

Geometry of Ricci Solitons

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Ricci Solitons

A complete Riemannian (M^n, g_{ij}) is a *Ricci soliton* if there exists a smooth function f on M such that

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}, \quad (1.1)$$

for some constant ρ . f is called a *potential function* of the Ricci soliton. $\rho = 0$: *steady* soliton; $\rho > 0$: *shrinking* soliton; $\rho < 0$: *expanding* soliton; $f = \text{Const.}$: Einstein metric.

Ricci solitons are

- natural extension of Einstein manifolds;
- self-similar solutions to the Ricci flow
- possible singularity models of the Ricci flow
- critical points of Perelman's λ -entropy and μ -entropy.

Thus it is important to understand the geometry/topology of Ricci solitons and their classification.

Some Basic Facts about Ricci Solitons

- Compact steady or expanding solitons are Einstein in all dimensions;
- Compact shrinking solitons in dimensions $n = 2$ and $n = 3$ must be of positive constant curvature (by Hamilton and Ivey respectively);
- In dimension $n \geq 4$, there are compact non-Einstein gradient shrinking solitons;
- There is no non-flat complete noncompact shrinking soliton in dimension $n = 2$;
- The only three-dimensional complete noncompact non-flat gradient shrinking gradient solitons are quotients of round cylinder $\mathbb{S}^2 \times \mathbb{R}$.
- Ricci solitons exhibit rich geometric properties.

Examples of gradient Shrinking Ricci Solitons

- *Positive Einstein manifolds such as round spheres*

Remark:

(i) Suppose $R_{ij} = \frac{1}{2}g_{ij}$. Then under the Ricci flow, we have

$$g_{ij}(t) = (1 - t)g_{ij},$$

which exists for $-\infty < t < 1$, and shrinks homothetically as t increases. Moreover, the curvature blows up like $1/(1 - t)$. (This is an example of Type I singularity).

(ii) Similarly, a shrinking gradient Ricci soliton satisfying the equation

$$R_{ij} + \nabla_i \nabla_j f - \frac{1}{2}g_{ij} = 0$$

corresponds to the self-similar Ricci flow solution $g_{ij}(t)$ of the form

$$g_{ij}(t) = (1 - t)\varphi_t^*(g_{ij}), \quad t < 1,$$

where φ_t are the 1-parameter family of diffeomorphisms generated by $\nabla f/(1 - t)$.

- *Round cylinders $\mathbb{S}^{n-1} \times \mathbb{R}$*

- *Gaussian shrinking solitons*

$(\mathbb{R}^n, g_0, f(x) = |x|^2/4)$ is a gradient shrinker:

$$Rc + \nabla^2 f = \frac{1}{2}g_0.$$

- *Gradient Kähler shrinkers on $\mathbb{C}P^2 \# (-\mathbb{C}P^2)$*

In early 90's Koiso, and independently by myself, constructed a gradient shrinking metric on $\mathbb{C}P^2 \# (-\mathbb{C}P^2)$. It has $U(2)$ symmetry and positive Ricci curvature. More generally, they found $U(n)$ -invariant Kähler-Ricci solitons on twisted projective line bundle over $\mathbb{C}P^{n-1}$ for all $n \geq 2$.

- *Gradient Kähler shrinkers on $\mathbb{C}P^2 \# 2(-\mathbb{C}P^2)$*

In 2004, Wang-Zhu found a gradient Kähler-Ricci soliton on $\mathbb{C}P^2 \# 2(-\mathbb{C}P^2)$ which has $U(1) \times U(1)$ symmetry. More generally, they proved the existence of gradient Kähler-Ricci solitons on all Fano toric varieties of complex dimension $n \geq 2$ with non-vanishing Futaki invariant.

- *Noncompact gradient Kähler shrinkers*

In 2003, Feldman-Ilmanen-Knopf found the first complete noncompact $U(n)$ -invariant shrinking gradient Kähler-Ricci solitons, which are cone-like at infinity.

