On tree–level higher order gravitational couplings in superstring theory

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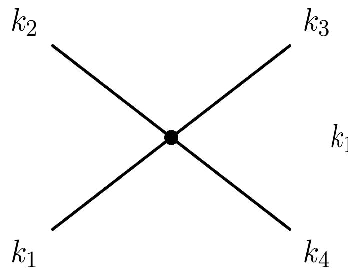
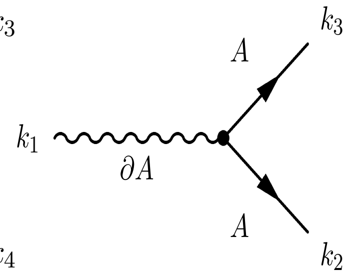
Field theory: Gauge vs. gravitational amplitudes

Perturbative Feynman rules in YM:



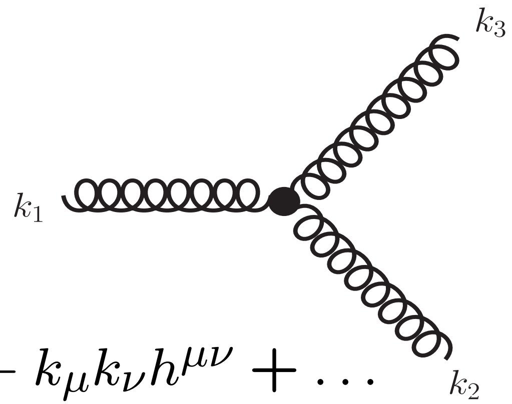


$$\mathcal{L} = -\frac{1}{4g_{YM}^2}\ F^a_{\mu\nu}F^{a\mu\nu}$$

only three– and four–vertices

$$V_3^{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) = g_{YM}\ \{\ \eta^{\mu_1\mu_2}\ k_{12}^{\mu_3} + \eta^{\mu_2\mu_3}\ k_{23}^{\mu_1} + \eta^{\mu_1\mu_3}\ k_{31}^{\mu_2}\ \}$$

Perturbative Feynman rules in Einstein gravity:



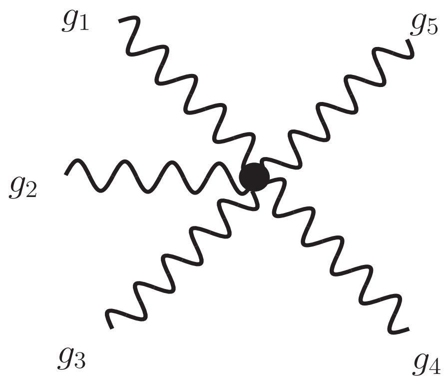
$$\mathcal{L} = -\frac{2}{\kappa_4^2}\ \sqrt{-g}\ R$$

arbitrary many external legs: $R = \Box h - k_\mu k_\nu h^{\mu\nu} + \ldots$

$$V_3^{\mu_1\nu_1,\mu_2\nu_2,\mu_3\nu_3}(k_1,k_2,k_3) = \kappa_4\ \{\ (k_1k_2)\ \eta^{\mu_1\nu_1}\ \eta^{\mu_2\nu_2}\ \eta^{\mu_3\nu_3} + k_1^{\mu_3}\ k_2^{\nu_3}\ \eta^{\mu_1\mu_2}\ \eta^{\nu_1\nu_2} + \ldots\ \}$$

Even more involved:

- higher order α' corrections
- new irreducible vertices at each order in α'

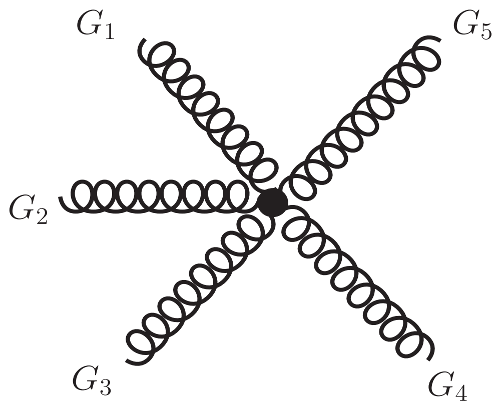


$\alpha'^2\,\zeta(2)\,F^4$

$\alpha'^3\,\zeta(3)\,\{F^5,\ D^2F^4\}$

$\alpha'^4\,\zeta(4)\,\{D^2F^5,\ D^4F^4\}$

$\vdots$



$\alpha'^3\,\zeta(3)\,R^4$

$\alpha'^4\,\zeta(4)\,\{R^5,\ D^2R^4\}$

$\alpha'^5\,\zeta(5)\,\{D^2R^5,\ D^4R^4\},\ \alpha'^5\,\zeta(2)\,\zeta(3)\,\{D^2R^5,\ D^4R^4\}$

