Advanced QFT

Spring 2016

Assignment 1

Total Marks 130

March 2, 2016

Date of submission: **16 March, 4 pm**

1. Consider the action of a free scalar field theory:

$$S = \frac{1}{2}\int d^4x \left[\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right] \tag{1}$$

where, $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ and $\eta_{\mu\nu} = \text{diag}[1,-1,-1,-1]$. Now introduce a new field χ through the relation:

$$\phi = \chi + \frac{1}{2}\lambda\chi^2. \tag{2}$$

(a) Express the action in terms of χ and its derivatives.

(b) Now forget about the description in terms of the ϕ field altogether, and treat this as a new field theory of the field χ. Derive the momentum space Feynman rules for calculating correlation functions for the χ field.

(c) Calculate the three and four point correlation functions:

$$< \tilde{\chi}(k_1)\tilde{\chi}(k_2)\tilde{\chi}(k_3) >_c$$

and 10+10+(5+10)+10

$$< \tilde{\chi}(k_1)\tilde{\chi}(k_2)\tilde{\chi}(k_3)\tilde{\chi}(k_4) >_c \tag{3}$$

to order λ and λ^2, respectively. Here

$$< \prod_{i=1}^{n}\tilde{\chi}(k_i) >_c \equiv \int \mathcal{D}[\tilde{\chi}(k)]e^{iS}\prod_{i=1}^{n}\tilde{\chi}(k_i)\Big/\mathcal{D}[\tilde{\chi}(k)]e^{iS} \tag{4}$$

and $<>_c$ denotes sum over only the connected Feynman diagrams.

(d) Calculate the S-matrix element for the scattering of two χ particles with momenta k_1 and k_2 to go into two χ particles of momenta p_1 and p_2 to order λ^2. In this calculation, you can ignore the forward scattering amplitude (i.e. the contribution from disconnected Feynman diagrams). You can simplify the answer by expressing the final result using Mandelstam variables:

$$s = (k_1 + k_2)^2, \ t = (k_1 - p_1)^2, \ u = (k_1 - p_2)^2. \quad (5)$$

Note that only two of these variables are independent since we have a relation:

$$s + t + u = 4m^2 \quad (6)$$

which you should be able to prove. Thus the final result should be expressed as a function of only two of these three variables (say of s and t).

2. Consider an action for a scalar field ϕ coupled to a fermionic field ψ:

$$S = \int d^4x \left[\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \lambda\bar{\psi}\gamma^\mu\psi\partial_\mu\phi\right] \quad (7)$$

   (a) Derive the Feynman rules for this theory.

   (b) Using the Feynman rules calculate — 10+10

$$< \tilde{\psi}_{\alpha_1}(k_1)\tilde{\psi}_{\alpha_2}(k_2)\tilde{\bar{\psi}}_{\beta_1}(p_1)\tilde{\bar{\psi}}_{\beta_2}(p_2) >_c \quad (8)$$

   to order λ^2. Here $<>_c$ denotes connected Green's function.

3. Consider a field theory of a scalar field ϕ and a fermionic field ψ with the action:

$$S = \int d^4x \left[\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}M^2\phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi\right] \quad (9)$$

   (a) Now introduce a new fermionic field χ through the relation:

$$\psi = e^{i\lambda\phi}\chi \quad (10)$$

   and express the action in terms of the fields ϕ and χ. Here λ is a constant. — 5+(10+10)

   (b) Now forget about the original action and work with this new action regarding ϕ and χ as independent fields, and λ as a small parameter. To order λ, calculate the S-matrix element relevant for computing the decay of a ϕ particle into two χ particles. Assume that $M > 2m$ such that this decay is energetically possible.

4. Consider the real Grassman numbers θ_i which are defined through

$$\theta_i\theta_j = -\theta_j\theta_i \quad (11)$$

   (a) Calculate $e^{A\theta_1\theta_2 + B\theta_3\theta_4}$ where A and B are simple constants.

(b) Integration over Grassman variables is defined through

$$\int d\theta_i \equiv 0, \qquad \int d\theta_i \theta_i \equiv 1 \tag{12}$$

$$\text{and} \quad d\theta_i d\theta_j = -d\theta_j d\theta_i, \qquad d\theta_i \theta_j = -\theta_j d\theta_i. \tag{13}$$

Find out

$$\int d\theta_1 d\theta_2 e^{A\theta_1\theta_2} \tag{14}$$

(c) Let us introduce complex Grassman variables (θ, η) through

$$\theta \equiv \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2), \qquad \theta^* \equiv \frac{1}{\sqrt{2}}(\theta_1 - i\theta_2). \tag{15}$$

Show that

$$(\theta\eta)^* \equiv \eta^*\theta^* = -\theta^*\eta^*. \tag{16}$$

(d) Calculate the Gaussian integral

5+5+5+5+5

$$\int d\theta^* d\theta e^{-\theta^* B\theta}. \tag{17}$$

Can you find any difference with the Gaussian integral of ordinary variables?

(e) Differentiation of Grassman variables is defined through

$$\frac{\partial}{\partial\theta_i}\theta_i \equiv 1, \qquad \frac{\partial}{\partial\theta_i}1 \equiv 0. \tag{18}$$

Do you think, the below relation holds true?

$$\frac{\partial}{\partial\theta_i}\left(f(\theta_1, \cdots \theta_n)g(\theta_1, \cdots \theta_n)\right) = \frac{\partial f}{\partial\theta_i}g + f\frac{\partial g}{\partial\theta_i} \tag{19}$$

Justify your answer.

5. In this problem I'm going to have you compute the path integral for a free particle moving in one dimension. This was discussed in your class, but I want to make sure you work through the details yourself at least once. So, we want to compute

$$Z = \int \mathcal{D}[x(t)] exp\left(\frac{i}{\hbar}\int_{t_i}^{t_f}\frac{1}{2}m\dot{x}^2\right) \tag{20}$$

where the initial and final positions and times (x_i, t_i), (x_f, t_f) are fixed. To do this, break up the time interval $[t_i, t_f]$ into N equal size pieces, and let $x_k = x(t_i + kT/N)$, where $T = t_f - t_i$. Then, we can write 15

$$Z = Lt_{N\to\infty} C^N \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_{N-1} exp\left(\frac{i}{\hbar}\frac{m}{2}\sum_{j=0}^{N-1}\left(\frac{x_{j+1}-x_j}{T/N}\right)^2 \frac{T}{N}\right) \tag{21}$$

where, C is chosen so as to make the result finite. Find the simplified expression of Z.

General Remarks

1. Follow Path integral formalism to derive all the Feynman rules.
2. Whenever you are writing down the Feynman rules, clearly draw the Feynman diagrams of the propagators as well as interacting vertices with all the momenta, conserved current and possible indices.

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