The Harmony of Superstring Amplitudes: Implications from/to Field-Theory

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$\Delta p \cdot \Delta q \geqslant \frac{1}{2} \hbar$

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Harmony of scattering amplitudes

"Scattering amplitudes:
the most perfect microscopic structures in the universe",

L. Dixon, `arXiv:1105.0771`

- Scattering amplitudes in gauge and gravity theory

  have a **remarkably rich yet simple structure**

- Allow to **develop** even more **powerful tools**

  to understand their behavior

Scattering amplitudes: field–theory vs. string theory

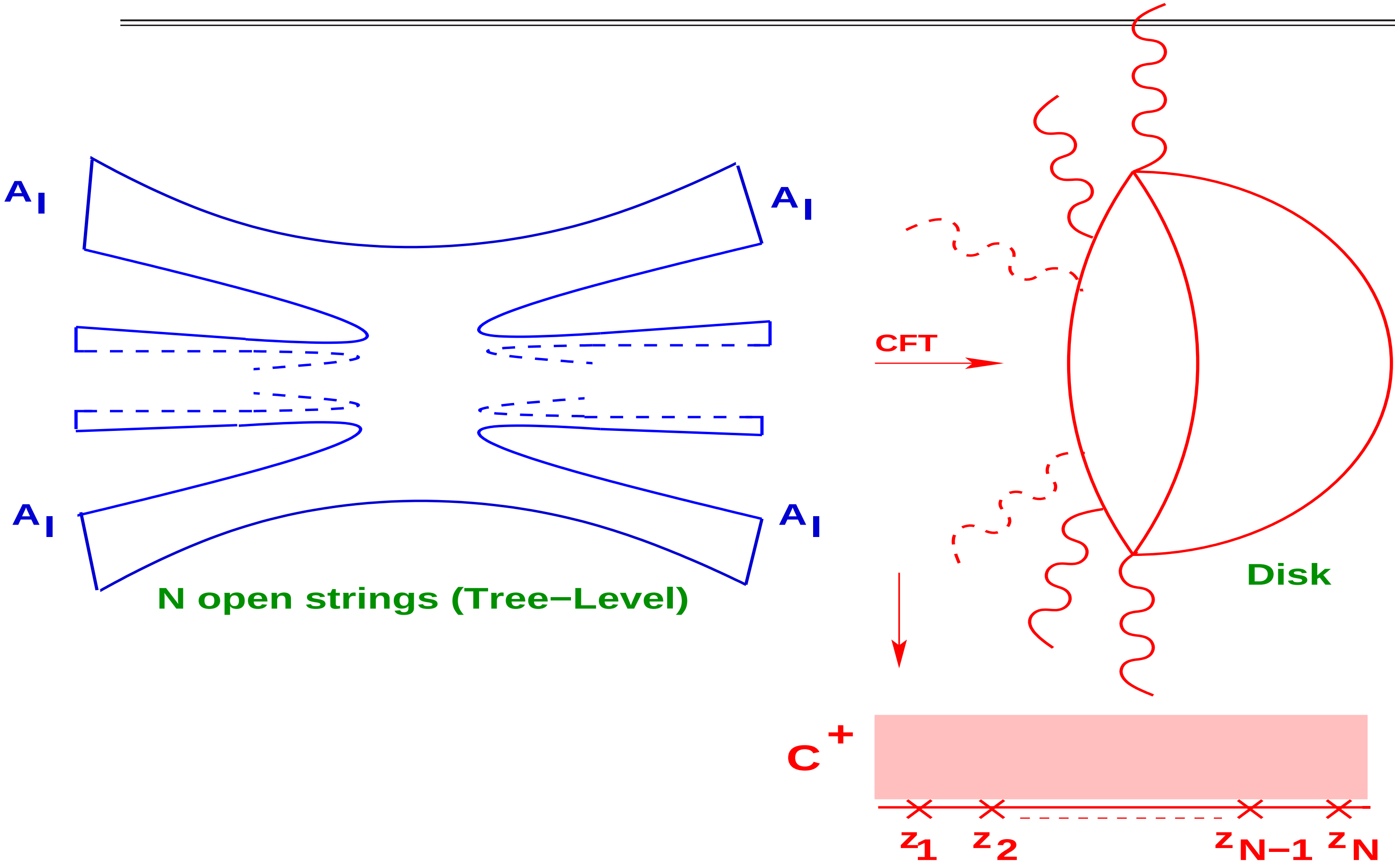
Various **relations within or between** gravity and gauge theory scattering amplitudes suggest a **unification** within or between these theories of the sort **inherent to string theory** !

E.g.:

- KLT: relation between gravity and gauge theory,
- BCJ: relations within gauge theory between color and kinematics

⋮

I. Disk scattering of open strings



$$\boxed{\mathcal{A}(1,2,\ldots,N)=g_{YM}^{N-2}\sum_{\sigma\in S_N}\mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}\ldots T^{a_{\sigma(N)}})\ A(\sigma(1),\sigma(2),\ldots,\sigma(N))}$$

$A(1,2,\ldots,N)$ tree–level color–ordered N–leg partial amplitude (helicity subamplitude)

Partial subamplitudes

$A(1,2,\ldots,N)$ tree–level color–ordered N–leg partial amplitude

The $(N-1)!$ subamplitudes are not all independent:

