(S_-)^{s-1}|0\rangle &\propto |s, s-s+1\rangle \equiv |s-1\rangle \\
(S_-)^{s}|0\rangle &\propto |s, s-s\rangle = |s, 0\rangle \equiv |s\rangle \\
(S_-)^{s+1}|0\rangle &\propto |s, 1\rangle \equiv |s+1\rangle \\
&. \\
(S_-)^{2s}|0\rangle &\propto |s, s-2s\rangle = |s, -s\rangle \equiv |2s\rangle
\end{aligned}$$

So, $|S_-\rangle^p \Rightarrow 0 \le p \le 2s$.

Hence,

$$\begin{aligned}
|c|^2 &= \langle l|S_+S_-|l\rangle \\
&= \langle l|S^2 - S_z^2 + S_z|l\rangle \\
&= s(s+1) - (s-l)^2 + (s-l) \\
\Rightarrow c &= \sqrt{(2s-l)(l+1)}
\end{aligned}$$

where, we have used $S_z|l\rangle = (s-l)|l\rangle$. Hence,

$$\begin{aligned}
S_-|l\rangle &= \sqrt{(2s-l)(l+1)}|l+1\rangle \\
\Rightarrow (S_-)^2|l\rangle &= \sqrt{(2s-l)(l+1)}s_-|l+1\rangle \\
\Rightarrow (S_-)^p|l\rangle &= \sqrt{(2s-l)(l+1)(2s-l-1)(l+2)..(2s-l-p+1)(l+p)}|l+p\rangle
\end{aligned}$$

Putting $l = 0$,

$$\boxed{(S_-)^p|0\rangle = \sqrt{\frac{p!(2s)!}{(2s-p)!}}|p\rangle} \Rightarrow \boxed{(S_-)^p|s,s\rangle = \sqrt{\frac{p!(2s)!}{(2s-p)!}}|s,s-p\rangle} \tag{56}$$

(b) <u>Defining Spin Coherent State</u>: Let's consider a state

$$\begin{aligned}
|\mu\rangle &\equiv N^{-\frac{1}{2}} \exp(\mu S_-)|0\rangle \\
&= N^{-\frac{1}{2}} \sum_p^{\infty} \frac{(\mu S_-)^p}{p!}|0\rangle \\
&= N^{-\frac{1}{2}} \sum_p^{2s} \frac{(\mu^p}{p!} \sqrt{\frac{p!(2s)!}{(2s-p)!}}|p\rangle
\end{aligned}$$

$$\boxed{|\mu\rangle = N^{-\frac{1}{2}} \sum_{p=0}^{2s} \mu^p \sqrt{\frac{(2s)!}{(2s-p)!p!}}|p\rangle} \tag{57}$$

(c) Properties:

- **Finding Normalization Constant:**

$$\begin{aligned}
\langle\mu|\mu\rangle &= N^{-1}\sum_p\sum_q \mu^{*q}\mu^p\sqrt{\frac{(2s)!}{p!(2s-p)!}}\sqrt{\frac{(2s)!}{q!(2s-q)!}}\langle q|p\rangle \\
&= N^{-1}\sum_{p=0}^{2s}|\mu|^{2p}\frac{(2s)!}{p!(2s-p)!} \\
&= N^{-1}\sum_{p=0}^{2s}{}^{2s}C_p(|\mu|^2)^p \\
1 &= N^{-1}(1+|\mu|^2)^{2s}
\end{aligned}$$

$$\boxed{|\mu\rangle = (1+|\mu|^2)^{-s}\exp(\mu S_-)|0\rangle = (1+|\mu|^2)^{-s}\sum_{p=0}^{2s}\mu^p\sqrt{\frac{(2s)!}{p!(2s-p)!}}|p\rangle} \tag{58}$$

- **Inner Product:** Just similar to the last calculation shows

$$\boxed{\langle\lambda|\mu\rangle = \frac{1}{(1+|\lambda|^2)^s(1+|\mu|^2)^s}(1+\lambda^*\mu)^{2s}} \tag{59}$$

and, hence

$$\begin{aligned}
|\langle\lambda|\mu\rangle|^2 &= [\frac{1+\lambda^*\mu+\lambda\mu^*+|\lambda\mu|^2}{(1+|\mu|^2)(1+|\lambda|^2)]}]^{2s} \\
&= [\frac{(1+|\mu|^2)(1+|\lambda|^2)-1-|\lambda|^2-|\mu|^2|\lambda\mu|^2+1+\lambda^*\mu+\lambda\mu^*+|\lambda\mu|^2}{(1+|\mu|^2)(1+|\lambda|^2)}]^{2s}
\end{aligned}$$

$$\boxed{|\langle\lambda|\mu\rangle|^2 = [1-\frac{|\lambda-\mu|^2}{(1+|\mu|^2)(1+|\lambda|^2)}]^{2s}} \tag{60}$$

- **Completeness Relation:** We require,

$$\int d^2\mu|\mu\rangle\langle\mu|m(|\mu|^2) = \sum_0^{2s}|p\rangle\langle p| = 1 \tag{61}$$

where, $m(|\mu|^2) \geqslant 0$ is a weight function and $d^2\mu = (dRe\mu)(dIm\mu)$
Now,

$$\int d^2\mu|\mu\rangle\langle\mu|m(|\mu|^2) = \sum_{p,q}|p\rangle\langle q|\sqrt{\frac{(2s)!}{p!(2s-p)!}}\sqrt{\frac{(2s)!}{q!(2s-q)!}} \int \rho d\rho d\theta(1+\rho^2)^{-2s}m(\rho^2)\rho^{p+q}\exp[i\theta(p-q)] \tag{62}$$

where, we have used the form (58) and assumed $\mu = \rho\exp(i\theta)$.
Now, using the orthogonality condition of the angular variable i.e. $\int_0^{2\pi} d\theta\exp(i\theta(p-q)) = 2\pi\delta_{p,q}$ we get:

$$\int d^2\mu|\mu\rangle\langle\mu|m(|\mu|^2) = \sum_{p=0}^{2s}|p\rangle\langle p|\frac{(2s)!}{p!(2s-p)!}I(p,s) \tag{63}$$

where,

$$I(p,s) \equiv 2\pi\int_0^\infty d\rho\rho\frac{\rho^{2p+1}}{(1+\rho^2)^{2s}}m(\rho^2) \tag{64}$$

Now, if we see the form (63), clearly we need to choose $m(\rho^2)$ in such a way that $I(p,s) = \frac{p!(2s-p)!}{(2s)!}$ to fulfil our requirement of the orthogonality condition (61).
Calculation shows,

$$\boxed{m(|\mu|^2) = \frac{2s+1}{\pi(1+|\mu|^2)^2}} \Rightarrow \boxed{\int d^2\mu|\mu\rangle\langle\mu|m(|\mu|^2) = \sum_0^{2s}|p\rangle\langle p| = 1} \tag{65}$$

This result can be proved by an another method which we will see shortly.

