A new class of Kähler metrics and Ricci flow

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M : a compact Kähler manifold.

g : A Kähler metric with the Kähler form $\omega = \omega_g$.

In local coordinates $z_1, \cdots, z_n$, the metric g is given by a Hermitian positive matrix-valued function $(g_{i\bar{j}})$. Then its Kahler form can be written as

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$$

g being Kähler $\Leftrightarrow d\omega = 0$

What is the most canonical metric on a given M?

One particular class of canonical metrics is that of Kähler-Einstein metrics.

Definition: A metric g is Kähler-Einstein if it is Kahler and

$$\mathrm{Ric}(g) = \lambda \omega_g, \quad \lambda = -1, 0, 1,$$

where $\mathrm{Ric}(g)$ is the Ricci curvature form. In local coordinates,

$$\mathrm{Ric}(g) = -\sqrt{-1}\partial\bar{\partial} \log det(g_{i\bar{j}}).$$

Clearly, it measures the deviation of volume form from the Euclidean one.

- E. Calabi studied these Kähler-Einstein metrics in 50's.

- Yau: M has a Ricci-flat Kähler metrics, now referred as Calabi-Yau metric, in each Kähler class if $c_1(M) = 0$.

- Aubin, Yau: Any M with $c_1(M) < 0$ has a unique Kähler-Einstein metric with $\lambda = -1$.

- Tian: Any complex surface with $c_1(M) > 0$ and reductive automorphism group has a Kähler-Einstein metric.

- One expects: Any M with $c_1(M) > 0$ should have a Kähler-Einstein metric if a certain stability condition is satisfied.

However, most Kähler manifolds do not have definite first Chern class, such as, elliptic surfaces in complex dimension two or a manifold obtained by blowing up a given manifold sufficiently many times.

Can one expect any canonical metric on such a manifold?

Or more likely, is there a way of deforming such a manifold towards a Kähler manifold with a certain canonical metric?

Now let us discuss a new class of metrics introduced by J. Song and myself.

Assume that M admits a holomorphic fibration $\pi : M \mapsto M'$ over a κ-dimensional Kähler manifold M' with a Kähler metric ω' and satisfying: $c_1(M) = \lambda[\pi^*\omega']$ for some rational number λ.

Assume that $0 < \kappa < n$. Then generic fibers of π have vanishing first Chern class, so they are Calabi-Yau manifolds of dimension $n - \kappa$. For simplicity, assume that fibers are connected.

Let $M'_0 \subset M'$ be a dense-open subset of M' over which the fibers of π are smooth. Then there is a holomorphic map $f : M'_0 \mapsto \mathcal{M}_{CY}$, where $\mathcal{M}_{CY}$ is the moduli of all smooth Calabi-Yau manifolds diffeomorphic to smooth fibers.

Note that $\mathcal{M}_{CY}$ is essentially smooth and admits a canonical metric, that is, the Weil-Peterson metric Ω_{WP}.

Consider the generalized Kähler-Einstein equation on M':

$$\mathrm{Ric}(g) = \lambda\,\omega_g + f^*\omega_{WP},$$

where ω_{WP} denotes the Weil-Peterson metric on $\mathcal{M}_{CY}$.

The metric encodes information of M through f.

Kähler metrics which arise from collapsing or large complex limits satisfy the generalized Kähler- Einstein equation. For instance, the metric on $S^2 \backslash \{x_1, \cdots, x_{24}\}$ which arises from degeneration of Calabi-Yau metrics on a K3 surface is a generalized Kähler-Einstein metric.

They arise naturally from compactifying the quotients of spaces of all Kähler metrics by diffeomorphism groups. They are as natural as those Kähler-Einstein metrics.

To establish existence of generalized Kähler-Einstein metrics on M', we reduce the equation to a scalar equation as one did for Kähler-Einstein metrics.

Since $c_1(M) = \lambda\pi^*\omega'$, we can choose a volume form Ω on M such that $\mathrm{Ric}(\Omega) = \lambda\pi^*\omega'$. We integrate along fibers of $\pi : M \mapsto M'$ to obtain a volume form $\pi_*\Omega$:

$$\pi_*\Omega(x) = \int_{\pi^{-1}(x)} \Omega,$$

which may have singularities because of singular fibers.
Define F by making

$$\pi_*\Omega = F\,(\omega')^\kappa.$$

There is another way of expressing F:

$$F = \frac{\Omega}{\Omega_{SF} \wedge (\omega')^\kappa},$$

where Ω_{SF} is a relative flat volume along fibers.