Properties of the string world-sheet

In addition to **cyclic symmetries** by applying **reflection** and **parity symmetries**

$$\boxed{A(1,2,\ldots,N) = (-1)^N\ A(N,\ldots,2,1)}$$

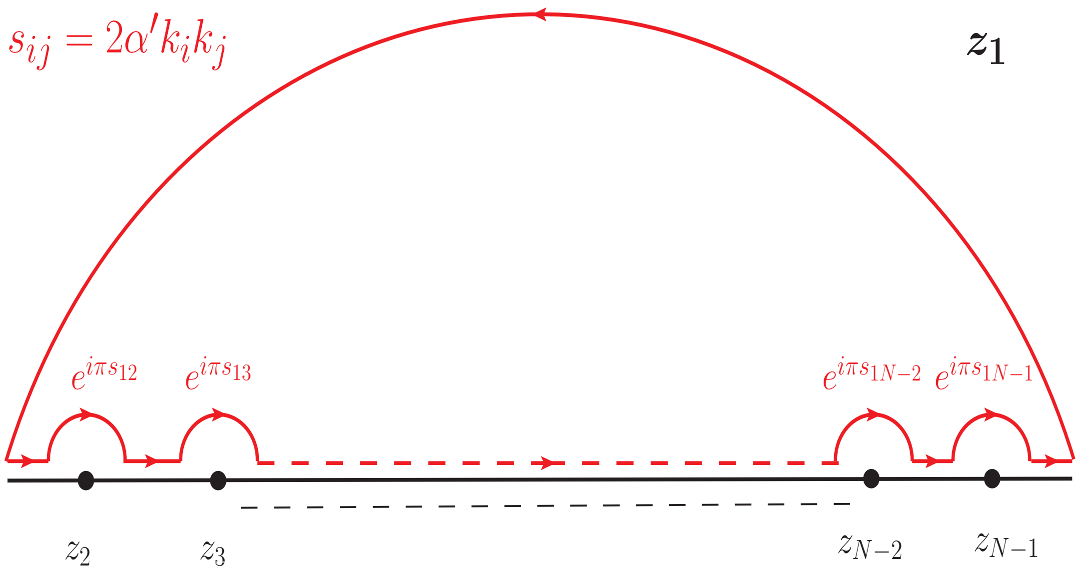
reduce the number of independent partial amplitudes

from $(N-1)!$ to $\frac{1}{2}(N-1)!$

World–sheet derivation of amplitude relations

By applying world–sheet string techniques $\Longrightarrow$ new algebraic identities

$$A(1,\ldots,N) = V_{\mathrm{CKG}}^{-1} \int\limits_{z_1<\ldots<z_N} \left(\prod_{j=1}^{N} dz_j \right) \sum_{\mathcal{K}_I} \mathcal{K}_I \prod_{i<j}^{N} |z_i-z_j|^{s_{ij}}\, (z_i-z_j)^{n^I_{ij}}$$



by analytically continuing
the z_1–integration to
the whole complex plane
and integrating z_1
along the contour integral

$$A(1,2,\ldots,N) \; + e^{i\pi s_{12}}\, A(2,1,3,\ldots,N-1,N) + e^{i\pi (s_{12}+s_{13})}\, A(2,3,1,\ldots,N-1,N)$$

$$+ \ldots + e^{i\pi (s_{12}+s_{13}+\ldots+s_{1N-1})}\, A(2,3,\ldots,N-1,1,N) = 0$$

St.St. `arXiv:0907.2211`

Subamplitude relations in string theory

- proof does not rely on **any kinematic properties** of subamplitudes
- for any open string state: **boson or fermion**
- these relations hold in **any space–time dimensions D**
- for **any amount of supersymmetry**

Take $\alpha' \to 0$ limit ($e^{i\pi s_{ij}} \to 1$):

$$\boxed{\begin{aligned} A_{FT}(1,2,\ldots,N) \;\; &+ A_{FT}(2,1,3,\ldots,N-1,N) + A_{FT}(2,3,1,\ldots,N-1,N) \\ &+ \ldots + A_{FT}(2,3,\ldots,N-1,1,N) = 0 \end{aligned}}$$

Subcyclic property (photon-decoupling identity: $T^{a_N} \to 1$):

$$\sum_{\sigma\in S_{N-1}} A_{FT}(\sigma(1),\sigma(2),\ldots,\sigma(N-1),N) = 0$$

Subamplitude relations in string theory

E.g. $N=4$:

$$\boxed{\frac{A(1,2,4,3)}{A(1,2,3,4)}=\frac{\sin(\pi u)}{\sin(\pi t)}\quad,\quad\frac{A(1,3,2,4)}{A(1,2,3,4)}=\frac{\sin(\pi s)}{\sin(\pi t)}}$$

As a result these relations allow to express all six partial amplitudes in terms of **one**, say $A(1,2,3,4)$:

$$\begin{aligned}A(1,4,3,2)&=A(1,2,3,4)\ ,\\ A(1,2,4,3)&=A(1,3,4,2)=\frac{\sin(\pi u)}{\sin(\pi t)}\ A(1,2,3,4)\ ,\\ A(1,3,2,4)&=A(1,4,2,3)=\frac{\sin(\pi s)}{\sin(\pi t)}\ A(1,2,3,4)\ .\end{aligned}$$

generic N :

These relations allow for a **complete reduction** of the full string subamplitudes to a **minimal basis of** $(N-3)!$ **subamplitudes**

E.g. $N = 5$: Relations:

$$\begin{aligned} & \sin[\pi(s_2 - s_4)]\ A(1,2,3,4,5) + \{\sin[\pi(s_1 + s_2 - s_4)] - \sin(\pi s_1)\}\ A(1,3,4,5,2) \\ + & \sin[\pi(s_2 - s_4)]\ A(1,4,5,2,3) + \{\sin(\pi s_5) + \sin[\pi(s_2 - s_4 - s_5)]\}\ A(1,5,2,3,4) = 0 \end{aligned}$$

$$\begin{aligned} & [\sin(\pi s_1) + \sin(\pi s_5)]\ A(1,2,3,4,5) + \sin[\pi(s_1 + s_5)]\ A(1,3,4,5,2) \\ + & \{\sin[\pi(s_1 + s_2 - s_4)] - \sin[\pi(s_2 - s_4 - s_5)]\}\ A(1,4,5,2,3) + \sin[\pi(s_1 + s_5)]\ A(1,5,2,3,4) = 0 \end{aligned}$$

As a result these relations allow to express all six partial amplitudes in terms of **two**, say $A(1,2,3,4,5)$ and $A(1,3,2,4,5)$, e.g.:

$$\begin{aligned} A(1,2,5,4,3) &= -A(1,3,4,5,2) = \sin[\pi(s_3 - s_1 - s_5)]^{-1} \\ & \times \ \{\ \sin[\pi(s_3 - s_5)]\ A(1,2,3,4,5) + \sin[\pi(s_2 + s_3 - s_5)]\ A(1,3,2,4,5)\ \}\ , \\ A(1,3,4,2,5) &= -A(1,5,2,4,3) = \sin[\pi(s_3 - s_1 - s_5)]^{-1} \\ & \times \ \{\ \sin(\pi s_1)\ A(1,2,3,4,5) - \sin[\pi(s_1 + s_2)]\ A(1,3,2,4,5)\ \}\ ,\ \ldots \end{aligned}$$

Clearly, in the field theory limit, these two relations boil down to the subcyclic identity:

$$A_{FT}(1,2,3,4,5) + A_{FT}(1,3,4,5,2) + A_{FT}(1,4,5,2,3) + A_{FT}(1,5,2,3,4) = 0.$$

Subamplitude relations: string theory vs. field theory

In STTH these relations hold to all orders in α' !