Examples of Steady Ricci Solitons

- *The cigar soliton* Σ

In dimension $n = 2$, Hamilton discovered the *cigar soliton* $\Sigma = (\mathbb{R}^2, g_{ij})$, where the metric g_{ij} is given by

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$

with potential function

$$f = -\log(1 + x^2 + y^2).$$

The cigar has positive (Gaussian) curvature and linear volume growth, and is asymptotic to a cylinder of finite circumference at ∞ .

- $\Sigma \times \mathbb{R}$: an 3-D steady Ricci soliton with nonnegative curvature.

- *The Bryant soliton on \mathbb{R}^n*

In the Riemannian case, higher dimensional examples of noncompact gradient steady solitons were found by Robert Bryant on \mathbb{R}^n ($n \geq 3$). They are rotationally symmetric and have positive sectional curvature. The volume of geodesic balls $B_r(0)$ grow on the order of $r^{(n+1)/2}$, and the curvature approaches zero like $1/s$ as $s \rightarrow \infty$.

- *Noncompact steady Kähler-Ricci soliton on \mathbb{C}^n*

I found a complete $U(n)$ -symmetric steady Ricci soliton on \mathbb{C}^n ($n \geq 2$) with positive curvature. The volume of geodesic balls $B_r(0)$ grow on the order of r^n , n being the complex dimension. Also, the curvature $R(x)$ decays like $1/r$.

- *Noncompact steady Kähler-Ricci soliton on $\mathbb{C}^n/\hat{\mathbb{Z}}_n$*

I also found a complete $U(n)$ symmetric steady Ricci soliton on the blow-up of $\mathbb{C}^n/\mathbb{Z}_n$ at 0, the same underlying space that Eguchi-Hanson ($n=2$) and Calabi ($n \geq 2$) constructed ALE Hyper-Kähler metrics.

Examples of 3-D Singularities in the Ricci flow.

- *3-manifolds with $Rc > 0$.*

According to Hamilton, any compact 3-manifold (M^3, g_{ij}) with $Rc > 0$ will shrink to a point in finite time and becomes round.

- *The Neck-pinching*

If we take a dumbbell metric on topological \mathbb{S}^3 with a neck like $\mathbb{S}^2 \times I$, we expect the neck will shrink under the Ricci flow because the positive curvature in the \mathbb{S}^2 direction will dominate the slightly negative curvature in the direction of interval I . We also expect the neck will pinch off in finite time. (In 2004, Angnents and Knopf confirmed the neck-pinching phenomenon in the rotationally symmetric case.)

- *The Degenerate Neck-pinching*

One could also pinch off a small sphere from a big one. If we choose the size of the little to be just right, then we expect a degenerate neck-pinching: there is nothing left on the other side. (This picture is confirmed by X.-P. Zhu and his student Gu in the rotationally symmetric case in 2006)

Singularities of the Ricci flow

In all dimensions, Hamilton showed that the solution $g(t)$ to the Ricci flow will exist on a maximal time interval $[0, T)$, where either $T = \infty$, or $0 < T < \infty$ and $|Rm|_{\max}(t)$ becomes unbounded as t tends to T . We call such a solution a *maximal solution*. If $T < \infty$ and $|Rm|_{\max}(t) \rightarrow \infty$ as $t \rightarrow T$, we say the maximal solution $g(t)$ *develops singularities* as t tends to T and T is a *singular time*. Furthermore, Hamilton classified them into two types:

$$\text{Type I: } \limsup_{t \rightarrow \infty} (T - t) |Rm|_{\max}(t) < \infty$$

$$\text{Type II: } \limsup_{t \rightarrow \infty} (T - t) |Rm|_{\max}(t) = \infty$$

Determine the structures of singularities

Understanding the structures of singularities of the Ricci flow is the first essential step. The parabolic rescaling/blow-up method was developed by Hamilton since 1990s' and further developed by Perelman to understand the structure of singularities. We now briefly outline this method.