Using a theorem I proved before, one can show

$$\sqrt{-1}\partial\overline{\partial} \log F = -\mathrm{Ric}(\Omega) + \mathrm{Ric}(\omega') + f^* \omega_{WP}.$$

If we change the metric ω' to $\omega' + \sqrt{-1}\partial\overline{\partial}\varphi$, then corresponding F changes to

$$Fe^{-\lambda\varphi}\frac{(\omega')^{\kappa}}{(\omega' + \sqrt{-1}\partial\overline{\partial}\varphi)^{\kappa}}.$$

It follows that the equation for generalized Kähler-Einstein metrics can be reduced to the scalar equation:

$$(\omega' + \sqrt{-1}\partial\overline{\partial}\varphi)^{\kappa} = Fe^{-\lambda\varphi}\omega'^{\kappa}.$$

Any of its solutions gives rise to a generalized Kähler-Einstein metric as above. Furthermore, $\omega = \omega' + \sqrt{-1}\partial\overline{\partial}\varphi$ extends to a positive closed current on M' such that

$$\mathrm{Ric}(\omega) = \lambda\omega + f^{*}\omega_{WP}.$$

It resembles the equation for Kähler-Einstein metrics except that F is usually singular along a subvariety. One needs some information on asymptotic behavior of F near such a subvariety.

- F is in $L^{1+\epsilon}$ for some $\epsilon > 0$.

Of course, one is able to get more precise information on its asymptotic behavior.

J.Song and I proved for $\lambda = 0$ or -1:

For any above M', there is a unique solution φ of

$$(\omega' + \sqrt{-1}\partial\bar{\partial}\varphi)^\kappa = Fe^{-\lambda\varphi}\omega'^\kappa.$$

In particular, it implies that there is a unique generalized Kähler-Einstein metric on any given M'.

We actually proved this theorem even if M' can have some canonical singularities.

The proof used the regularity theory of Kolodziej and its extension to singular varieties.

Now let us see how these generalized Kähler-Einstein metrics arise from the study of Ricci flow.

The Ricci flow was introduced by Hamilton in early 80's:

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2R_{ij} \\ g(0) = \text{a given metric} \end{cases}$$

• For any initial metric, there is a unique solution $g(t)$ on $M \times [0,T)$ (Hamilton, DeTurck).

If $g(0)$ is Kähler, so is every $g(t)$. Hence, we can consider the Kähler-Ricci flow:

$$\begin{cases} \dfrac{\partial}{\partial t}\omega_{g(t)} = -\mathrm{Ric}(g(t)) \\ \omega_{g(0)} = \text{a given Kähler form } \omega_0 \end{cases}$$

We expect: The Kähler-Ricci flow on M deforms any Kähler metric to a generalized Kähler-Einstein metric on a certain canonical model M_{can} of M.

However, Ricci flow may develop singularity at finite time or may not have a global solution. Therefore, we first need to justify why we can expect a global solution for Kähler-Riic flow in a suitable sense.

Tian and Z. Zhang proved:

For any initial Kähler metric ω_0, there is a unique maximal solution $g(t)$ the Kähler-Ricci flow on $[0, T)$, where T is the maximum of t such that $[\omega_0] - t\, c_1(M)$ is a Kähler class.

Note that if $c_1(M) \leq 0$ or more generally, K_M is nef., then the Ricci flow has a global solution. Otherwise, the flow develops singularity at T.

Special cases due to Cao, Tsuji etc.. A version of this was also given by Carcini-LaNave assuming certain technical estimates.

What happens to $g(t)$ when t tends to $T < \infty$?

There are two cases:

1. $[\omega_0] - Tc_1(M) = 0$, so $(M, g(t))$ collapses to a point as t tends to T, i.e., the flow becomes extinct at finite time. It follows that M is Fano.

2. $[\omega_0] - Tc_1(M) \neq 0$, then $(M, g(t))$ converges to a metric space $(M_0, g(T))$ (of possibly lower dimension) in a suitable weak topology. We expect that M_0 is a variety and $g(T)$ is a smooth metric on its smooth part.