$\vdots$

String theory: Gauge vs. gravitational amplitudes

However at tree–level:

gravity = gauge theory ⊗ gauge theory

- Spectrum:

$$\left|\begin{matrix}\mathcal{N}{=}8\\ SUGRA\end{matrix}\right\rangle = \left|\begin{matrix}\mathcal{N}{=}4\\ SYM\end{matrix}\right\rangle \otimes \left|\begin{matrix}\mathcal{N}{=}4\\ SYM\end{matrix}\right\rangle$$

$$256 = 16 \times 16$$

E.g.: in $D = 4$, N=8: Fock space decomposition of the 256 states of the $\mathcal{N} = 8$ supergravity multiplet

- Vertex operators:

$$V_G(\epsilon, \bar{z}, z) \simeq V_g(\bar{\epsilon}, \eta) \otimes V_g(\epsilon, \xi)$$

$$\epsilon_{\mu\nu} = \bar{\epsilon}_\mu \otimes \epsilon_\nu$$

with $R_{\mu\nu\rho\sigma} = \kappa\ k_{[\mu} k_{[\rho}\ \bar{\epsilon}_{\nu]} \otimes \epsilon_{\sigma]}$ linearized Riemann tensor

String theory: Gauge vs. gravitational amplitudes

- Amplitudes (on–shell S–matrix): KLT relations

$$
\begin{aligned}
M_4(1,2,3,4)_{S^2} &= (2\alpha'\pi)^{-1}\ \sin(\pi s_{12})\ \overline{A}_4(1,2,3,4)_{D_2}\ A_4(1,2,4,3)_{D_2} \\
M_5(1,2,3,4,5)_{S^2} &= (2\alpha'\pi)^{-2}\ \sin(\pi s_{12})\ \sin(\pi s_{34})\ \overline{A}_5(1,2,3,4,5)_{D_2}\ A_5(2,1,4,3,5)_{D_2} \\
&+\ \text{permutations of (23)} \\
M_6(1,2,3,4,5,6)_{S^2} &= (2\alpha'\pi)^{-3}\ \sin(\pi s_{12})\ \sin(\pi s_{45})\ \overline{A}_6(1,2,3,4,5,6)_{D_2} \\
&\times\ \{\ \sin(\pi s_{35})\ A_6(2,1,5,3,4,6)_{D_2} + \sin[\pi(s_{34}+s_{35})]\ \ A_5(2,1,5,4,3,6)_{D_2}\} \\
&+\ \text{permutations of (234)} \\
&\ \vdots
\end{aligned}
$$

Field–theory amplitudes are obtained (reproduced) for $\alpha' \to 0$

E.g.: $$M(1^-,2^-,3^+,4^+) = \left(\frac{\kappa}{2}\right)^2 \frac{\langle 12\rangle^8\ [12]}{N(4)\ \langle 34\rangle}\,\frac{B(s_{12},s_{14})}{B(-s_{12},-s_{14})} \to \left(\frac{\kappa}{2}\right)^2 \frac{\langle 12\rangle^8\ [12]}{N(4)\ \langle 34\rangle}$$

with: $\langle ij\rangle[ij] = s_{ij} = \alpha' k_i k_j, \quad N(n) = \prod_{i=1}^{n-1}\prod_{j=i+1}^{n}\langle ij\rangle$

Tree–level higher order gravitational couplings

KLT: Open string amplitudes give rise to closed string amplitudes

$\Longrightarrow$ (color ordered) gluon amplitudes give rise to graviton amplitudes

E.g.:

$$A(1,2,3,4,5) = C_1\ A_{YM}(1,2,3,4,5) + C_2\ A_{F^4}(1,2,3,4,5)$$

$$A_{YM}(1^-,2^-,3^+,4^+,5^+) = i\ \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\ ,$$

$$A_{F^4}(1^-,2^-,3^+,4^+,5^+) = \alpha'^2\ (\ \langle 12\rangle[23]\langle 34\rangle[41] - \{3,4\}\{4,5\} - \{1,2\}\{1,5\}\ )$$
$$\times\ A_{YM}(1^-,2^-,3^+,4^+,5^+)$$