3.2 Analogies with Harmonic Oscillator

For the moment let's write:

$$\boxed{S_- \to \sqrt{2s}a^\dagger} \quad \& \quad \boxed{\mu \to \frac{\alpha}{\sqrt{2s}}} \tag{66}$$

Then, (58) becomes

$$|\mu\rangle \rightarrow |\alpha\rangle_s = (1 + \frac{|\alpha|^2}{2s}) \exp(\alpha a^\dagger)|0\rangle \tag{67}$$

Using $\lim_{x\rightarrow\infty}(1 + \frac{|\alpha|^2}{2s})^{2s} = \exp(\frac{|\alpha|^2}{2})$
we get,

$$\boxed{\lim_{s\rightarrow\infty} |\alpha\rangle_s = \exp(-\frac{|\alpha|^2}{2}) \exp(\alpha a^\dagger)|0\rangle} \tag{68}$$

This is precisely the harmonic oscillator coherent state. So,**higher spin limit and our identification(66) gives us the coherent state**.

3.3 Some Matrix Elements

Let's define an operator,

$$\boxed{A \equiv S - S_z} \tag{69}$$

- Using the form of $|\mu\rangle$ (58) :

$$\begin{aligned}
\langle\mu|A|\mu\rangle &= (1 + |\mu|^2)^{-2s} \sum_{p,q} \mu^{*p}\mu^q \sqrt{\frac{(2s)!}{p!(2s-p)!}} \sqrt{\frac{(2s)!}{q!(2s-q)!}} \langle p|(S - S_z)|n\rangle \\
&= (1 + |\mu|^2)^{-2s} \sum_{p=0}^{2s} |\mu|^{2p} \frac{(2s)!}{p!(2s-p)!} p
\end{aligned}$$

Now,

$$\begin{aligned}
(1 + x)^p &= C_0 + C_1 x + C_2 x^2 + ... + C_p x^p \\
\Rightarrow p(1 + x)^{p-1} &= C_1 + 2C_2 x + 3C_3 x^2 + ... + pC_p x^{p-1} \\
\Rightarrow xp(1 + x)^{p-1} &= C_1 x + 2C_2 x^2 + 3C_3 x^3 + ... + pC_p x^p
\end{aligned}$$

Therefore,

$$\sum_{p=0}^{2s} |\mu|^{2p} \frac{(2s)!}{p!(2s-p)!} p = \sum_{p=0}^{2s} {}^{2s}C_p |\mu|^{2p} p = |\mu|^2 2s(1 + |\mu|^2)^{2s-1} \tag{70}$$

Hence,

$$\boxed{\langle\mu|A|\mu\rangle = \frac{2s|\mu|^2}{1+|\mu|^2}} \tag{71}$$

- We want to calculate $\langle\mu|S_+|\mu\rangle$.
  Now using $S_-|p\rangle = \sqrt{(2s-p)(p+1)}|p+1\rangle$, we get

$$\langle\mu|S_+|\mu\rangle = (1+|\mu|^2)^{-2s}\sum_{p,q}\sqrt{\frac{(2s)!(2s-p)(p+1)}{p!(2s-p)!}}\sqrt{\frac{(2s)!}{p!(2s-q)!}}\mu^{*p}\mu^q\langle p+1|q\rangle$$

$$\Rightarrow \langle\mu|S_+|\mu\rangle = (1+|\mu|^2)^{-2s}\sum_{p=0}^{2s-1}\frac{(2s)!}{p!(2s-p-1)!}\mu^{*p}\mu^{p+1}$$

$$\begin{aligned}\Rightarrow \langle\mu|S_+|\mu\rangle &= (1+|\mu|^2)^{-2s}\sum_{p=0}^{2s-1}\frac{(2s)!}{p!(2s-p-1)!}(|\mu|^2)^p\mu \\ &= (1+|\mu|^2)^{-2s}2s\sum_{p=0}^{2s-1}{}^{2s-1}C_p(|\mu|^2)^p\mu\end{aligned}$$

$$\Rightarrow \boxed{\langle\mu|S_+|\mu\rangle = \frac{2s\mu}{1+|\mu|^2}} \tag{72}$$

Similar type of straightforward calculations show

- $$\langle\lambda|A|\mu\rangle = \frac{2s\lambda^*\mu}{1+\lambda^*\mu}\langle\lambda|\mu\rangle \tag{73}$$

- $$\langle\lambda|S_+|\mu\rangle = \frac{2s\mu}{1+\lambda^*\mu}\langle\lambda|\mu\rangle \tag{74}$$

and

- $$\langle\lambda|S_+|\mu\rangle = \frac{1}{\lambda^*}\langle\lambda|A|\mu\rangle \tag{75}$$

3.4 An Alternative Parametrization

We will introduce another useful parametrization (θ, ϕ) instead of (μ_R, μ_I) by the relation

$$\boxed{\mu = \exp(i\phi)\tan(\theta/2)} \tag{76}$$

with, $0 < \theta < \pi$ & $0 \leqslant \phi \leqslant 2\pi$.

So, $|\mu|^2 = \tan^2(\theta/2)$ $1 + |\mu|^2 = \sec^2(\theta/2)$.Hence,

$$\boxed{|\mu\rangle = (1 + |\mu|^2)^{-s}\exp(\mu S_-)|0\rangle} \tag{77}$$

$$\Downarrow \tag{78}$$

$$\boxed{|\theta, \phi\rangle \equiv |\Omega\rangle = \cos^{2s}(\theta/2)\exp[\tan(\theta/2)\exp(i\phi)S_-]|0\rangle} \tag{79}$$

- **Completeness Relation:** Using this:

$$d\mu_R d\mu_I = \begin{vmatrix} \frac{\partial\mu_R}{\partial\theta} & \frac{\partial\mu_I}{\partial\phi} \\ \frac{\partial\mu_I}{\partial\theta} & \frac{\partial\mu_I}{\partial\phi} \end{vmatrix} d\theta d\phi$$

we get,

$$d^2\mu = \frac{1}{2}\sec^2(\theta/2)\tan(\theta/2)d\theta d\phi \tag{80}$$

$$\boxed{\frac{2s+1}{\pi}\int d^2\mu\frac{1}{(1+|\mu|^2)^2}|\mu\rangle\langle\mu|} \tag{81}$$

$$\Downarrow \tag{82}$$

$$\boxed{(2s+1)\int d\Omega\frac{1}{4\pi}|\Omega\rangle\langle\Omega| = 1} \tag{83}$$

- **Geometrical Interpretation:**

  The point μ is the projection onto the $\mu-$plane of the point (θ, ϕ) on the sphere from the opposite pole.