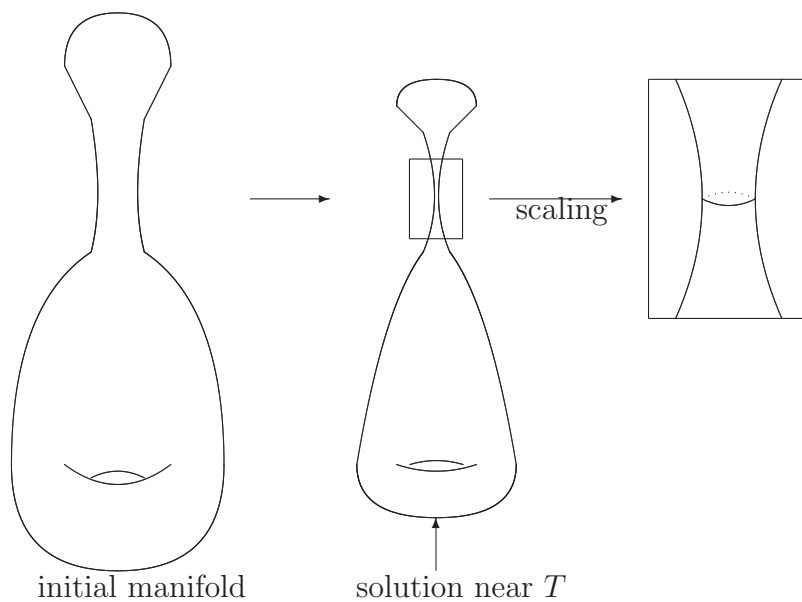


Figure 1: Rescaling

The Rescaling Argument:

- **Step 1:** Take a sequence of (almost) maximum curvature points (x_k, t_k) , where $t_k \rightarrow T$ and $x_k \in M$, such that for all $(x, t) \in M \times [0, t_k]$, we have

$$|Rm|(x, t) \leq CQ_k, \quad Q_k = |Rm|(x_k, t_k).$$

- **Step 2:** rescale $g(t)$ around (x_k, t_k) (by the factor Q_k and shift t_k to new time zero) to get the rescaled solution to the Ricci flow $\tilde{g}_k(t) = Q_k g(t_k + Q_k^{-1}t)$ for $t \in [-Q_k t_k, Q_k(T - t_k)]$ with

$$|Rm|(x_k, 0) = 1, \quad \text{and} \quad |Rm|(x, t) \leq C$$

on $M \times [-Q_k t_k, 0]$.

By Hamilton's **compactness theorem** and Perelman's **non-collapsing estimate**, rescaled solutions $(M^n, \tilde{g}_k(t), x_k)$ converges to $(\tilde{M}, \tilde{g}(t), \tilde{x})$, $-\infty < t < \Omega$, which is a complete *ancient solution* with bounded curvature and is κ -*noncollapsed on all scales*.

Hamilton's Compactness Theorem:

For any sequence of marked solutions $(M_k, g_k(t), x_k)$, $k = 1, 2, \dots$, to the Ricci flow on some time interval $(A, \Omega]$, if for all k we have

- $|Rm|_{g_k(t)} \leq C$, and
- $\text{inj}(M_k, x_k, g_k(0)) \geq \delta > 0$,

then a subsequence of $(M_k, g_k(t), x_k)$ converges in the C_{loc}^∞ topology to a complete solution $(M_\infty, g_\infty(t), x_\infty)$ to the Ricci flow defined on the same time interval $(A, \Omega]$.

Remark: In $n = 3$, by imposing an injectivity radius condition, Hamilton obtained the following structure results:

Type I: spherical or necklike structures;

Type II: either a steady Ricci soliton with positive curvature or $\Sigma \times \mathbb{R}$, the product of the cigar soliton with the real line.

Perelman's No Local Collapsing Theorem

Given any solution $g(t)$ on $M^n \times [0, T)$, with M compact and $T < \infty$, there exist constants $\kappa > 0$ and $\rho_0 > 0$ such that for any point $(x_0, t_0) \in M \times [0, T)$, $g(t)$ is κ -noncollapsed at (x_0, t_0) on scales less than ρ_0 in the sense that, for any $0 < r < \rho_0$, whenever

$$|Rm|(x, t) \leq r^{-2}$$

on $B_{t_0}(x_0, r) \times [t_0 - r^2, t_0]$, we have

$$\text{Vol}_{t_0}(B_{t_0}(x_0, r)) \geq \kappa r^n.$$

Corollary: If $|Rm| \leq r^{-2}$ on $B_{t_0}(x_0, r) \times [t_0 - r^2, t_0]$, then

$$\text{inj}(M, x_0, g(t_0)) \geq \delta r$$

for some positive constant δ .