$C_1, C_2 =$ Gaussian hypergeometric functions encoding the α'–dependence

St.St., Taylor, hep-th/0607184, hep-th/0609175

Tree–level higher order gravitational couplings

Type I or Type II superstring:

$$\mathcal{L}_{\rm tree} = \frac{1}{2\kappa^2}\, R + \frac{\alpha'^3}{2^9\, 4!\, \kappa^2}\, \zeta(3)\, t_8 t_8\, R^4$$

Gross, Witten, 1986
Gross, Sloan, 1987

$$\mathcal{L}'_{\rm tree} = \kappa^{-2} \sum_{n\geq 4}^{\infty} \sum_{m=0}^{\infty} \alpha'^{n-1+m} \sum_{\substack{i_r\in \mathbf{N}, i_1>1 \\ i_1+\ldots+i_d=n-1+m}}{}' \zeta(i_1,\ldots,i_d)\, c_{m,n,\vec{i}}\, \left(t^{\vec{i}}_{m,n}\, D^{2m} R^n\right)$$

$\hookrightarrow$ constraints and transcendentality properties of couplings

$$\zeta(i_1,\ldots,i_d)\, c_{m,n,\vec{i}}$$

Multi zeta values (MZVs)

$$\zeta(i_1,\ldots,i_d) = \sum_{n_1>\ldots>n_d>0} \prod_{r=1}^{d} n_r^{-i_r}, \quad i_r \in \mathbf{N},\ i_1 > 1$$

transcendentality degree $\sum_{r=1}^{d} i_r = n - 1 + m$ and depth d

Many relations over $\mathbf{Q}$, e.g.:

$$\begin{aligned}
\zeta(2,1) &= 2\,\zeta(3) \\
\zeta(4,1) &= 2\,\zeta(5) - \zeta(2)\,\zeta(3) \\
\zeta(5,3) &= -\frac{5}{2}\,\zeta(6,2) - \frac{21}{25}\,\zeta(2)^4 + 5\,\zeta(3)\,\zeta(5) \\
&\vdots
\end{aligned}$$

Multi zeta values (MZVs)

The set of integral linear combinations of MZVs is a ring

Zagier: For a given weight $w \in \mathbf{N}$ the dimension d_w of the space spanned by MZVs: $\boldsymbol{d_w = d_{w-2} + d_{w-3}}$, $d_0, d_1 = 0$,

w	d_w	basis
2	1	$\zeta(2)$
3	1	$\zeta(3)$
4	1	$\zeta(2)^2$
5	2	$\zeta(5),\ \zeta(2)\zeta(3)$
6	2	$\zeta(2)^3,\ \zeta(3)^2$
7	3	$\zeta(7),\ \zeta(2)\zeta(5),\ \zeta(3)\zeta(2)^2$
8	4	$\zeta(2)^4,\ \zeta(2)\zeta(3)^2,\ \zeta(3)\zeta(5),\ \zeta(5,3)$

Field redefinitons, Bianchi identities

- Ricci tensors and Ricci scalars can always be eliminated on–shell
- Weyl tensor can always be written as Riemann tensor

- $D^2 R \simeq R^2$ (etc.)

We stick to the prescription to write all terms
with the highest possible number of Riemann tensors.

E.g.: $D^2 R^4 \simeq R^5$

moreover: $D^2 R^4 \Big|_{\substack{\text{4-point} \\ \text{on-shell}}} = 0$

$\Longrightarrow$ one needs to compute 5–graviton amplitude to probe $\alpha'^4 \, \{D^2 R^4, R^5\}$ terms

Conclusion: Only "true terms" can be probed on–shell

Gravitational amplitudes in superstring theory

Task: Compute graviton amplitudes:

$$M(1,2,3,4,5) \quad , \quad M(1,2,3,4,5,6)$$

and extract their power series expansion in α'

St.St. arXiv:0910.0180

ingredients: gluon amplitudes

- generalized Euler integrals
- multiple Gaussian hypergeometric functions
- multiple Euler–Zagier sums $\longrightarrow$ MZVs