In FT similar relations found by:

- Kleiss, Kuijf, 1989 $(N-2)!$
  Del Duca, Dixon, Maltoni, 2000
- Bern, Carrasco, Johannson, 2008 $(N-3)!$

In the FT limit our relations simply reduce to the well–known identities:

$$\frac{A_{FT}(1,2,4,3)}{A_{FT}(1,2,3,4)}=\frac{u}{t}\quad,\quad\frac{A_{FT}(1,3,2,4)}{A_{FT}(1,2,3,4)}=\frac{s}{t}$$

Subcyclic property $A_{FT}(1,2,3,4)+A_{FT}(1,3,4,2)+A_{FT}(1,4,2,3)=0$

Reproduce Kleiss–Kuijf and Bern–Carrasco–Johannson identitities in field–theory limit $\alpha'\to 0$

II. Higher–point closed superstring amplitudes

(Color ordered) gluon amplitudes give rise to graviton amplitudes

At tree–level:

$$\boxed{\textbf{gravity} = \textbf{gauge theory} \otimes \textbf{gauge theory}}$$

- Spectrum:

$$\boxed{\begin{aligned} \left|\begin{matrix}\mathcal{N}{=}8\\ SUGRA\end{matrix}\right\rangle &= \left|\begin{matrix}\mathcal{N}{=}4\\ SYM\end{matrix}\right\rangle \otimes \left|\begin{matrix}\mathcal{N}{=}4\\ SYM\end{matrix}\right\rangle \\ 256 &= 16 \times 16 \end{aligned}}$$

E.g.: in $D = 4$, N=8: Fock space decomposition
of the 256 states of the $\mathcal{N} = 8$ supergravity multiplet

- Vertex operators:

$$\boxed{\begin{aligned} V_G(\epsilon, \overline{z}, z) &\simeq V_g(\overline{\epsilon}, \eta) \otimes V_g(\epsilon, \xi) \\ \epsilon_{\mu\nu} &= \overline{\epsilon}_\mu \otimes \epsilon_\nu \end{aligned}}$$

with $R_{\mu\nu\rho\sigma} = \kappa\ k_{[\mu} k_{[\rho}\ \overline{\epsilon}_{\nu]} \otimes \epsilon_{\sigma]}$
linearized Riemann tensor

String theory: Gauge vs. gravitational amplitudes

- Amplitudes (on–shell S–matrix): KLT relations: closed $=$ open$\otimes$open

$$\boxed{\begin{aligned} M_4(1,2,3,4)_{S^2} &= (2\alpha'\pi)^{-1}\ \sin(\pi s_{12})\ \overline{A}_4(1,2,3,4)_{D_2}\ A_4(1,2,4,3)_{D_2} \\ M_5(1,2,3,4,5)_{S^2} &= (2\alpha'\pi)^{-2}\ \sin(\pi s_{12})\ \sin(\pi s_{34})\ \overline{A}_5(1,2,3,4,5)_{D_2}\ A_5(2,1,4,3,5)_{D_2} \\ &+ \ \text{permutations of }(23) \\ M_6(1,2,3,4,5,6)_{S^2} &= (2\alpha'\pi)^{-3}\ \sin(\pi s_{12})\ \sin(\pi s_{45})\ \overline{A}_6(1,2,3,4,5,6)_{D_2} \\ &\times\ \{\ \sin(\pi s_{35})\ A_6(2,1,5,3,4,6)_{D_2}+\sin[\pi(s_{34}+s_{35})]\ \ A_5(2,1,5,4,3,6)_{D_2}\} \\ &+ \ \text{permutations of }(234) \\ &\ \vdots \end{aligned}}$$

Field–theory amplitudes are obtained (reproduced) for $\alpha' \to 0$

E.g.: $$M(1^-,2^-,3^+,4^+) = \left(\frac{\kappa}{2}\right)^2 \frac{\langle 12\rangle^8\ [12]}{N(4)\ \langle 34\rangle}\ \frac{B(s_{12},s_{14})}{B(-s_{12},-s_{14})} \longrightarrow \left(\frac{\kappa}{2}\right)^2 \frac{\langle 12\rangle^8\ [12]}{N(4)\ \langle 34\rangle}$$

with: $\langle ij\rangle[ij] = s_{ij} = \alpha' k_i k_j, \quad N(n) = \prod_{i=1}^{n-1}\prod_{j=i+1}^{n}\langle ij\rangle$

Tree–level higher order gravitational couplings

Type I or Type II superstring:

$$\mathcal{L}_{\text{tree}} = \frac{1}{2\kappa^2}\, R + \frac{\alpha'^3}{2^9\, 4!\, \kappa^2}\, \zeta(3)\ t_8 t_8\ R^4$$

Gross, Witten, 1986
Gross, Sloan, 1987

$$\mathcal{L}'_{\text{tree}} = \kappa^{-2} \sum_{n\geq 4}^{\infty} \sum_{m=0}^{\infty} \alpha'^{\,n-1+m} \sum_{\substack{i_r\in \mathbf{N},\, i_1>1 \\ i_1+\ldots+i_d=n-1+m}}{}' \zeta(i_1,\ldots,i_d)\ c_{m,n,\vec{i}}\ (t^{\vec{i}}_{m,n}\ D^{2m} R^n)$$