- **Some Relations In Terms of New Parameters:**

  Using the relation (59)

$$\begin{aligned}
\langle \Omega' | \Omega \rangle \equiv \langle \lambda | \mu \rangle &= \frac{1}{(1+|\lambda|^2)^s (1+|\mu|^2)^s} (1+\lambda^* \mu)^{2s} \\
&= [\cos(\theta/2)\cos(\theta'/2) + \sin(\theta/2)\sin(\theta'/2)\exp(i(\phi-\phi'))]^2 s \\
&\equiv (z_1+z_2)^{2s} \\
&= ((z_1+z_2)(z_1^*+z_2^*))^s \\
&. \\
&. \\
|\langle \Omega' | \Omega \rangle| &= (\frac{1+\hat{r}.\hat{r}'}{2})^s
\end{aligned}$$

where,

$$\hat{r}(\theta,\phi) = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k} \tag{84}$$

is the unit vector in the direction specified by (θ, ϕ).
Simple calculations show:

- $$\begin{aligned}
\langle \Omega | A | \Omega \rangle &= s(1-\cos\theta) \\
\langle \Omega | S_+ | \Omega \rangle &= s\sin\theta\exp(i\phi) \\
\langle \Omega | S_- | \Omega \rangle &= s\sin\theta\exp(-i\phi)
\end{aligned}$$

From these relations it's clear that

$$\begin{aligned}
\langle S_z \rangle &= s\cos\theta \\
\langle S_x \rangle &= s\sin\theta\cos\phi \\
\langle S_y \rangle &= s\sin\theta\sin\phi
\end{aligned}$$

In general,

$$\boxed{\langle \Omega | S | \Omega \rangle = s\hat{r}} \tag{85}$$

3.5 The Effect of Changing the Ground State

From (77)

$$|\mu\rangle = (1+|\mu|^2)^{-s}\exp(\mu S_-)|0\rangle \equiv B(\mu)|0\rangle \tag{86}$$

where, $S_z|0\rangle = s|0\rangle$. Now, if we make a rotate our quantization axis z to a new z', then we **demand**

$$|a\rangle' = B'(a)|0'\rangle \tag{87}$$

where, $S_{z'}|0'\rangle = s|0'\rangle$ and $B'(a) \equiv (1+|b|^2)^{-s}\exp(aS'_-)$

That means,

$$\boxed{|\mu\rangle = B(\mu)|0\rangle} \Rightarrow \boxed{|a\rangle' = B'(a)|0'\rangle} \tag{88}$$

Here,changing of basis representation: $\{\mu, \lambda, ..., 0\} \Rightarrow \{a, b, ..., 0'\}$.

AIM: Expressing $|a\rangle'$ in terms of $|\mu\rangle$.

Using the completeness property of $\{|\mu\rangle\}$ we get,

$$|a\rangle' = \frac{2s+1}{\pi}\int \frac{d^2\mu}{(1+|\mu|^2)^2}|\mu\rangle\langle\mu|a\rangle' \tag{89}$$

Therefore, clearly **our aim boils down to calculate the probability amplitude** $\langle\mu|a\rangle'$.

Let a unitary rotation operator which carries $|0\rangle$ to $|0\rangle'$ be denoted by R, s.t.

$$\boxed{|0\rangle' = R|0\rangle} \tag{90}$$

Now, eigenvalues should not change with our change of basis.Hence,

$$\begin{aligned} {}'\langle 0|S_{z'}|0\rangle' &= \langle 0|S_z|0\rangle \\ \Rightarrow \langle 0|R^\dagger S_{z'}R|0\rangle &= \langle 0|S_z|0\rangle \\ \Rightarrow R^\dagger S_{z'}R &= S_z \end{aligned}$$

Hence,

$$\boxed{S_{z'} = RS_zR^\dagger} \quad \& \quad \boxed{(S_+)' = RS_+R^\dagger} \tag{91}$$

Using above relations:

$$\boxed{|a\rangle' = R|a\rangle} \tag{92}$$

So, **the rotation operator transforms $|a\rangle$ to $|a\rangle'$.** Hence, we need to calculate

$$\langle\mu|a\rangle' = \langle\mu|R|a\rangle' \tag{93}$$

From **Euler's Theorem**,

$$\boxed{R = \exp(-i\alpha S_z)\exp(-i\beta S_y)\exp(-i\gamma S_z)} \tag{94}$$

where, α, β, γ are Euler's angles describing the rotation.

Now, expressing $\langle\mu|, |a\rangle$ in terms of $\langle p|, |p'\rangle$

$$\langle\mu|R|a\rangle = (1+|\mu|^2)^{-s}(1+|a|^2)^{-s}\sum_{p,p'=0}^{2s}\sqrt{\frac{(2s)!}{p!(2s-p)!}}\sqrt{\frac{(2s)!}{p'!(2s-p')!}}\mu^{*p}a^{p'}\langle p|R|p'\rangle$$

$$\begin{aligned}\Rightarrow \langle\mu|R|a\rangle = (1+|\mu|^2)^{-s}(1+|a|^2)^{-s}\sum_{p,p'=0}^{2s}\sqrt{\frac{(2s)!}{p!(2s-p)!}}\sqrt{\frac{(2s)!}{p'!(2s-p')!}}\\ \mu^{*p}a^{p'}\langle p|\exp(-i\alpha S_z)\exp(-i\beta S_y)\exp(-i\gamma S_z)|p'\rangle\end{aligned} \tag{95}$$

$$\begin{aligned}\Rightarrow \langle\mu|R|a\rangle = (1+|\mu|^2)^{-s}(1+|a|^2)^{-s}\sum_{p,p'=0}^{2s}\sqrt{\frac{(2s)!}{p!(2s-p)!}}\sqrt{\frac{(2s)!}{p'!(2s-p')!}}\\ \mu^{*p}a^{p'}\exp\{-i\alpha(s-p)\}\exp\{-i\gamma(s-p')\}\langle p|\exp(-i\beta S_y)|p'\rangle\end{aligned} \tag{96}$$

where, we have used $S_z|p\rangle = (s-p)|p\rangle \equiv (s-p)|s, s-p\rangle$ which follows from (56).