Remark: There is also a stronger version: only require the scalar curvature $R \leq r^{-2}$ on $B_{t_0}(x_0, r)$.

The Proof of Perelman's No Local Collapsing Theorems

Perelman proved two versions of “no local collapsing” property, one with a new entropy functional, and the other the reduced volume associated to a space-time distance function obtained by path integral analogous to what Li-Yau did in 1986.

- The \mathcal{W} -functional and μ -entropy:

$$\mu(g, \tau) = \inf \left\{ \mathcal{W}(g, f, \tau) \mid f \in C^\infty(M), \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV = 1 \right\},$$

$$\text{where } \mathcal{W}(g, f, \tau) = \int_M [\tau(R + |\nabla f|^2) + f - n](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV.$$

- Perelman's reduced distance l and reduced volume $\tilde{V}(\tau)$:

For any space path $\sigma(s)$, $0 \leq s \leq \tau$, joining p to q , define the action $\int_0^\tau \sqrt{s}(R(\sigma(s), t_0 - s) + |\dot{\sigma}(s)|_{g(t_0-s)}^2) ds$, the L -length $L(q, \tau)$ from (p, t_0) to $(q, 0)$, $l(q, \tau) = \frac{1}{2\sqrt{\tau}} L(q, \tau)$, and

$$\tilde{V}(\tau) = \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-l(q,\tau)} dV_\tau(q).$$

- Monotonicity of μ and \tilde{V} under the Ricci flow:

Perelman showed that under the Ricci flow $\partial g(\tau)/\partial\tau = 2Rc(\tau)$, $\tau = t_0 - t$, both $\mu(g(\tau), \tau)$ and $\tilde{V}(\tau)$ are nonincreasing in τ .

Remark: $\mathbb{S}^1 \times \mathbb{R}$ is **NOT** κ -noncollapsed on large scales for any $\kappa > 0$ and neither is the cigar soliton Σ , or $\Sigma \times \mathbb{R}$. In particular, $\Sigma \times \mathbb{R}$ cannot occur in the limit of rescaling! (However, $\mathbb{S}^2 \times \mathbb{R}$ is κ -noncollapsed on all scales for some $\kappa > 0$.)

Magic of 3-D: The Hamilton-Ivey Pinching Theorem

In dimension $n = 3$, we can express the curvature operator $Rm : \Lambda^2(M) \rightarrow \Lambda^2(M)$ as

$$Rm = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \nu \end{pmatrix},$$

where $\lambda \geq \mu \geq \nu$ are the principal sectional curvatures and the scalar curvature $R = 2(\lambda + \mu + \nu)$.

The Hamilton-Ivey pinching theorem *Suppose we have a solution $g(t)$ to the Ricci flow on a three-manifold M^3 which is complete with bounded curvature for each $t \geq 0$. Assume at $t = 0$ the eigenvalues $\lambda \geq \mu \geq \nu$ of Rm at each point are bounded below by $\nu \geq -1$. Then at all points and all times $t \geq 0$ we have the pinching estimate*

$$R \geq (-\nu)[\log(-\nu) + \log(1 + t) - 3]$$

whenever $\nu < 0$.

Remark: This means in 3-D if $|Rm|$ blows up, the positive sectional curvature blows up faster than the (absolute value of) negative sectional curvature. As a consequence, **any limit of parabolic dilations at an almost maximal singularity has $Rm \geq 0$**

Ancient κ -Solution

An **ancient κ -solution** is a *complete ancient solution with nonnegative and bounded curvature, and is κ -noncollapsed on all scales.*

Recap: Whenever a maximal solution $g(t)$ on a compact M^n develop singularities, parabolic dilations around any (maximal) singularity converges to some limit ancient solution $(\tilde{M}, \tilde{g}(t), \tilde{x})$, which has bounded curvature and is κ -noncollapsed. Moreover, if $n = 3$, then the ancient solution has nonnegative sectional curvature, thus an ancient κ -solution.

