Mass, quasi-local mass, and the flow of static metrics

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Quasilocal mass

- QLM “usefully” characterizes bounded \( n \)-dim submanifolds of Riemannian \( n \)-manifolds; or bounded \( n \)-dim spacelike submanifolds of \( (n + 1) \)-spacetimes.

- Examples:
  - Brown-York mass: \( \int_{\Sigma} (H_0 - H) \, dA \),
  - \( H = \) mean curvature of \( \Sigma \),
  - \( H_0 = \) mean curvature of isometrically embedded copy of \( \Sigma \) in flat space.
  - Hawking mass: \( \frac{1}{16\pi} \sqrt{A(\Sigma)} \left( 1 - \int_{\Sigma} H^2 \, dA \right) \).
  - Bartnik mass.
Bartnik’s quasi-local mass: time symmetric case

Definition:
• Consider a bounded domain $B$ with compact boundary $\partial B$ embedded isometrically in an 
  • asymptotically flat,
  • scalar curvature $R \geq 0$
  • Riemannian manifold that admits no stable minimal sphere outside $B$.
These are the *allowed extensions* of $B$.
• Compute ADM mass of each allowed extension.
• The infimum of the ADM mass over all allowed extensions is the quasi-local mass $m_B$ of $B$. 
Remarks on QLM Definition

Note that:

- \( B \subseteq B' \Rightarrow m_{B'} \geq m_B \) (monotonicity).
- \( m_B \geq 0 \) (positivity).

Note also:

- Extensions need not be smooth.
- “Geometric boundary conditions”: Induced metrics and mean curvatures must match across \( \partial B \) (but extrinsic curvature not required to match).
- That is, Bartnik permits distributional stress-energy on boundary.
The conjecture

- Question: When (if ever) is $m_B > 0$?
- Conjecture: The mass $m_B$ is realized as the ADM mass of some extension of $B$ (i.e., the infimum in Bartnik’s definition is a minimum).
- Furthermore, the extension that realizes $m_B$ is a solution of the static Einstein equations:

$$R^g_{ij} = k_n^2 \nabla_i u \nabla_j u .$$

- Bianchi identity implies $\Delta u = 0$.
- For $k_n^2 = \frac{n-1}{n-2}$ the static Einstein equations imply that the locally static spacetime metric

$$ds^2 = -e^{2u} dt^2 + e^{-\frac{2u}{n-2}} g_{ij} dx^i dx^j$$

is Ricci-flat.
Flow of Static Metrics (B List; PhD thesis under G Huisken)

\[
\frac{\partial g_{ij}}{\partial \lambda} = -2 \left( R_{ij} - k_n^2 \nabla_i u \nabla_j u \right)
\]
\[
\frac{\partial u}{\partial \lambda} = \Delta u
\]

Fixed points yield locally static, Ricci-flat spacetime metrics

\[
ds^2 = -e^{2u} dt^2 + e^{-\frac{2u}{n-2}} g_{ij} dx^i dx^j ,
\]

where we have taken \( k_n^2 = \frac{n-1}{n-2} \).
Begin with metric
\[dS^2 = G_{\mu\nu} dx^\mu dx^\nu = e^{2k_n u} dt^2 + g_{ij} dx^i dx^j\]
with
\[\frac{\partial u}{\partial t} = 0, \quad \frac{\partial g_{ij}}{\partial t} = 0.\]

Define vector field
\[X := -k_n \nabla^i u \frac{\partial}{\partial x^i}.\]

Apply Hamilton-DeTurck flow
\[\frac{\partial G_{\mu\nu}}{\partial \lambda} = -2R_{\mu\nu}^G + \mathcal{L}_X G_{\mu\nu}.\]

This yields the static metric flow equations of the last slide.
Aside: Ricci solitons from scalar fields