- constraints and transcendentality properties of curvature couplings

  St. St., `arXiv:0910.0180`

- information on candidate counter terms satisfying SUSY Ward identities

  Beisert, Elvang, Freedman, Kiermaier, Morales, St. St. `arXiv:1009.1643`

Multi zeta values (MZVs)

$$\boxed{\zeta(i_1,\ldots,i_d) = \sum_{n_1>\ldots>n_d>0} \prod_{r=1}^{d} n_r^{-i_r}, \quad i_r \in \mathbf{N},\ i_1 > 1}$$

transcendentality degree $\sum_{r=1}^{d} i_r = n - 1 + m$ and depth d

Many relations over $\mathbf{Q}$, e.g.:

$$\begin{aligned}
\zeta(2,1) &= 2\,\zeta(3)\\
\zeta(4,1) &= 2\,\zeta(5) - \zeta(2)\,\zeta(3)\\
\zeta(5,3) &= -\frac{5}{2}\,\zeta(6,2) - \frac{21}{25}\,\zeta(2)^4 + 5\,\zeta(3)\,\zeta(5)\\
&\ \ \vdots
\end{aligned}$$

Multi zeta values (MZVs)

The set of integral linear combinations of MZVs is a ring
e.g.: $\zeta(m)\ \zeta(n) = \zeta(m,n) + \zeta(n,m) + \zeta(m+n)$

Zagier: For a given weight $w \in \mathbf{N}$ the dimension d_w of the space spanned by MZVs: $\boldsymbol{d_w = d_{w-2} + d_{w-3}}$, $d_0, d_1 = 0$,

w	d_w	basis
2	1	$\zeta(2)$
3	1	$\zeta(3)$
4	1	$\zeta(2)^2$
5	2	$\zeta(5),\ \zeta(2)\zeta(3)$
6	2	$\zeta(2)^3,\ \zeta(3)^2$
7	3	$\zeta(7),\ \zeta(2)\zeta(5),\ \zeta(3)\zeta(2)^2$
8	4	$\zeta(2)^4,\ \zeta(2)\zeta(3)^2,\ \zeta(3)\zeta(5),\ \zeta(5,3)$
9	5	$\zeta(2)^3\zeta(3),\ \zeta(3)^3,\ \zeta(2)^2\zeta(5),\ \zeta(2)\zeta(7),\ \zeta(9)$
10	7	$\zeta(2)^5,\ \zeta(2)^2\zeta(3)^2,\ \zeta(2)\zeta(3)\zeta(5),\ \zeta(5)^2,\ \zeta(3)\zeta(7),\ \zeta(2)\zeta(5,3),\ \zeta(7,3)$

Field redefinitons, Bianchi identities

- Ricci tensors and Ricci scalars can always be eliminated on–shell
- Weyl tensor can always be written as Riemann tensor

- $D^2 R \simeq R^2$ (etc.)

We stick to the prescription to write all terms
with the highest possible number of Riemann tensors.

E.g.: $D^2 R^4 \simeq R^5$

moreover: $D^2 R^4 \Big|_{\substack{\text{4–point} \\ \text{on-shell}}} = 0$

$\Longrightarrow$ one needs to compute 5–graviton amplitude to probe $\alpha'^4 \; \{D^2 R^4, R^5\}$ terms

Conclusion: Only "true terms" can be probed on–shell

Gravitational amplitudes in superstring theory

Recall:

$$M(1,2,3,4) \sim \frac{B(s,u)}{B(-s,-u)} = -e^{-2\sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{2n+1}(s^{2n+1}+t^{2n+1}+u^{2n+1})}$$

Task: Compute graviton amplitudes:

$$M(1,2,3,4,5) \quad , \quad M(1,2,3,4,5,6)$$

and extract their power series expansion in α'

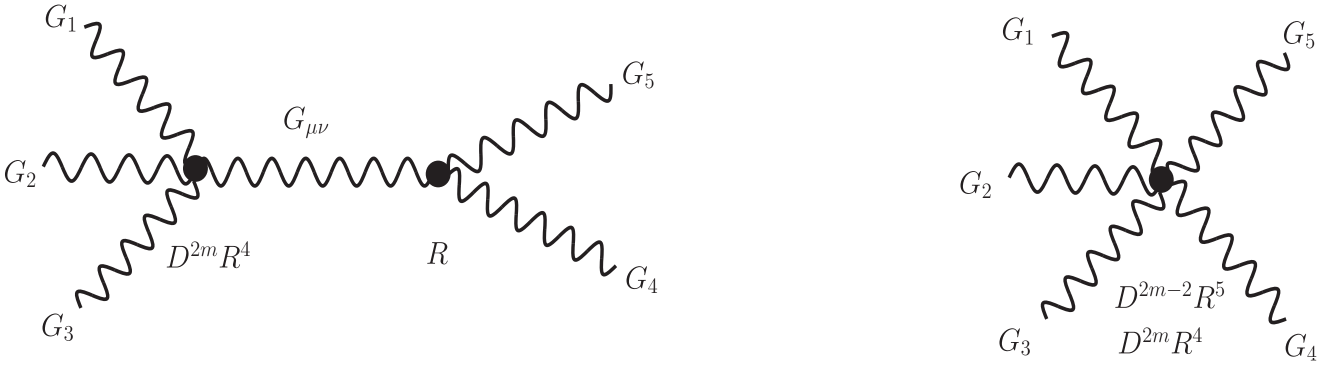
St.St. `arXiv:0910.0180`

- completely model independent
- universal to all string compactifications
- any numbers of supersymmetries or dimensions D

Five–graviton amplitude:

The α'–expansion has no $\left\{\begin{array}{ll} \alpha'^4 & \zeta(4) \\ \alpha'^5 & \zeta(2)\zeta(3) \\ \alpha'^6 & \zeta(6) \\ \alpha'^7 & \zeta(3)\zeta(4),\ \zeta(2)\zeta(5) \\ \alpha'^8 & \zeta(8),\ \zeta(2)\zeta(3)^2,\ \zeta(5,3) \end{array}\right.$