$$\begin{aligned}\Rightarrow \langle\mu|R|a\rangle = (1+|\mu|^2)^{-s}(1+|a|^2)^{-s}\sum_{p,p'=0}^{2s}\sqrt{\frac{(2s)!}{p!(2s-p)!}}\sqrt{\frac{(2s)!}{p'!(2s-p')!}}\\ \mu^{*p}a^{p'}\exp\{-i\alpha(s-p)\}\exp\{-i\gamma(s-p')\}\\ \sum_t(-1)^t\frac{\sqrt{(2s-p)!p!(2s-p')!p'!}}{(2s-p-t)!(p'-t)!t!(t+p-p')!}\cos^{2s+p'-p-2t}(\beta/2)\sin^{2t+p-p'}(\beta/2)\end{aligned} \tag{97}$$

Now, **let's evaluate the same thing i.e.** $\langle \mu | R | a \rangle$ **in the new** (θ, ϕ)**basis which is** $\langle \theta, \phi | R | \theta', \phi' \rangle$. To do this we have to use the pre-defined transformation :

$$\mu \equiv \exp(i\phi) \tan(\theta/2), \qquad a \equiv \exp(i\phi') \tan(\theta'/2) \tag{98}$$

Then after doing the calculation we get

$$\begin{aligned}\langle \theta, \phi | R | \theta', \phi' \rangle = {} & (\cos(\beta/2) \cos(\theta/2) \cos(\theta'/2))^{2s} \exp\{-i(\alpha + \gamma)\} \\ & \sum_{p,p'=0}^{2s} \sum_{t} \frac{(2s)!}{(2s - p - t)!(p' - t)!t!(t + p - p')!} \\ & \tan^{2t+p-p'}(\beta/2) \tan^{p}(\theta/2) \tan^{p'}(\theta'/2) \exp\{-ip()\phi - \alpha\} \\ & \exp\{ip'(\phi' + \gamma)\}\end{aligned} \tag{99}$$

After performing a bit long calculation we get finally

$$\begin{aligned}\langle \theta, \phi | R | \theta', \phi' \rangle = {} & \exp(-i\alpha s) \exp(-i\gamma s)(\cos \beta/2 \cos \theta/2 \cos \theta'/2)^{2s} \\ & [1 + \tan \theta/2 \exp\{-i(\phi - \alpha)\} \tan \theta'/2 \exp\{i(\phi' - \gamma)\} - \tan \theta'/2 \\ & \exp\{i(\phi' + \gamma)\} \tan \beta/2 + \tan \theta/2 \exp\{-i(\phi - \alpha)\} \tan \beta/22]^{2s}\end{aligned} \tag{100}$$

So, (97) & (100) are same thing in two different basis.
See if we put R as identity operator and equate Euler angles to zero($\alpha = \beta = \gamma = 0$) then we get,

$$\langle \theta, \phi | \theta', \phi' \rangle = [\cos \theta/2 \cos \theta'/2 + \sin \theta/2 \sin \theta'/2 \exp\{i(\phi' - \phi)\}]^{2s} \tag{101}$$

which is in agreement with our expectation.

3.6 Some Applications

(a) **Partition Function for a Single Spin in a Magnetic Field:**

With a suitable choice of the zero of the energy, we can write the partition function in the form

$$Z = Tr\{\exp(-Ah)\} \equiv \sum_{p=0}^{2s} \exp(-ph) = \frac{(\exp{(-h)})^{2s+1} - 1}{\exp{(-h)} - 1} \tag{102}$$

where, $h \equiv \gamma_m H/k_B T$, γ_m is particle's magnetic moment, H is the magnetic field.

Now, it can be written as

$$\begin{aligned} Z &= (2s+1)\int \frac{d\Omega}{4\pi}[\frac{1}{2}(1+e^{-h}) + \frac{1}{2}(1-e^{-h})\cos\theta]^{2s} \\ &= (2s+1)\int \frac{d\Omega}{4\pi}\langle\Omega|\exp(-\beta H)|\Omega\rangle \end{aligned}$$

Now, mean value of $\langle A\rangle = -Z^{-1}\frac{\partial Z}{\partial H}$, then after performing this calculation we find

$$\begin{aligned} \langle A\rangle &= \frac{2s(2s+1)}{Z}\int \frac{d\Omega}{4\pi}(\frac{\sin^2(\theta/2)e^{-h}}{\cos^2(\theta/2) + \sin^2(\theta/2)e^{-h}})[\frac{1}{2}(1+e^{-h}) + \frac{1}{2}(1-e^{-h})\cos\theta]^{2s} \\ &= (2s+1)\int \frac{d\Omega}{4\pi}\langle p(\Omega)\rangle\langle\Omega|\hat{\rho}|\Omega\rangle \end{aligned}$$

where, $\hat{\rho} \equiv \exp(-\beta H)/Z$ is the density matrix and $p(\Omega) = (\frac{2s\sin^2(\theta/2)e^{-h}}{\cos^2(\theta/2)+\sin^2(\theta/2)e^{-h}})$

Now, see in the limit $\hbar \to 0$ we get

$$\langle A\rangle = \frac{2s(2s+1)}{Z}\int \frac{d\Omega}{4\pi}\sin^2(\theta/2) = ... = s \tag{103}$$

and, in the limit $\hbar\rangle\rangle 1$ we get $\langle A\rangle \to e^{-h}$.