- Solutions \((u, g_{ij})\) of the static equations (= Euclidean-signature Einstein-free-scalar equations) produce local steady Ricci solitons

\[dS^2 = e^{2k_n u} dt^2 + g_{ij} dx^i dx^j, \quad \mathcal{X} = -k_n \nabla^i u \frac{\partial}{\partial x^i}.\]

- Local solutions \((u, g_{ij})\) of the (Euclidean-signature) free-scalar equations with cosmological term

\[R^g_{ij} = k_n^2 \nabla^i u \nabla^j u + \kappa g_{ij},\]

produce local scaling Ricci solitons

\[dS^2 = e^{2k_n u} dt^2 + g_{ij} dx^i dx^j, \quad \mathcal{X} = -\kappa t \frac{\partial}{\partial t} - k_n \nabla^i u \frac{\partial}{\partial x^i}.\]
Examples

- A local soliton from the Schwarzschild metric:

\[ ds^2 = \left(1 - \frac{2m}{r}\right)^{\sqrt{2}} dt^2 + r (r - 2m) (d\theta^2 + \sin^2 \theta d\phi^2) . \]

- From the Buchdahl (Janis-Newman-Winnicour, etc) solution:

\[
\begin{align*}
    ds^2 &= \left(1 - \frac{2m}{r}\right)^{3\sqrt{1-\delta}/2} dt^2 + \left(1 - \frac{2m}{r}\right)^{\delta} d\tau^2 \\
    &\quad + \left(1 - \frac{2m}{r}\right)^{-\delta} dr^2 + r^2 \left(1 - \frac{2m}{r}\right)^{1-\delta} (d\theta^2 + \sin^2 \theta d\phi^2) .
\end{align*}
\]

- A complete, expanding soliton:

\[
    ds^2 = \frac{e^{2r}}{\left(1 + e^{\sqrt{2}r}\right)} dt^2 + dr^2 + \left(1 + e^{\sqrt{2}r}\right) (d\theta^2 + \sinh^2 \theta d\phi^2) .
\]
List’s flow: strategy

“If you can’t solve a problem, then there is an easier problem you can solve: Find it!”—G Polya (source: Wikipedia), and probably a great many other thesis supervisors as well.

• Specialize to flow on $\mathbb{R}^n$: no inner boundary $\partial B$.
  - List (PhD thesis): Flow exists for some interval $\lambda \in [0, T)$.
  - Can extend beyond $T$ unless $\lim_{\lambda \to T} \sup_{x \in \mathbb{R}^n} |\text{Riem}| = \infty$.

• Specialize to “strongly asymptotically flat” initial data.
  - List (PhD thesis): Flow preserves asymptotic flatness, mass remains constant.

• Specialize to rotational symmetry.

• Further specializations if necessary (initial data, dimension,...).
List’s flow on complete manifolds (no boundary)

- \( \frac{\partial}{\partial \lambda} |\nabla u|^2 = \Delta |\nabla u|^2 - 2|\nabla \nabla u|^2 - 4 \left(|\nabla u|^2\right)^2. \)

- Max principle: \( |\nabla u|^2 < \frac{\text{const}}{1 + \lambda}. \) “Deviation from \((n\text{-dim})\) Ricci flow dies out.”

- \( \frac{\partial S}{\partial \lambda} = \Delta S + 2S_{ij}S^{ij} + 4\left(\Delta u\right)^2, \)
  where \( S_{ij} := R_{ij} - k_n^2 \nabla_i u \nabla_j u \) and \( S := R - k_n^2 |\nabla u|^2 \)

- Minimum principle: \( S = R - k_n^2 |\nabla u|^2 \geq -\frac{\text{const}}{1 + \lambda}. \)

- Bad news: \( R > 0 \) is not preserved, but “almost”:
  (i) \( R \geq -\frac{\text{const}}{1 + \lambda} \) and
  (ii) \( R \geq k_n^2 |\nabla u|^2 \geq 0 \) is preserved.
Rotational Symmetry