Indeed, Tian and Z. Zhang proved:

If $([\omega_0] - T\, c_1(M))^n > 0$, then $g(t)$ converges to a unique smooth metric $g(T)$ outside a subvariety. In particular, if M has non-negative Kodaira dimension which takes values $-\infty, 0, 1, \cdots, n$, then $([\omega_0] - T\, c_1(M))^n > 0$ holds if $T < \infty$ and $g(t)$ converges to a unique smooth metric $g(T)$ outside a subvariety.

We believe that the same is true even if $g(t)$ may collapse to a lower dimensional space. We want to examine how to extend $g(T)$ and continue the Ricci flow.

Of course, $g(t)$ develops singularities at T. Let S_0 be its singular set of $g(T)$. Let M_1 be the metric completion of $(M \backslash S_0, g(T))$.

Conjecture: M_1 is an analytic variety. It might be a flip of M, or a variety obtained by certain standard algebraic procedure.

Recently, G. Lanave and I found an approach to studying connection between flips and Ricci flow on Kähler manifolds.

Note that M_1 might have singularities as a variety. Even if M_1 is smooth, it is unclear if $g(T)|_{M\setminus S_0}$ extends to a smooth metric on M_1. Nevertheless, we expect:

Conjecture: M_1's singularities are not so bad that one can still run a generalized Kähler-Ricci flow on M_1.

If M_1 is smooth, using a result of Kolodziej, one can show that the extension of $g(T)$ as a current has locally bounded potential and has bounded volume form in a suitable sense.

Chen, Tian and Z. Zhang proved the local existence for the Kähler-Ricci flow on M_1 with the initial potential which may arise from the extension of $g(T)$ as above.

Recently, Song-Tian can prove that if M_1 is singular and embedded into some projective space, then the Kähler-Ricci flow has a unique solution starting from any the initial metric which arises from restriction of the Fubini-Study metrics on the projective space.

It is possible that one can solve the 2nd conjecture by using the above.

Suppose that such a procedure can be achieved and the Kähler- Ricci flow can continue on M_1, then we can further apply the above procedure to get $M_1, M_2, ..., M_{N-1}$ and a solution $g(t)$ of the Kähler-Ricci flow on $[0, t_N)\backslash\{t_i\}$ for $t_0 = 0 < t_1 < \cdots t_N$ satisfying:

(1) On each interval $[t_{i-1}, t_i)$, $g(t)$ is a regular solution of the Kähler-Ricci flow on M_i ($i = 1, \cdots, N$);

(2) $(M_i, \lim_{t\to t_i+} g(t))$ is obtained from $(M_{i-1}, \lim_{t\to t_i-} g(t))$ as described above.

Such a $g(t)$ can be referred as a Kähler-Ricci flow with surgery.

Conjecture: There is an $N < \infty$ such that M_N with $-c_1(M_N) \geq 0$ (possibly lower dimensional if collapsing occurs in previous surgeries) or M_{N-1} collapse to a point as t tends to t_N.

We say that the Kähler-Ricci flow with surgery $g(t)$ with initial metric g_0 on M becomes extinct at $t_N < \infty$ if the above $(M_{N-1}, g(t))$ collapses to a point as t tends to t_N.

If M is a Fano manifold and the initial Kähler class is proportional to the first Chern class, one can show that the Kähler-Ricci flow becomes extinct at a finite time. One can construct many more examples of manifolds bi-rational to Fano manifolds on which the flow become extinct at finite time.

Conjecture: A Kähler-Ricci flow becomes extinct at finite time if and only if the underlying algebraic manifold is birational to a Fano manifold or something very close.

In fact, I also expect that a Kähler-Ricci flow collapses to a lower dimensional variety if and only if the underlying manifold is uni-ruled.

Now we assume that above M_N exists. Then there should be a global solution $g(t)$ for $t \geq t_N$ as shown by using results of Chen-Tian-Zhang and Tian-Zhang when M_N is smooth. We need to examine what is the limit of $(M_N, g(t))$ as $t \to \infty$. For simplicity, we write M for M_N.

Now the generalized Kähler-Einstein metrics come in. They are exactly the limiting metrics we need.