(b) **Ferromagnetic Spin Waves:**

The ground state $|0\rangle$ of the ferromagnetic has $p_i = 0$ for all spins i $(i = 1, 2, ..., N)$.In the $\{|\mu\rangle\}$ representation

$$|0\rangle = \int dM(\mu)|\mu\rangle\langle\mu|0\rangle \tag{104}$$

where $|\mu\rangle$ is shorthand for $|\mu_1, \mu_2, ..., \mu_N$ and $\int dM(\mu)$ for the expression

$$(\frac{2s+1}{\pi})^N \prod_{i=1}^{N} \int d^2\mu_i (1+|\mu_i|^2)^{-2} \quad (105)$$

The wavefunction of the ground state in the $\{|\mu\rangle\}$ representation is

$$\langle 0|\mu\rangle \equiv \phi_0(\mu) = \prod_i (1+|\mu_i|^2)^{-s} \quad (106)$$

In the p representation a state containing a single spin wave of wave-vector $|k\rangle$ is given by

$$|k\rangle = N^{-\frac{1}{2}} \sum_i \exp(i\vec{k}.\vec{R}_i)|0,0,...,p_i=1,0,0..\rangle \quad (107)$$

Hence, in the μ space we have

$$\begin{aligned} |k\rangle &= \int dM(\mu)|\mu\rangle\langle\mu|k\rangle \\ &= \sqrt{2s}\int dM(\mu)|\mu\rangle\{N^{-(1/2)}\sum_i \exp(i\vec{k}.\vec{R}_i)\mu_i^*\}\langle\mu|0\rangle \end{aligned}$$

So, the amplitude of the spin-wave state in μ space is

$$\phi_k(\mu) = (2s)^{(1/2)} N^{(-1/2)} \exp(i\vec{k}.\vec{R}_i)\mu_i^*\phi_0(\mu) \equiv \mu_k^*\phi_0(\mu) \quad (108)$$

(c) **Two Spin 1/2 Particles Interacting via the Heisenberg Hamiltonian**

Hamiltonian:

$$H = -2JS_1.S_2 \quad (109)$$

It can be shown that the diagonal elements of the density operator $\rho \equiv \exp(-\beta H)/Tr(\exp(-\beta H))$ [where $\beta = \frac{1}{k_B T}$] are

$$\langle\mu_1\mu_2|\rho|\mu_1\mu_2\rangle = \frac{1}{3e^{2\beta J}+1}\frac{1}{1+|\mu_1|^2}\frac{1}{1+|\mu_2|^2}$$
$$\{e^{2\beta J}(1+\frac{1}{2}|\mu_1+\mu_2|^2+|\mu_1\mu_2|^2)+\frac{1}{2}|\mu_1-\mu_2|^2\} \quad (110)$$

After doing some manipulation, we end up with

$$\Rightarrow \langle\mu_1\mu_2|\rho|\mu_1\mu_2\rangle = \frac{1}{3+e^{-2\beta J}}(1-\frac{(1-e^{-2\beta J})|\mu_1-\mu_2|^2}{2(1+|\mu_1|^2)(1+|\mu_2|^2)}) \quad (111)$$

In terms of the angular variables(θ,ϕ) it turns out to be

$$\langle\Omega_1\Omega_2|\rho|\Omega_1\Omega_2\rangle = \frac{1}{4}(1+\frac{1-e^{-2\beta J}}{3+e^{-2\beta J}}\hat{r_1}.\hat{r_2}) \quad (112)$$

These show:

(i) For β J $\rangle 0$, i.e., ferromagnetic coupling, the spins are correlated and tend to align parallel(i.e. with $\hat{r_1}.\hat{r_2} > 0$)

(ii) For β J $\langle 0$, i.e., anti-ferromagnetic coupling, the spins are tend to align anti-parallel(i.e. with $\hat{r_1}.\hat{r_2} < 0$)

(iii) The density matrix is invariant under rotations.

Although we have shown here some simple problems but this view point may be more useful for some non-trivial problems.

4 Semi-Classical Quantization of Integrable Systems

Though CM as well as QM are well explored but the intermediate state of these two fields are still not clearly understood. The Semi-classical quantization mainly deals with the connection between the classical and quantum mechanical equations of motion. Our aim of this section will be as follows:

- We will review the Canonical Transformation,Hamilton-Jacobi formalism of CM and then see the similarity with Schroedinger Wave Equation.

- WKB approximation for 1D quantum problems and the generalization to the higher dimensions.
- Torus Quantization
- Speciality of the closed periodic orbits in quantization.

Let's start with a short review on the basics.

4.1 Canonical Transformation(CT)

- A canonical transformation is one which preserves the the Poisson bracket of any two functions defined over the phase space i.e.

$$\{f, g\}_{(q,p)} = \{f, g\}_{(Q,P)} \tag{113}$$

  The consequence of this definition of the canonical transformation turns out to be the condition as

$$MJM^T = J \tag{114}$$

  where, M is the matrix of the general transformation on phase space and J is the Symplectic matrix.

- In a canonical theory, one may take any two of the variables from (q,p,Q,P) as the independent ones. $F_2(q, P)$ is one of the generating functions of the CT with the choice of (q,P) as the independent ones. This satisfies

$$\frac{\partial F_2}{\partial q} = p(q, P) \qquad \& \qquad \frac{\partial F_2}{\partial P} = Q(q, P) \tag{115}$$

  This $F_2(q, P)$ is related to another generating function $F_1(q, Q)$ of the CT through the following Legendre Transformation

$$F_1(q, Q) = F_2(q, P) - PQ \tag{116}$$

  and, It generates through the following equations:

$$\frac{\partial F_1}{\partial q} = p(q, Q) \qquad \& \qquad \frac{\partial F_1}{\partial Q} = -P(q, Q) \tag{117}$$

- In the extended phase space(EPS) (p, q, t)

$$(pdq - Hdt) - (PdQ - H'dt) \tag{118}$$

is a total differential in the EPS for any CT $(p, q) \rightarrow (P, Q)$. Hence, for an arbitrary closed contour C in the phase space(for an autonomous system)

$$\oint (pdq - PdQ) = \oint (pdq + QdP) = 0 \tag{119}$$

- According to Miller, the significance of the generating functions can be explored easily if we consider the classical limit of QM using Dirac's transformation theory. We know,

$$\langle q|p\rangle = \frac{1}{2\pi\hbar}\exp(ipq/\hbar)$$
$$\langle Q|P\rangle = \frac{1}{2\pi\hbar}\exp(iPQ/\hbar)$$

Let, the form of the wave functions in the mixed representation are

$$\langle q|Q\rangle = A_1(q, Q)\exp(if_1(q, Q)/\hbar)$$
$$\langle q|P\rangle = A_2(q, P)\exp(if_2(q, P)/\hbar)$$

with real amplitude and phase.