- Symmetry: \( ds^2 = f^2(\lambda, r)dr^2 + r^2g(S^{n-1}, \text{can}), \ u = u(\lambda, r) \).
- Flow equations acquire DeTurck (diffeomorphism) term, which can be solved for and eliminated.
- The \( \frac{\partial g_{ij}}{\partial \lambda} \) equation reduces to a single PDE. System becomes:

\[
\frac{\partial f}{\partial \lambda} = \Delta f + X \cdot \nabla f - \frac{(n - 2)}{r^2f} \left( f^2 - 1 \right) + k^2_n f|\nabla u|^2,
\]

\[
X := \left( \frac{n - 2}{r} - \frac{n}{rf^2} - \frac{1}{f^3} \frac{\partial f}{\partial r} \right) \frac{\partial}{\partial r},
\]

\[
\frac{\partial}{\partial \lambda} |\nabla u|^2 = \Delta |\nabla u|^2 - 2|\nabla \nabla u|^2 - 4 \left( |\nabla u|^2 \right)^2.
\]
Intuition from Ricci flow

- If $\lambda \in \mathbb{R}$, system is equivalent to Hamilton-DeTurck flow on $\mathbb{R}^{n+1}$ with $\mathbb{R} \times \text{SO}(n)$ symmetry.

- If $t \in S^1$, system is equivalent to Hamilton-DeTurck flow on $S^1 \times \mathbb{R}^n$ with $U(1) \times \text{SO}(n)$ symmetry (perturbed “hot flat space”).

- Good news: Hot flat space (with finite boundary rather than asymptotic flatness: $S^1 \times B^n$) attracts flows in numerical simulations [Headrick-Wiseman arxiv:hep-th/0606086].

- Good news: Ricci flow on $\mathbb{R}^{n+1}$ with SO($n+1$) symmetry exhibits long-time existence and convergence for asymptotically flat initial data that does not contain any minimal sphere [Oliynyk-Woolgar arxiv:math/0607438].
Expanding Ricci soliton on $\mathbb{R}^2$ given by

$$ds^2 = \lambda \left( f^2(r)dr^2 + r^2\zeta^2 d\theta^2 \right), \quad \zeta = \text{const}.$$ 

Changing coordinates to $\rho = r\sqrt{\lambda}$, we get

$$ds^2 = f^2(\rho/\sqrt{\lambda})d\rho^2 + \rho^2\zeta^2 d\theta^2.$$ 

$f(x)$ is given implicitly by

$$\left(\frac{1}{\zeta} - 1\right) \exp \left(\frac{1}{\zeta} - 1 - \frac{x^2}{2\alpha'}\right) = \left(\frac{1}{f(x)} - 1\right) \exp \left(\frac{1}{f(x)} - 1\right)$$

$f(x) \to \zeta$ for $x \searrow 0$.

$f(x) \to 1$ for $x \nearrow \infty$.

$f$ is monotonic on $(0, \infty)$. 
Graph of $f(x)$ for $\zeta = 1/3$

$$ds^2 = \lambda (f^2(r)dr^2 + r^2\zeta^2d\theta^2)$$

$$= f^2(\rho/\sqrt{\lambda})d\rho^2 + \rho^2\zeta^2d\theta^2$$

- Asymptotic deficit angle is 2D mass (hint: think of isoperimetric deficit).
- At fixed $\lambda$, the $\rho \to \infty$ limit is flat cone of deficit angle $\delta = 2\pi(1 - \zeta)$.
- At fixed $\rho$, the $\lambda \to \infty$ limit is flat space.
Lessons Learned

- Mass (deficit angle) constant along flow.
  - Oliynyk-Woolgar; Dai-Ma: True in $n$ dimensions.

- Geometric limit is flat space; mass jumps to zero.
  - Oliynyk-Woolgar: True in $n$ dimensions.