These terms cannot be generated by any reducible diagram (with five external gravitons)



diagrams contributing for $N=5$ at order α'^{3+m}

$\Longrightarrow$ absence of contact interactions with coefficients:

$$\alpha'^5\zeta(2)\zeta(3),\alpha'^6\zeta(6),\ \alpha'^7\zeta(3)\zeta(4),\alpha'^7\zeta(2)\zeta(5),\ \alpha'^8\zeta(8),\alpha'^8\zeta(2)\zeta(3)^2,\alpha'^8\zeta(5,3)$$

Tree–level higher order gravitational couplings

	$N=4$	$\boldsymbol{N=5}$	$\boldsymbol{N=6}$	$N=7$	$N=8$
$\alpha'^3\ \zeta(3)$	R^4				
$\alpha'^4\ \cancel{\zeta(4)}$	$\cancel{D^2R^4}$	$\cancel{\boldsymbol{R^5}}$			
$\alpha'^5\ \zeta(5)$	D^4R^4	$\boldsymbol{D^2R^5}$	$\boldsymbol{R^6}$		
$\alpha'^5\ \cancel{\zeta(2)\zeta(3)}$	$\cancel{D^4R^4}$	$\cancel{\boldsymbol{D^2R^5}}$	$\cancel{\boldsymbol{R^6}}$		
$\alpha'^6\ \zeta(3)^2$	D^6R^4	$\boldsymbol{D^4R^5}$	$\boldsymbol{D^2R^6}$	R^7 ?	
$\alpha'^6\ \zeta(6)$	$\cancel{D^6R^4}$	$\cancel{\boldsymbol{D^4R^5}}$	$\cancel{\boldsymbol{D^2R^6}}$	R^7 ?	
$\alpha'^7\ \zeta(7)$	D^8R^4	$\boldsymbol{D^6R^5}$	$\boldsymbol{D^4R^6}$	D^2R^7 ?	R^8 ?
$\alpha'^7\ \zeta(3)\zeta(4)$	$\cancel{D^8R^4}$	$\cancel{\boldsymbol{D^6R^5}}$	$\cancel{\boldsymbol{D^4R^6}}$	D^2R^7 ?	R^8 ?
$\alpha'^7\ \zeta(2)\zeta(5)$	$\cancel{D^8R^4}$	$\cancel{\boldsymbol{D^6R^5}}$	$\cancel{\boldsymbol{D^4R^6}}$	D^2R^7 ?	R^8 ?
$\alpha'^8\ \zeta(3)\zeta(5)$	$D^{10}R^4$	$\boldsymbol{D^8R^5}$	$\boldsymbol{D^6R^6}$	D^4R^7 ?	D^2R^8 ?
$\alpha'^8\ \zeta(8)$	$\cancel{D^{10}R^4}$	$\cancel{\boldsymbol{D^8R^5}}$	$\cancel{\boldsymbol{D^6R^6}}$	D^4R^7 ?	D^2R^8 ?
$\alpha'^8\ \zeta(2)\zeta(3)^2$	$\cancel{D^{10}R^4}$	$\cancel{\boldsymbol{D^8R^5}}$	$\cancel{\boldsymbol{D^6R^6}}$	D^4R^7 ?	D^2R^8 ?
$\alpha'^8\ \zeta(5,3)$	$\cancel{D^{10}R^4}$	$\cancel{\boldsymbol{D^8R^5}}$	$\cancel{\boldsymbol{D^6R^6}}$	D^4R^7 ?	D^2R^8 ?

$\Longrightarrow$ refined transcendentality property: only MZVs of odd weight appear !

Concluding remarks

Constraints on higher order gravitational couplings:
Vanishing and transcendentality properties:
only MZVs of odd weight appear

Note: $\zeta(5,3) = -\frac{5}{2}\zeta(6,2) - \frac{21}{25}\zeta(2)^4 + 5\zeta(3)\zeta(5)$

- Results restrict (together with one–loop results) the ring of possible modular forms describing the perturbative and non–perturbative completion of the higher order terms in $D = 10$ type IIB superstring theory

- Results constrain candidate counter terms in N=8 SUGRA:
  D^4R^4, D^6R^4 have non–vanishing single–soft scalar limits
  $\Longrightarrow$ operators violate continous $E_{7(7)}$–symmetry
  $\Longrightarrow$ no counter terms at 5– and 6–loop

Beisert, Elvang, Freedman, Kiermaier, Morales, St. St. `arXiv:1009.1643`

III. Full $N-$point superstring amplitude

Compact and short expression in terms of a *minimal basis* of $(N-3)!$ *building blocks*

$$\boxed{A(1,2,\ldots,N;\alpha') = \sum_{\sigma\in S_{N-3}} A_{YM}(1,2_\sigma,\ldots,(N-2)_\sigma,N-1,N)\ F^{\sigma}_{(1,\ldots,N)}(\alpha')}$$

Mafra, Schlotterer, St.St., `arXiv:1105.xxxx` to appear

A_{YM}	Yang–Mills subamplitudes
$F^{\sigma}(\alpha')$	generalized Euler integral multiple Gaussian hypergeometric functions

Pure spinor formalism allows for remarkable **simplifications** to package kinematics and $\alpha'-$dependence

Structure of $N-$point superstring amplitude

Remarks:

- $N=4$: $\left\{\begin{array}{rcl} A_{YM}(1,2,3,4) & \rightarrow & 2g_{YM}^2\,\frac{1}{su}\,t_8(\xi_1,k_1,\xi_2,k_2,\xi_3,k_3,\xi_4,k_4) \\ F_{1234} & \rightarrow & \frac{\Gamma(1-s)\ \Gamma(1-u)}{\Gamma(1-s-u)} \end{array}\right.$ Green, Schwarz, 1982

- $D=4$, MHV

$$A_{YM}(1^-,2^-,3^+,\ldots,N^+)=g_{YM}^{N-2}\,\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle N1\rangle}\ ,$$