Our **aim** is to find the semi-classical limit of the above wave-functions as $\hbar \rightarrow 0$. Using completeness relation

$$\delta(P - P') = \int_{-\infty}^{+\infty} dq\langle P|q\rangle\langle q|P'\rangle$$
$$= \int_{-\infty}^{+\infty} dqA_2(q, P)A'_2(q, P')\exp\{i[f'_2(q, P') - f_2(q, P)]/\hbar\}$$

In the limit $\hbar \rightarrow 0$, the exponent varies rapidly, and the dominant contribution to the integral comes only from P' in the nbd of P.Now,

$$f_2'(q,P') - f_2(q,P) \simeq \frac{\partial f_2(q,P)}{\partial P}(P'-P) \equiv \omega(q,P)(P'-P)$$

Putting this in the previous expression we get,

$$|A_2(q,P)| = \frac{1}{\sqrt{2\pi\hbar}}|\frac{\partial^2 f_2(q,P)}{\partial P \partial q}|^{1/2} \tag{120}$$

Now, consider $\langle q|Q\rangle$:

$$\begin{aligned}\langle q|Q\rangle &= \oint_{-\infty}^{\infty} dP \langle q|P\rangle\langle P|Q\rangle \\ &= \frac{1}{\sqrt{2\pi\hbar}}\oint_{-\infty}^{\infty} dP A_2(q,P)\exp\{i[f_2(q,P) - PQ]/\hbar\}\end{aligned}$$

In the limit $\hbar \to 0$ if we use saddle point approximation then,

$$\boxed{[\frac{\partial f_2(q,P)}{\partial P}]_{P=\tilde{P}} = Q} \tag{121}$$

where, $\tilde{P}$ is the saddle point.
clearly if we see (115) then clearly **at the saddle point the function $f_2(q,P)$ appearing in the quantum wave function obeys the same partial differential equation as the classical generator** $F_2(q,P)$. This equivalence can be proved for the variable q also. Therefore,**we may identify $f_2(q,P)$ with the generator $F_2(q,P)$ in the limit** $\hbar \to 0$ and write (120) as

$$|A_2(q,P)| = \frac{1}{\sqrt{2\pi\hbar}}|\frac{\partial^2 F_2(q,P)}{\partial P \partial q}|^{1/2} \tag{122}$$

So, we get the stetement of the **Correspondence Principle** as

$$\boxed{\langle q|P\rangle = \frac{1}{\sqrt{2\pi\hbar}}|\frac{\partial^2 f_2(q,P)}{\partial P \partial q}|^{1/2}\exp(if_2(q,P)/\hbar)} \tag{123}$$

$$Classical limit \Downarrow \hbar \to 0 \tag{124}$$

$$\boxed{\langle q|P\rangle = \frac{1}{\sqrt{2\pi\hbar}}|\frac{\partial^2 F_2(q,P)}{\partial P \partial q}|^{1/2}\exp(iF_2(q,P)/\hbar)} \tag{125}$$

4.2 Hamilton-Jacobi Formalism & the Classical Limit

In spite of the remarkable difference between the classical & quantum philosophy, Hamilton-Jacobi formalism of classical mechanics can give us a possible connection between classical mechanical equations of motion and that of QM. Also, we will see WKB approximation in QM is basically a semi-classical approximation which matches with H-J equations.

- **Time-dependent HJ Equation:** We ask the following question that could there be some CT which makes the dynamics trivial, namely is there any (Q, P) in terms of which $H' = 0$?
  Then $Q = constant$ & $P = constant$ are the equations of motion, which is of course trivial.
  Cosidering the generating function of such kind namely $F_1(q, Q, t) = R(q, Q, t)$ the,

$$\begin{aligned} H' &= H + \frac{\partial F_1}{\partial t} \\ \Rightarrow 0 &= H + \frac{\partial R}{\partial t} \\ \Rightarrow & \boxed{H(q, \frac{\partial R}{\partial q}) = -\frac{\partial R}{\partial t}} \end{aligned}$$

  which is called the Hamilton-Jacobi equation(HJ).Here, R is called the Hamilton's Principal Function.

- **Time-independent form of the HJ Equations of Motion(EqM):** If the Hamiltonian doesn't have explicit time dependence then we can perform the separation of variables through the relation

$$R(q, t) = S(q, E) - Et \tag{126}$$

  where E is a constant of motion.Then we get the time-independent HJ EqM as

$$\Rightarrow \boxed{H(q, \frac{\partial S}{\partial q}) = E} \tag{127}$$

- **Wave Nature of the HJ EqM:** The HJ EqM actually describes a whole family of orbits in the configuration space. For eg. in a two dimensional configuration space described by two coordinates (q_1, q_2), $S(q_1, q_2, E) = S_i$ is satisfied by a curve on the configuration space(with E=constant).Trajectory of a particle, starting on an initial point on S_i, is along a ray that is perpendicular to these curves of constant S, since its direction of motion is determined by $\dot{q} = p/m = (\nabla_q S)/m$. In this sense there is an implicit wave picture in the family of the trajectories of the particle, with the wave-fronts defined by the the lines of constant phase S, and the propagating rays describing the particle trajectories as in geometrical optics.

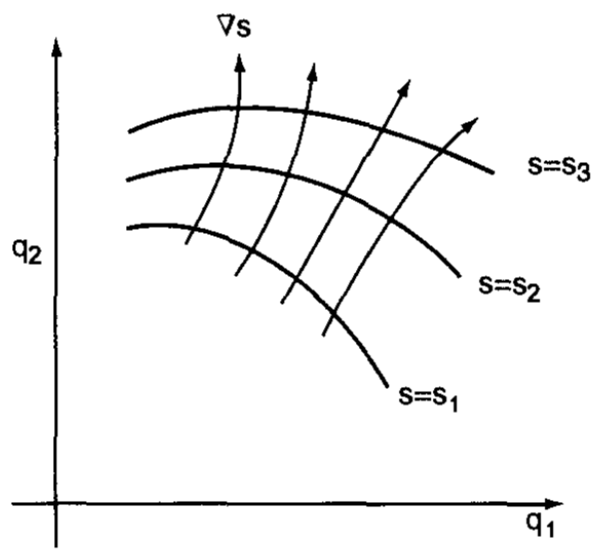

Figure 2: An ensemble of rays each representing a classical trajectory

4.3 The WKB Method

Our **AIM** of this section is to see the similarity between the HJ equation and the Schroedinger Wave Mechanicswhich will lead us to the WKB approximation. Most importantly we will see how a matching of the wave function across a turning point introduces an extra phase in it.