- Brown-York quasilocal mass evaporates smoothly; evaporation occurs primarily on distance scales $r < \sqrt{\lambda}$.
  - Oliynyk-Woolgar: True in $n$ dimensions.

- Agrees with qualitative “rolling tachyon” predictions in string theory.
Rotationally symmetric List flow: quantities of interest

- Must ensure coordinates stay good:
  - $H = \frac{n-1}{rf} = \text{mean curvature of orbits of } SO(n)$. Coordinates good if no minimal sphere forms (at infinity; $f \to \infty$).
  - $X = \left[ \frac{1}{f^3} \frac{\partial f}{\partial r} + \frac{(n-2)}{r} \left(1 - \frac{1}{f^2}\right) \right] \frac{\partial}{\partial r} = \text{DeTurck vector field.}$

- Must bound the curvatures:
  - $\kappa_1 = \frac{1}{rf^3} \frac{\partial f}{\partial r} = \text{sectional curvature in planes containing } \frac{\partial}{\partial r}.$
  - $\kappa_2 = \frac{1}{r^2} \left(1 - \frac{1}{f^2}\right) = \text{sectional curvature in planes } \perp \frac{\partial}{\partial r}.$
  - $R = (n-1) \left[ \frac{2}{rf^3} \frac{\partial f}{\partial r} + \frac{(n-2)}{r^2} \left(1 - \frac{1}{f^2}\right) \right]$
Getting started

- Estimate $w(\lambda, r) := f^2(\lambda, r) - 1$:

$$\frac{\partial w}{\partial \lambda} = \Delta w + Y \cdot \nabla w - \frac{2(n - 2)}{r^2} w + 2k_n^2 f^2 |\nabla u|^2$$

- Lower bound, prevents degeneration of coordinates:

  \begin{align*}
  n > 2 & : \quad w(\lambda, r) \geq \inf_r \{w(0, r)\} > -1 \\
  n = 2 & : \quad w(\lambda, r) \geq \min \left\{ w_\infty, \inf_r \{w(0, r)\} \right\}
  \end{align*}

- Upper bound, using $|\nabla u|^2 \leq C_1/(1 + \lambda)$:

  $$w(\lambda, r) \leq C_2(1 + \lambda)^p, \quad p = 2k_n^2 C_1$$

- Note: No minimal spheres form in finite time; can form in limit.
(a) \( \text{const} \leq f^2 \leq \text{const} \cdot (1 + \lambda)^p \).

(b) \( \frac{1}{r^2} (f^2 - 1) \geq -\text{const} \cdot (1 + \lambda)^{p+1} \).

(c) \( |\nabla u|^2 \leq \frac{\text{const}}{1+\lambda} \).

(d) \( R(\lambda, x) \geq -\frac{\text{const}}{1+\lambda} \) “Sum of sectional curvatures \(> -\infty \).”

(e) \( R(\lambda, x) \geq k_n^2 |\nabla u|^2 \ \forall (\lambda, x) \) if true \( \forall (0, x) \).
Further estimates

If we assume that

\[(f)\quad \frac{1}{r} |\nabla u| \leq \alpha = \text{const},\]

then we also deduce that

\[(g)\quad \frac{1}{r^2} \left( f^2 - 1 \right) \leq \text{const} \cdot \left[ 1 + \alpha^2 (1 + \lambda)^{p+1} \right] \leq \text{const} \cdot e^{\text{const} \cdot \lambda},\]

\[(h)\quad \left| \frac{\partial}{\partial r} (|\nabla u|) \right| \leq \text{const} \cdot e^{\text{const} \cdot \lambda}, \text{ and}\]

\[(i)\quad \kappa_1 + \frac{2}{f^2 (1+f)} \frac{1}{r^2} \left( f^2 - 1 \right) \leq \text{const} \cdot e^{\text{const} \cdot \lambda},\]