Parke, Taylor, 1986
Berends, Giele, 1988

- $(N-3)!$ dimensional minimal basis of hypergeometric functions F^σ

- For any external state of SYM VM (FY = STTH Ward identities)

$$B_N[n] \;=\; \int_0^1 dx_1 \ldots \int_0^1 dx_{N-3} \prod_{a=1}^{N-3} x_a^{1+a-N+n_a} \prod_{b=a}^{N-3} x_a^{\,2\alpha' k_{b+3}\left(k_1+\sum\limits_{j=a+3}^{b+2} k_j\right)}$$

$$\times \;\left(1-\prod_{j=a}^{b} x_j\right)^{2\alpha' k_{2+a}k_{3+b}+n_{ab}} \qquad , \qquad \begin{gathered} b \geq a = 1,2,\ldots,N-3\;, \\ n_a, n_{ab} = 0, \pm 1 \end{gathered}$$

$\frac{1}{2}N(N-3)$ Laurent polynomials = number of kinematic invariants

Structure of integrals is analyzed:

- *multiple pole* structure (dual channels)
- *transcendentality* properties
- *Gröbner basis* analysis to account for various relations

N	dimension	function F^σ	reference
4	1	${}_2F_1$	Green, Schwarz, et al., 1982
5	2	${}_3F_2$	Medina, et al., `hep-th/0208121`
6	6	triple hypergeometric function $F^{(3)}$	Oprisa, St.St., `hep-th/0509042`
7	24	multiple hypergeometric function	St.St., Taylor, `arXiv:0708.0574`
N	$(N-3)!$	multiple hypergeometric function	Mafra, Schlotterer, St.St.

α'–expansion $\Longleftrightarrow$ multiple Euler–Zagier sums

E.g. N=7:

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \; \frac{x^{s_2}\ (1-x)^{s_3}\ y^{t_2}\ (1-y)^{s_4}\ z^{t_6}\ (1-z)^{s_5}\ w^{s_7}\ (1-w)^{s_6}}{(1-xy)\ (1-wz)\ (1-yz)}\ (1-wxyz)^{s_1-t_1+t_4-t_7}$$

$$\times\ (1-xy)^{-s_3-s_4+t_3}\ (1-wz)^{-s_5-s_6+t_5}\ (1-yz)^{-s_4-s_5+t_4}\ (1-wyz)^{s_5+t_1-t_4-t_5}\ (1-xyz)^{s_4-t_3-t_4+t_7}$$

$$= \mathcal{I}_0\ +\ \mathcal{I}_{1a}\ (s_1+s_2+s_3+s_4+s_5+s_6+s_7)\ +\ \mathcal{I}_{1b}\ (t_1+t_2+t_3+t_4+t_5+t_6+t_7) + \mathcal{O}(\alpha'^2)$$

with **Multiple Euler–Zagier sums** $\mathcal{I}_0,\ \mathcal{I}_{1a},\ \mathcal{I}_{1b}$:

$$\mathcal{I}_0 = \int \frac{1}{(1-xy)(1-wz)(1-yz)} = \sum_{\substack{n_1,n_2=0\\ n_3=1}}^{\infty} \frac{1}{n_3\ (1+n_1)\ (n_1+n_2+1)\ (n_2+n_3)} = \frac{27}{4}\zeta(4)$$

$$\mathcal{I}_{1a} = \int \frac{\ln w}{(1-xy)(1-wz)(1-yz)} = \sum_{\substack{n_1,n_3=1\\ n_2=0}}^{\infty} \frac{1}{n_1\ n_3^2\ (n_1+n_2)\ (n_2+n_3)} = \frac{7}{2}\zeta(5) - 4\zeta(2)\zeta(3)$$

$$\mathcal{I}_{1b} = \int \frac{\ln y}{(1-xy)(1-wz)(1-yz)} = \sum_{\substack{n_1,n_2=1\\ n_3=0}}^{\infty} \frac{1}{n_1\ n_2\ (n_2+n_3)\ (n_1+n_3)^2} = -\frac{9}{2}\zeta(5) + \zeta(2)\zeta(3)$$

Effective D–brane action (α'–expansion)

Series of higher derivative terms (α'–corrections to SYM):

$$\mathcal{L}^{Dp}_{\text{effective}} = \text{Tr} \sum_{n\geq 4}^{\infty} \sum_{m=0}^{\infty} \alpha'^{\,n-2+m} \sideset{}{'}\sum_{\substack{i_r\in \mathbb{N},\, i_1>1 \\ i_1+\ldots+i_d=n-2+m}} \zeta(i_1,\ldots,i_d)\, d_{m,n,\vec{i}}\ (t^{\vec{i}}_{m,n}\, D^{2m}F^{n})$$

$\alpha'^0\ 1$	$\mathbf{F^2}$
$\alpha'\ 0$	$F^3 \quad D^2F^2$
$\alpha'^2\ \zeta(2)$	$\mathbf{F^4} \quad D^2F^3 \quad D^4F^2$
$\alpha'^3\ \zeta(3)$	$\mathbf{F^5} \quad \mathbf{D^2F^4} \quad D^6F^2$
$\alpha'^4\ \zeta(4)$	$\mathbf{F^6} \quad \mathbf{D^4F^4} \quad \mathbf{D^2F^5}$
$\alpha'^5\ \zeta(2)\zeta(3),\ \zeta(5)$	$\mathbf{F^7} \quad \mathbf{D^6F^4} \quad \mathbf{D^4F^5} \quad \mathbf{D^2F^6}$
$\vdots$	$\ldots \quad \ldots \quad \ldots \quad \ldots$

Degree of transcendentality $\Longleftrightarrow$ order in α'–expansion

Arrange higher derivative terms

$$A(1,2,\ldots,N;\alpha') = \sum_{\sigma\in S_{N-3}} \underbrace{A_{YM}(1,2_\sigma,\ldots,(N-2)_\sigma,N-1,N)}_{\text{all kinematics}} \underbrace{F^{\sigma}_{(1,\ldots,N)}(\alpha')}_{\substack{\text{expansion}\\ \text{coefficients}}}$$