Consider Schroedinger equation:

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{r},t) + V(\vec{r})\psi(\vec{r},t) = i\hbar\frac{\partial\psi}{\partial t} \tag{128}$$

Let,

$$\psi(\vec{r},t) = A(\vec{r},t)\exp[iR(\vec{r},t)/\hbar] \tag{129}$$

with A & R real. Then putting this into the SE and equating the real & imaginary part we get,

$$\boxed{-\frac{\partial R}{\partial t} = \frac{(\nabla R)^2}{2m} + V(\vec{r}) - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}} \tag{130}$$

&

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla.(\rho\frac{\nabla R}{m}) = 0} \tag{131}$$

In the limit $\hbar \to 0$ (**WKB Approximation**)(130) becomes

$$\boxed{-\frac{\partial R}{\partial t} = \frac{(\nabla R)^2}{2m} + V(\vec{r})} \tag{132}$$

which is the **same as HJ EqM**.[Here $\rho = A^2$]

- **WKB in One Dimension:**

  (126)$\Rightarrow$ $R(x,t) = S(x,E) - Et$. So,

$$\frac{\partial R}{\partial t} = -E$$
$$\& \qquad \frac{\partial R}{\partial x} = \frac{\partial S}{\partial x}$$

Putting these in (132),

$$E = \frac{1}{2m}(\frac{\partial S}{\partial x})^2 + V(x) \tag{133}$$

$$\boxed{S(x,x_1,E) = \pm\int_{x_1}^{x}\sqrt{2m[E-V(x')]dx'}} \tag{134}$$

and,

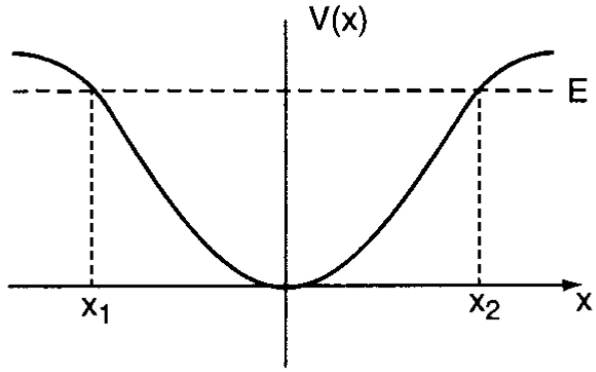

Figure 3: An Arbitrary 1-dimensional Potential

$$\boxed{S(x_2, x, E) = \pm \int_{x}^{x_2} \sqrt{2m[E - V(x')]dx'}} \tag{135}$$

For Time-independent potential the solution of the SE is

$$\begin{aligned}\psi(x,t) &= \psi(x)e^{-\frac{-Et}{\hbar}} \\ &= A(x)e^{iS(x,E)}e^{-\frac{-Et}{\hbar}}\end{aligned}$$

where, A and S are real.
So, $\rho = A^2(x) \Rightarrow \frac{\partial \rho}{\partial t} = 0$
Then (131) $\Rightarrow \rho = \frac{constant, K}{\frac{\partial S}{\partial x}} \equiv \frac{K}{p(x)} = \frac{K}{\sqrt{2m(E - V(x))}}$

<u>Case-I</u>: Solutions in the **classically allowed** region ($x_1 \leqslant x \leqslant x_2$) are oscillatory.Solution around the turning point x_2 is

$$\boxed{\psi(x,t) = \frac{1}{\sqrt{p(x)}}[A_1 e^{\frac{i}{\hbar}\int_x^{x_2} p(x')dx'} + A_2 e^{-\frac{i}{\hbar}\int_x^{x_2} p(x')dx'}]e^{-\frac{iEt}{\hbar}}} \tag{136}$$

Case-II: To get the solution in the **classically forbidden** region we can analytically continue (136) and then we will get for $x > x_2$

$$\boxed{\psi(x,t) = \frac{1}{\sqrt{p(x)}} C \exp[-\frac{1}{\hbar}\int_{x_2}^{x} |p(x')|dx'] e^{-\frac{iEt}{\hbar}}} \tag{137}$$

The solution blows up at the turning point. So, instead of looking at the turning point we would focus very near to the turning point. We expand potential around the turning point and keeping only upto the 1*st* order term if we put that in SE we get a differential equation known as Airy DE. This solution is valid only at the vicinity of the turning point.
we use these solutions to connect with the oscillatory solutions within the classically allowed region. So we get some matching conditions. Using those the solution in the **interior region** becomes:

$$\boxed{\psi(x) = \frac{2C}{\sqrt{p(x)}}[\sin\{\frac{1}{\hbar}\int_{x}^{x_2} p(x')dx' + \frac{\pi}{4}\}]} \tag{138}$$

and, in the **classically forbidden region**,e.g. in the region $x > x_2$ we get

$$\boxed{\psi(x) = \frac{C}{\sqrt{p(x)}} \exp[\frac{1}{\hbar}\int_{x}^{x_2} |p(x')|dx']} \tag{139}$$

Therefore, **a phase factor of $\pi/4$ is introduced in the wave function at the turning point of a smoothly varying potential.**

From here after a bit calculation we get the **WKB quantization condition** as

$$\boxed{\int_{x_1}^{x_2} p(x')dx' = (n + \frac{1}{2})\pi\hbar} \tag{140}$$

where, $n = 0, 1, 2, ...$
If one of the walls is infinite then the relation modifies as

$$\boxed{\int_{x_1}^{x_2} p(x')dx' = (n + \frac{3}{4})\pi\hbar} \tag{141}$$

So, an extra $\pi/2$ phase is introduced at the step wall.

- **WKB for Radial Motion:A Particle Moving in a 2-dimensional Central Potential V(r)** If we take square-well of depth V_0 then after a bit calculation we end up with the quantization condition

$$\boxed{\int_{r_1}^{r_2} Q_l(\tilde{r'})dr' = (n_r + \frac{1}{2})\pi} \tag{142}$$

where,

$$Q_l(r) = \sqrt{\frac{2m}{\hbar^2}[E - V(r) - \frac{\hbar^2}{2mr^2}(l^2 - \frac{1}{4})]} \tag{143}$$

is the local radial wave number in the l-th partial wave.
In general, the quantization condition is

$$\boxed{I_r \equiv \frac{1}{2\pi}S_r \equiv \oint p_r dr = (n_r + \frac{\mu}{4})\hbar} \tag{144}$$

where, μ is called the Maslov index which just counts the number of classical turning points, where the amplitude of the wave function diverges.Note thatin this case WKB wave function acquires a factor $\pm 2\pi(n_r + \mu/4)$ over a complete cycle.
In case the particle encounters a hard wall b times, the wave function goes to zero under Dirichlet boundary condition at every encounter with the wall, and picks up an extra phase of $b\pi$ for b reflections.Hence, the acquired phase becomes $2\pi(n_r + \frac{\mu}{4} + \frac{b}{2})$.On the other hand, under the Neumann BC,there is no change in the phase of the wave at the wall,and $b = 0$.Thus,the quantization condition (144) gets modified to

$$\boxed{I_r \equiv \frac{1}{2\pi}S_r \equiv \oint p_r dr = (n_r + \frac{\mu}{4} + \frac{b}{2})\hbar} \tag{145}$$