where \( \kappa_1 := \frac{1}{rf^3} \frac{\partial f}{\partial r} \) = sectional curvature in planes tangent to \( \frac{\partial}{\partial r} \).
Long time existence

\[(j) \quad (b)+(i) \Rightarrow \kappa_1 \leq const \cdot e^{\text{const} \cdot \lambda}.\]

\[(k) \quad (a)+(g) \Rightarrow \kappa_2 \leq const \cdot e^{\text{const} \cdot \lambda}.\]

\[(d) \quad \text{Recall } R \equiv (n-1) [2\kappa_1 + (n-2)\kappa_2] \geq -\frac{\text{const}}{1+\lambda}.\]

\[(l) \quad \text{Thus } (j)+(k)+(d) \Rightarrow \kappa_1 \geq -\text{const} \cdot e^{\text{const} \cdot \lambda}\]

\[(m) \quad \text{and } \kappa_2 \geq -\text{const} \cdot e^{\text{const} \cdot \lambda} \text{ when } n > 2.\]

Proposition: If \(\frac{1}{r} |\nabla u| \leq const\) for all \(\lambda \geq 0\), then sectional curvatures do not diverge for any \(\lambda \in [0, T]\) and any \(T > 0\). Then [List, PhD thesis] the flow exists for all \(\lambda \geq 0\).
Open questions

- Can we prove $\frac{1}{r} |\nabla u| \leq \text{const}$?
  - Yes! ...if $n = 2$.

Proposition: If $n = 2$ the flow exists for all $\lambda \geq 0$.

- Can we prove convergence of the flow to a limit, say flat space with $u = 0$?
  - No, not even close, but as Polya said: “If you can’t solve a problem, then there is an easier problem you can solve: Find it!”
  - We did.
Closing remark: Stationary metrics

- A stationary metric is a Lorentzian metric with timelike Killing vector field which is not necessarily hypersurface-orthogonal.
- The stationary metric

\[ ds^2 = -e^{2u} \left( dt + B_i dx^i \right)^2 + e^{-\frac{2u}{n-2}} g_{ij} dx^i dx^j \]

is Ricci flat if \((u, B_k, g_{ij})\) obey

\[ 0 := \Delta u + \frac{1}{4} e^{2 \left( \frac{n-1}{n-2} \right) u} |F|^2 , \]

\[ 0 = -\nabla^j F_{ij} - 2 \left( \frac{n-1}{n-2} \right) F_{ij} \nabla^j u , \]

\[ 0 = -2R_{ij} + 2 \left( \frac{n-1}{n-2} \right) \nabla_i u \nabla_j u - e^{2 \left( \frac{n-1}{n-2} \right) u} \left[ F_{ik} F_j^k - \frac{1}{2(n-2)} g_{ij} |F|^2 \right] . \]
Flow of stationary metrics

The flow

\[
\frac{\partial u}{\partial \lambda} := \Delta u + \frac{1}{4} e^{2\left(\frac{n-1}{n-2}\right)} u |F|^2 ,
\]

\[
\frac{\partial B_i}{\partial \lambda} = -\nabla^j F_{ij} - 2 \left(\frac{n-1}{n-2}\right) F_{ij} \nabla^j u ,
\]

\[
\frac{\partial g_{ij}}{\partial \lambda} = -2 R_{ij} + 2 \left(\frac{n-1}{n-2}\right) \nabla_i u \nabla_j u
\]

\[
- e^{2\left(\frac{n-1}{n-2}\right)} u \left[ F_{ik} F^k_j - \frac{1}{2(n-2)} g_{ij} |F|^2 \right] .
\]

- is as well-behaved as Yang-Mills-Ricci flow,
- contains List’s flow as the \( B_k = 0 \) case,
- has fixed points that correspond to Ricci-flat stationary metrics,
- but generally does not appear to arise as Hamilton-DeTurck flow.

Is there a Hamilton-DeTurck flow whose fixed points are Ricci-flat stationary metrics?