Exercise for Scattering Amplitude (F, T6)
Problem set 2, due to 22 May, 2019

1) We choose the massless four–momentum as

$$p^\mu = (E, E\ \sin\theta\cos\phi, E\ \sin\theta\sin\phi, E\ \cos\theta)\ ,$$

and the polarization vectors as:

$$\epsilon^\mu_\pm(p) = \frac{e^{\pm i\phi}}{\sqrt{2}}\ (0, \cos\theta\cos\phi \pm i\sin\phi, \cos\theta\sin\phi \mp i\cos\phi, -\sin\theta)$$

a) Determine $\lambda_\alpha,\ \lambda^\alpha, \tilde\lambda^{\dot\beta}$ and $\tilde\lambda_{\dot\beta}$.

b) How does the polarization vectors look like for $\theta = 0$ and $\phi = 0$ and how is this basis called ?

c) Prove, that $\epsilon_\pm(p)^2 = 0$ and $p_\mu\epsilon^\mu_\pm(p) = 0$. What is $(\epsilon_\pm(p))^*$?

d) Since $\epsilon_\pm(p)^2 = 0$ the expression $(\epsilon_\pm(p))_{\alpha\dot\beta} := \sigma_{\mu\alpha\dot\beta}\ \epsilon^\mu_\pm(p)$ can be written as product of a square $[p|_{\dot\beta} \equiv (0, \tilde\lambda_{\dot\beta})$ and angle spinor $|r\rangle_\alpha \equiv \binom{\tau_\alpha}{0}$. Calculate $(\epsilon_+(p))_{\alpha\dot\beta}$ and then find an angle spinor $|r\rangle_\alpha$ such that $(\epsilon_+(p))_{\alpha\dot\beta} = \tau_\alpha\tilde\lambda_{\dot\beta}$, with $\langle pr\rangle = -\sqrt{2}$. For the sequel we define: $(\epsilon_+(p;r))_{\alpha\dot\beta} := (\epsilon_+(p))_{\alpha\dot\beta}$.

e) On the other hand, in the lecture we have introduced $(\epsilon_+(p;q))_{\alpha\dot\beta} = -\sqrt{2}\ \frac{\mu_\alpha\tilde\lambda_{\dot\beta}}{\langle pq\rangle}$, with $|q\rangle_\alpha \equiv \binom{\mu_\alpha}{0}$. Compute $(\epsilon_+(p;q))_{\alpha\dot\beta} - (\epsilon_+(p;r))_{\alpha\dot\beta}$ and show that $\epsilon^\mu_+(p;q) - \epsilon^\mu_+(p;r) = \frac{\langle rq\rangle}{\langle pq\rangle}\ p^\mu$.

f) Repeat d) and e) for $(\epsilon_-(p;r))_{\alpha\dot\beta} := (\epsilon_-(p))_{\alpha\dot\beta}$.

2) Completeness relation

Prove the following relation:

$$\sum_{\sigma=\pm} \epsilon^\mu_\sigma(p;q)\ \epsilon^\nu_\sigma(p;q)^* = -\eta^{\mu\nu} + \frac{1}{p_\rho q^\rho}\ (q^\mu p^\nu + q^\nu p^\mu)\ .$$

For this the following relation is useful:

$$\sigma_\nu\overline{\sigma}_\tau\sigma_\mu + \sigma_\mu\overline{\sigma}_\tau\sigma_\nu = 2\ (\eta_{\tau\nu}\ \sigma_\mu + \eta_{\tau\mu}\ \sigma_\nu - \eta_{\mu\nu}\ \sigma_\tau)\ ,$$

which also may be proven.