4.4 Torus Quantization:From WBK to EBK

We have seen in (134) & (135) that the action and its first derivative (with respect to coordinates) is double valued function of coordinates with having the same value of the two branches at the turning points.Hence, for the one dimensional case we can draw two branches of the momentum ($p_x = \frac{\partial S(x,E)}{\partial x}$) as a function of the position showing that they join together at the turning points to form a single closed curve.This curve is the phase plot having unique value of the momentum at every point on the curve.Here the semi-classical wave function is

$$\psi(x,E) = \frac{1}{\sqrt{2\pi\hbar}} \left| \frac{\partial^2 S(x,E)}{\partial E \partial x} \right|^{1/2} \exp(iS(x,E)/\hbar) \tag{146}$$

which is same as (125).
Suppose, S changes by ΔS in one cycle i.e. phase changes by $\Delta S/\hbar$. Also above wave function shows amplitude is inversely proportional to the square root of the momentum, which itself changes sign at each turning point.A change in sign is equivalent to a change in phase π.So, each turning point gives an additional phase of $-\pi/2$. Therefore, the single-valuedness of the wave function demands

$$\frac{1}{\hbar}\Delta S - 2\frac{\pi}{2} = 2n\pi \tag{147}$$

where, n is an integer.
Now, since $\Delta S = 2\int_{x_1}^{x_2} p(x)dx$, hence the condition becomes

$$\boxed{\oint pdq = (n + \frac{\mu}{4})2\pi\hbar} \tag{148}$$

- **Transformation to Action-angle variables:** We may canonically transform from the variables $\{x, p\}$ to $\{I, \phi\}$ where ϕ is the cyclic coordinates such that Hamiltonian is just function of I only. This is done by defining for a given energy E by

$$\oint Id\phi = 2\pi I = \oint pdx \tag{149}$$

Then the quantization condition becomes

$$\boxed{I(E_n) = (n + \frac{\mu}{4})\hbar} \tag{150}$$

where, $n = 0, 1, 2, ...$

- For an integrable system with N DOF, there are N constants of motion $A_i(i = 1, 2, ..., N)$ .Hence, the trajectory in the phase space is confined to a N-dimensional manifold. According to Poincare-Hopf theorem if in an N-dimensional manifold one can construct N-independent (and commuting) vector fields, then the manifold has a structure of an N-torus. In an N-torus, there are N-topologically independent closed curves, with N independent branches of the action S. the most convenient choice of the basis is N-closed curves, each wound around the angle variable ϕ_j.An arbitrary closed curve in this topology may be expressed as a linear combination of these N windings in this basis. Single valuedness of the wave function demands that we get N-independent quantization conditions just like (150).

4.5 Connection to Classical Periodic Orbits

The study says that there is a closed connection between the semi-classical quantization and the periodic orbits of the classical system.For integrable systems Berry and Tabor have shown that it is always possible to derive a "trace-formula" for the level density whose oscillating part is expressed in terms of the classical periodic orbits.After a long mathematical calculation we get an important conclusion that **in the saddle point approximation, only those orbits on the torus with commensurate frequencies i.e. closed periodic orbits contribute to the density of states.** This establishes the connection between the oscillating part of the trace formula and the classical orbits.

When the frequencies are not commensurate, the orbits do not close although the motion is still confined to the torus.Such orbits are termed as multiply periodic.

A periodic orbit is specified by (M_1, M_2); it closes after M_1 turns around the angle ϕ_1 and M_2 turns around ϕ_2.The topology is thus determined by the indices $\vec{M}$. This is why the sum over $(M_!, M_2)$ is called a "topological sum".

5 Propagator, Path Integral and Partition Function

As we know Feynman's Path Integral approach is an another way of looking quantum mechanics which is based on the fundamental idea that a probability amplitude is associated with every possible path joining initial & final positions, and all the path contributes to the net probability amplitudes unlike classical mechanics. Also all the paths contribute equally in magnitudes but with different phase.
Now the propagator(transition amplitude) between two states comes out to be

$$\boxed{K(q_i, q_f; T = t_f - t_i) \equiv \langle q_f|U(t_f, t_i)|q_i\rangle = \int_{q_i}^{q_f} d[p]d[q]e^{iS_H[p,q]}} \tag{151}$$

where,

$$\boxed{S_H[p, q] = \int_{t_i}^{t_f} dt[p(t)\dot{q}(t) - H(p(t), q(t)]} \tag{152}$$

and(we have used $\hbar = 1$),

$$\boxed{d[p] \equiv \prod_{n=0}^{N-1} \frac{dp_n}{2\pi}} \quad \& \quad \boxed{d[q] \equiv \prod_{n=0}^{N-1} dq_n} \tag{153}$$

- **Partition Function:** In statistical mechanics, at a temperature T, all the information is stored within the partition function Z.

$$\begin{aligned} Z[\beta] &= tr(e^{-\beta H}) \\ &= \int dq\langle q|(e^{-\beta H})|q\rangle \\ &= \int dq\langle q|(e^{\frac{-iH}{\hbar}\frac{\hbar\beta}{i}})|q\rangle \\ &= \int dq_i K(i, i; -i\beta\hbar) \end{aligned}$$

Therefore, the partition function is the integration over the initial state of the propagator that goes around a loop for a 'time' $t = i\beta\hbar$.
So, in terms of the path integral formalism, we can find the partition function by inserting N resolution of unity $\int |p\rangle\langle p|q\rangle\langle q|\frac{dqdp}{2\pi\hbar}$ in $e^{-\beta\hbar}$.After doing the path integral calculation we get,

$$\boxed{Z[\beta] = \int d[q]d[p]exp\{-\int_0^\beta (\frac{i}{\hbar}p\dot{q} + H(p.q))d\tau\}} \tag{154}$$

where, $\dot{q} = \lim_{N\to\infty}(\frac{q_{j+1}-q_j}{\beta/N})$ is a 'temperature derivative', and $q(0) = q(\beta)$, a loop in the q space for a 'time' β

6 Conclusion

We have seen these coherent states are very useful, not only to explore the physics of the semi-classical quantization but it also has many other applications. Also, the path integral approach which we have followed partially but we can derive all the properties of the spin-coherent states following this method. This path integral quantization has many more significant contribution to quantum field theory.

7 References

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