Higher derivative terms

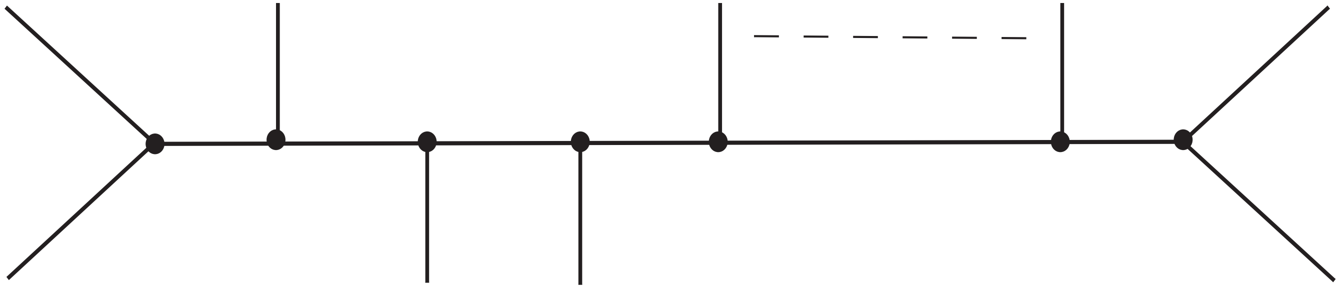
- *organized* according to the *YM amplitudes* A_{YM}
- *constructed* by YM amplitudes A_{YM} (serve as *building blocks* in the effective action) with the expansion coefficients encoded in the functions F.
- *entirely described* in terms of the fundamental *YM three-vertices* (as a consequence of BCJ).

Recursion relations

$$A(1,2,\ldots,N;\alpha') = \sum_{\sigma\in S_{N-3}} \underbrace{A_{YM}(1,2_\sigma,\ldots,(N-2)_\sigma,N-1,N)}_{\text{CSW, BCFW recursions in FT}} \quad \underbrace{F^{\sigma}_{(1,\ldots,N)}(\alpha')}_{\text{recursions in } B_N}$$

$$\boxed{B_N = \sum B_{n_1}\ B_{n_2}\cdot\ldots\cdot B_{n_k} \quad , \quad \sum_{l=1}^{k} n_l = N+3\ (k-1)}$$

with some partition $\{n_1,\ldots,n_k\}$ into k smaller amplitudes B_{n_l}.



allows to write B_N in terms of products of $(N-3)$ functions B_4.

Duality between color and kinematics in string theory

monodromy relations	relations for **different color orderings** at **one specific kinematics** σ	basis of $(N-3)!$ amplitudes $A(1,\ldots,N)$

$\longrightarrow$ BCJ relations from string theory

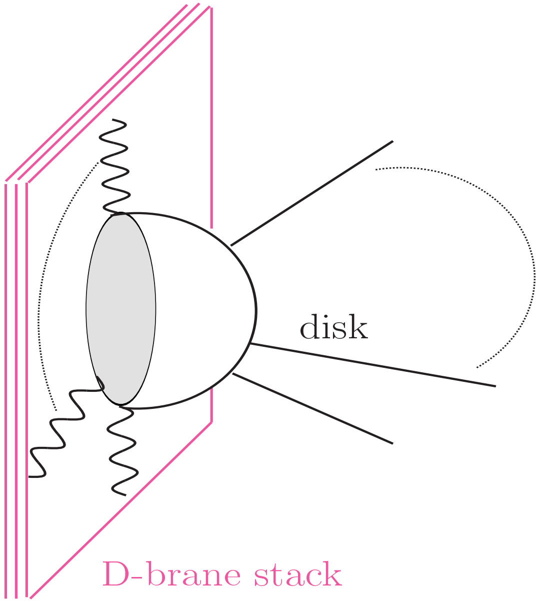
partial integration relations	relations for **different kinematics** at **one specific color ordering**	basis of $(N-3)!$ functions $F^{\sigma}_{(1,\ldots,N)}$

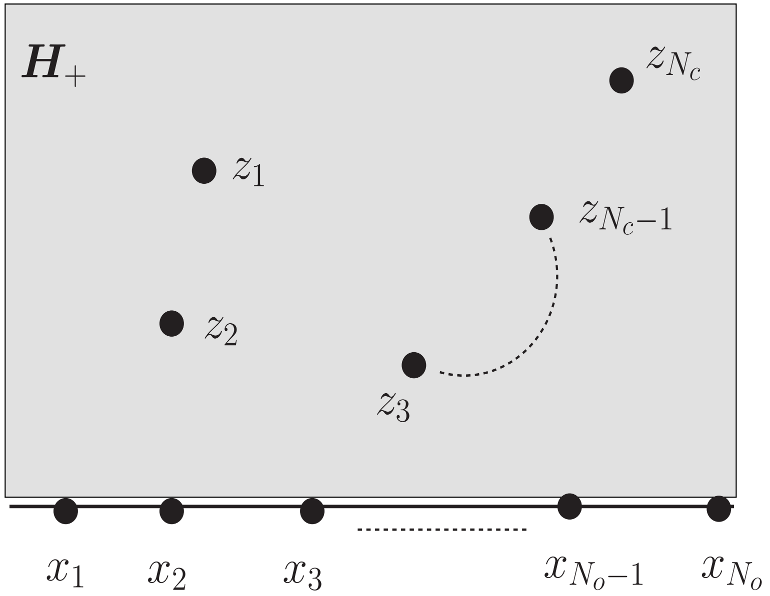
$\longrightarrow$ role of color and kinematics swapped

string theory derivation of Bern, Dennen, `arXiv:1103.0312` ?