Structure of Ancient κ -solutions in 3-D

Canonical Neighborhood Theorem (Perelman):

$\forall \varepsilon > 0$, every point (x_0, t_0) on an orientable nonflat ancient κ -solution $(\tilde{M}^3, \tilde{g}(t))$ has an open neighborhood B , which falls into one of the following three categories:

- (a) B is an ε -neck of radius $r = R^{-1/2}(x_0, t_0)$; (i.e., after scaling by the factor $R(x_0, t_0)$, B is ε -close, in $C^{[\varepsilon^{-1}]}$ -topology, to $\mathbb{S}^2 \times [-\varepsilon^{-1}, \varepsilon^{-1}]$ of scalar curvature 1.)

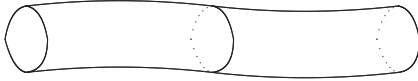


Figure 2: ε -neck

- (b) B is an ε -cap; (i.e., a metric on \mathbb{B}^3 or $\mathbb{RP}^3 \setminus \bar{\mathbb{B}}^3$ and the region outside some suitable compact subset is an ε -neck).



Figure 3: ε -cap

- (c) B is compact (without boundary) with positive sectional curvature (hence diffeomorphic to the 3-sphere by Hamilton).

Classification of 3-d κ -shrinking solitons

- Classification of 3-d κ -shrinking solitons (Perelman): they are either round \mathbb{S}^3/Γ , or round cylinder $\mathbb{S}^2 \times \mathbb{R}$, or its \mathbb{Z}_2 quotients $\mathbb{S}^2 \times \mathbb{R}/\mathbb{Z}_2$. In particular, *there is no 3-d complete noncompact κ -shrinking solitons with $Rm > 0$.*

Improvements in 3-D

- Complete noncompact non-flat shrinking gradient soliton with bounded Rm and $Rc \geq 0$ are quotients of $\mathbb{S}^2 \times \mathbb{R}$. (Naber, 2007)
- Complete noncompact non-flat shrinking gradient soliton with $Rc \geq 0$ and with curvature growing at most as fast as $e^{ar(x)}$ are quotients of $\mathbb{S}^2 \times \mathbb{R}$. (Ni-Wallach, 2007)
- Complete noncompact non-flat shrinking gradient soliton are quotients of $\mathbb{S}^2 \times \mathbb{R}$ (Cao-Chen-Zhu, 2007).

Further Extensions

I. 4-D:

- Any complete gradient shrinking soliton with $Rm \geq 0$ and PIC, and satisfying additional assumptions, is either a quotient of \mathbb{S}^4 or a quotient of $\mathbb{S}^3 \times \mathbb{R}$. (Ni-Wallach, 2007)
- Any non-flat complete noncompact shrinking Ricci soliton with bounded curvature and $Rm \geq 0$ is a quotient of either $\mathbb{S}^3 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}^2$. (Naber, 2007)

II. $n \geq 4$ with Weyl tensor $W = 0$: Let (M^n, g) be a complete gradient shrinking soliton with vanishing Weyl tensor.

- Ni-Wallach (2007): Assume (M^n, g) has $Rc \geq 0$ and that $|Rm|(x) \leq e^{a(r(x)+1)}$ for some constant $a > 0$, where $r(x)$ is the distance function to some origin. Then its universal cover is \mathbb{R}^n , \mathbb{S}^n or $\mathbb{S}^{n-1} \times \mathbb{R}$. (also by X. Cao and B. Wang)
- Petersen-Wylie (2007): *Assume*

$$\int_M |Rc|^2 e^f dvol < \infty.$$

Then (M^n, g) is a finite quotient of \mathbb{R}^n , \mathbb{S}^n or $\mathbb{S}^{n-1} \times \mathbb{R}$.

- Z.-H. Zhang (2008): *Any complete gradient shrinking soliton with vanishing Weyl tensor must be a finite quotients of \mathbb{R}^n , \mathbb{S}^n or $\mathbb{S}^{n-1} \times \mathbb{R}$.*