All the above conjectures can be affirmed in the case of complex surfaces.

Observe that by scaling metrics and reparametrizing time, we get the normalized Kahler-Ricci flow:

$$\begin{cases} \frac{\partial}{\partial t}\omega_{g(t)} = -(\mathrm{Ric}(g(t)) - \lambda\,\omega_{g(t)}) \\ \omega_{g(0)} = \text{a given Kähler form } \omega_0 \end{cases}$$

Here λ is determined by the first Chern class $c_1(M)$, or more precisely, by the Kodaira dimension in general cases.

The case $\lambda = -1$ is what we will use.

Recall the definition of so called canonical model M_{can} of M.

Suppose that K_M^l is base-point free for sufficiently large ℓ, that is, there are no common zeroes for any basis of $H^0(M, K_M^\ell)$. Then there is an induced holomorphic fibration by any such a basis:

$$\pi : M \mapsto M_{can} \subset \mathbf{CP}^N,$$

such that $-\mu c_1(M) = \pi^*[\omega_{FS}]$ for some integer $\mu > 0$, where $[\omega_{FS}]$ is the positive generator of $\mathbf{CP}^N$. Here M_{can} denotes the image of π (the canonical model of M). It can have singularity and its dimension κ is called Kodaira dimension of M.

Note that the Kodaira dimension $\kappa = -\infty, 0, 1, \cdots, n$.

If $\kappa = 0$, we can show the Kähler-Ricci flow converges to a (possibly singular) Calabi-Yau metric on M.

If $\kappa = n$, then M is minimal and of general type. One can show that there is a canonical Kähler-Einstein metric (possibly singular) on M_{can}. Furthermore, the expanding Kähler-Ricci flow converges to this Kähler-Einstein metric. (cf. Tsuji, Tian-Zhang)

Now assume that $0 < \kappa < n$. Then we are in the situation for generalized Kähler-Einstein metrics.

Consider the generalized Kähler-Einstein equation on M_{can}:

$$\operatorname{Ric}(g) = \lambda\,\omega_g + f^*\omega_{WP},$$

where ω_{WP} denotes the Weil-Peterson metric on $\mathcal{M}_{CY}$.

Recall that J.Song and I have proved ($\lambda = -1$):

For any canonical model M_{can}, there is a unique generalized Kähler-Einstein metric ω_{can} on any canonical model M_{can}.

Furthermore, recently, Song and I proved:

If M is a compact Kähler manifold with nef K_M and Kodaira dimension $0 < \kappa < n$, then for any initial Kähler metric ω_0, the expanding Kähler-Ricci flow has a global solution $\omega(s)$ which converges to $\pi^*\omega_{can}$ as currents and converges in the L^∞-topology outside singular fibers.

In the case of finite time extinction, i.e., solution $g(t)$ of Ricci flow collapses to a point as t tends to T, we can further study finer structure of how $g(t)$ approaches to the point.

First we notice that $c_1(M) > 0$, so by scaling $g(t)$ and reparametrizing $t = t(s)$, we can reduce to studying the normalized Kähler-Ricci flow:

$$\begin{cases} \dfrac{\partial}{\partial s}\omega_{\tilde{g}(s)} = -\mathrm{Ric}(\tilde{g}(s)) + \omega_{\tilde{g}(s)} \\ \omega_{g(0)} = \text{a given Kähler form } \omega_0 \end{cases}$$

It has been known for long that this normalized flow has a global solution $\tilde{g}(s) = \lambda(s)g(t(s))$ for $s \geq 0$. Then we want to study if $\tilde{g}(s)$ has a limit as s goes to ∞.

It is believed that $\tilde{g}(s)$ should converge to a Kähler-Ricci soliton (possibly with mild singularity).

Here are some cases one can check the convergence:

• R. Hamilton-B. Chow: If M is a Riemann surface, $\tilde{g}(s)$ converges to a constant curvature metric.

• X.X. Chen-Tian: If the initial metric has positive bisectional curvature, then $\tilde{g}(s)$ converges to a Kähler-Einstein metric.

• Perelman, Tian-Zhu: If M has a Kähler- Einstein metric, then $\tilde{g}(s)$ converges to the Kähler-Einstein metric.

Thanks!