"in the color decompostion of an amplitude the role of color and kinematics may be swapped"

IV. Disk scattering of open and closed strings





$$\boxed{\mathcal{A} = \sum_{\sigma\in S_{N_o}} V_{\mathrm{CKG}}^{-1} \left(\int_{\mathcal{I}_\sigma} \prod_{j=1}^{N_o} dx_j \prod_{i=1}^{N_c} \int_{H_+} d^2 z_i \right) \langle \prod_{j=1}^{N_o} : V_o(x_j) : \prod_{i=1}^{N_c} : V_c(\overline{z}_i, z_i) : \rangle}$$

$V_o(x_i) =$ open string vertex operators inserted at x_i on the boundary of the disk

$V_c(\overline{z}_i, z_i) =$ closed string vertex operators inserted at z_i inside the disk

Disk scattering of open and closed strings

$$\boxed{\mathcal{A}(N_o,N_c)=\sum_{\sigma\in S_{N_o}}\sum_I \mathcal{K}_I\ \mathcal{I}_I^\sigma}$$

$\mathcal{I}_I^\sigma =$ complex disk integral involving open and closed positions

Result: Integrand looks like a pure open string amplitude

involving N_o+2N_c open strings,

with integrations over N_o+2N_c-3 real positions x_l,ξ_i,η_j:

$$\boxed{\begin{array}{lll} X_1=-\infty\ , & X_l=x_l\ , & l=2,\ldots,N_o\ , \\ X_{N_o+2i-1}=\xi_i\ , & X_{N_o+2i}=\eta_i\ , & i=1,\ldots,N_c \end{array}}$$

Disk scattering of open and closed strings

Introduce phase to render the integral single–valued:

$$\begin{aligned} \Pi(x_l,\xi_i,\eta_j) &= e^{2\pi i p_l q_i \theta[-(x_l-\xi_i)(x_l-\eta_i)]}\ e^{i\pi q_i q_j \theta[-(\xi_i-\xi_j)(\eta_i-\eta_j)]} \\ &\times\ e^{i\pi q_i D q_j \theta[-(\xi_i-\eta_j)(\eta_i-\xi_j)]}\ e^{2\pi i q_{i\|}^2 \theta(\eta_i-\xi_i)} \end{aligned}$$

Phase factor $\Pi(x_l,\xi_i,\eta_j)$ accounts for correct branch of the integrand

$$\mathcal{A}(N_o,N_c)=\left(\frac{i}{2}\right)^{N_c} \sum_{\Sigma\in\mathcal{P}} \Pi(\Sigma)\ \ A(1,\Sigma(2),\ldots,\Sigma(N_0+2N_c))$$

St.St, `arXiv:0907.2211`

Open & closed vs. pure open string disk amplitude

$\Longrightarrow$ Basic ingredients of open & closed disk amplitude:

(color) ordered open string amplitudes $A(1,\ldots,N)$

After inspecting phase $\Pi(x_l,\xi_i,\eta_j)$:

- many different contributions (open string orderings) $A(1,\ldots,N)$
- many striking relations between open string subamplitudes

$\longrightarrow$ Investigate pure open string sector:

- basis of $(N-3)!$ partial subamplitudes, with $N=N_o+2N_c$

General:

Disk amplitude involving N_o open and N_c closed strings is mapped to disk amplitudes of $N_o + 2N_c$ open strings

Examples:

$$
\begin{aligned}
\mathcal{A}(1,2) &= A(1,2,3,4)\ ,\\
\mathcal{A}(1,2;3) &= \sin(\pi t)\ A(1,2,3,4)\ ,\\
\mathcal{A}(1,2,3;4) &= \sin(\pi t)\ A(1,5,2,4,3) + \sin(2\pi q_{\|}^2)\ A(1,2,3,4,5)\ ,\\
\mathcal{A}(1,2;3,4) &= \sin\left(\frac{\pi s}{2}\right)\ \sin(\pi s)\ A(1,6,3,5,4,2)\\
&- \sin\left(\frac{\pi s}{2}\right)\ \sin(\pi t)\ A(1,3,5,4,2,6)\ ,\\
\mathcal{A}(1,2,3,4;5) &= \sin(\pi s_4)\ A(1,6,4,5,3,2)\\
&- \sin\pi\left(\frac{s_1}{2} - \frac{s_3}{2} + s_5\right)\ A(1,4,3,5,2,6)\ ,\\
&\ldots\ .
\end{aligned}
$$

St.St, `arXiv:0907.2211`

Sort of generalized KLT on the disk

This map reveals relations between
open & closed string disk amplitudes
and pure open string disk amplitudes !

Two vectors and two massless Neveu–Schwarz closed string states:

$$\langle A_{\mu_1}(x_1)\ A_{\mu_2}(x_2)\ G_{\mu_3\mu_4}(\overline{z}_1,z_1)\ G_{\mu_5\mu_6}(\overline{z}_2,z_2)\rangle\ \zeta^{\mu_1}\ \zeta^{\mu_2}\ \epsilon_1^{\mu_3\mu_4}\ \epsilon_2^{\mu_5\mu_6}$$
$$\simeq \langle A_{\mu_1}(x_1)\ A_{\mu_2}(x_2)\ A_{\mu_3}(x_3)\ A_{\mu_4}(x_4)\ A_{\mu_5}(x_5)\ A_{\mu_6}(x_6)\rangle\ \xi^{\mu_1}\xi^{\mu_2}\xi^{\mu_3}\xi^{\mu_4}\xi^{\mu_5}\xi^{\mu_6}$$

with identifications and assignment: $\xi_1=\zeta_1,\ \xi_2=\zeta_2,\ \xi_3\otimes\xi_4=\epsilon_1,\ \xi_5\otimes\xi_6=\epsilon_2$

Two vectors and two massless Ramond $p-$ and $q-$forms:

$$\langle A_{\mu_1}(x_1)\ A_{\mu_2}(x_2)\ F_{\alpha\beta}(\overline{z}_1,z_1)\ F_{\gamma\delta}(\overline{z}_2,z_2)\rangle$$
$$\times\zeta^{\mu_1}\ \zeta^{\mu_2}\ f^1_{\mu_0\ldots\mu_p}\ \left(P^+\Gamma^{[\mu_0}\ldots\Gamma^{\mu_p]}\right)^{\alpha\beta}\ f^2_{\nu_0\ldots\nu_q}\ \left(P^+\Gamma^{[\nu_0}\ldots\Gamma^{\nu_q]}\right)^{\gamma\delta}$$
$$\simeq \langle A_{\mu_1}(x_1)\ A_{\mu_2}(x_2)\ \chi_\alpha(x_3)\ \chi_\beta(x_4)\ \chi_\gamma(x_5)\ \chi_\delta(x_6)\rangle\ \xi^{\mu_1}\xi^{\mu_2}\ u^\alpha\ v^\beta\ u^\gamma\ v^\delta$$