Oprisa, St.St, hep-th/0509042

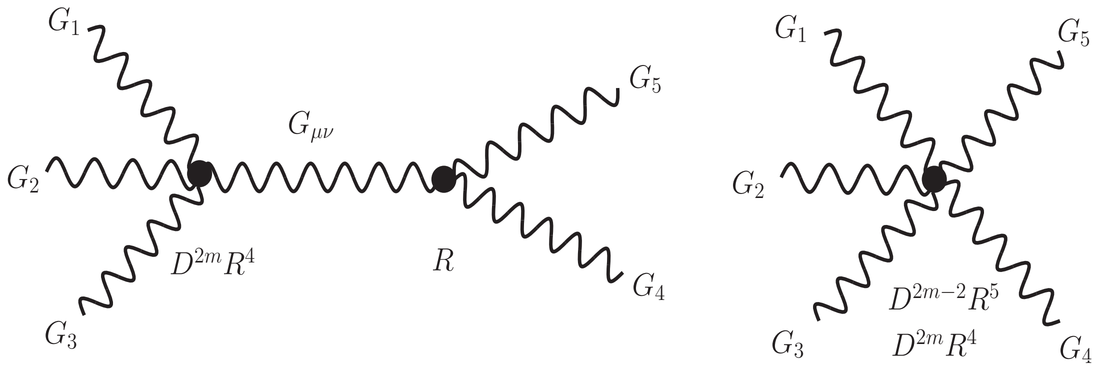
St.St, Taylor, hep-th/0609175, arXiv:0708.0574, arXiv:0711.4354

- completely model independent
- universal to all string compactifications
- any numbers of supersymmetries or dimensions D

Five–graviton amplitude:

The α'–expansion has no $\begin{cases} \alpha'^4 & \zeta(4) \\ \alpha'^5 & \zeta(2)\zeta(3) \\ \alpha'^6 & \zeta(6) \\ \alpha'^7 & \zeta(3)\zeta(4),\ \zeta(2)\zeta(5) \\ \alpha'^8 & \zeta(8),\ \zeta(2)\zeta(3)^2,\ \zeta(5,3) \end{cases}$

These terms cannot be generated by any reducible diagram (with five external gravitons)



diagrams contributing for $N=5$ at order α'^{3+m}

$\Longrightarrow$ absence of contact interactions with coefficients:

$$\alpha'^5\zeta(2)\zeta(3), \alpha'^6\zeta(6),\ \alpha'^7\zeta(3)\zeta(4), \alpha'^7\zeta(2)\zeta(5),\ \alpha'^8\zeta(8), \alpha'^8\zeta(2)\zeta(3)^2, \alpha'^8\zeta(5,3)$$

Tree–level higher order gravitational couplings

	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$
$\alpha'^3\ \zeta(3)$	R^4				
α'^4 ~~$\zeta(4)$~~	~~D^2R^4~~	~~R^5~~			
$\alpha'^5\ \zeta(5)$	D^4R^4	D^2R^5	R^6		
α'^5 ~~$\zeta(2)\zeta(3)$~~	~~D^4R^4~~	~~D^2R^5~~	~~R^6~~		
$\alpha'^6\ \zeta(3)^2$	D^6R^4	D^4R^5	D^2R^6	R^7 ?	
$\alpha'^6\ \zeta(6)$	~~D^6R^4~~	~~D^4R^5~~	~~D^2R^6~~	R^7 ?	
$\alpha'^7\ \zeta(7)$	D^8R^4	D^6R^5	D^4R^6	D^2R^7 ?	R^8 ?
$\alpha'^7\ \zeta(3)\zeta(4)$	~~D^8R^4~~	~~D^6R^5~~	~~D^4R^6~~	D^2R^7 ?	R^8 ?
$\alpha'^7\ \zeta(2)\zeta(5)$	~~D^8R^4~~	~~D^6R^5~~	~~D^4R^6~~	D^2R^7 ?	R^8 ?
$\alpha'^8\ \zeta(3)\zeta(5)$	$D^{10}R^4$	D^8R^5	D^6R^6	D^4R^7 ?	D^2R^8 ?
$\alpha'^8\ \zeta(8)$	~~$D^{10}R^4$~~	~~D^8R^5~~	~~D^6R^6~~	D^4R^7 ?	D^2R^8 ?
$\alpha'^8\ \zeta(2)\zeta(3)^2$	~~$D^{10}R^4$~~	~~D^8R^5~~	~~D^6R^6~~	D^4R^7 ?	D^2R^8 ?
$\alpha'^8\ \zeta(5,3)$	~~$D^{10}R^4$~~	~~D^8R^5~~	~~D^6R^6~~	D^4R^7 ?	D^2R^8 ?

$\Longrightarrow$ refined transcendentality property: only MZVs of odd weight appear !

Concluding remarks

Constraints on higher order gravitational couplings:
Vanishing and transcendentality properties:
only MZVs of odd weight appear

Note: $\zeta(5,3) = -\frac{5}{2}\zeta(6,2) - \frac{21}{25}\zeta(2)^4 + 5\zeta(3)\zeta(5)$

- Results take over for scattering
other members of the supergravity multiplet
(via SUSY Ward identities in the gauge sector)

St.St., Taylor, `arXiv:0708.0574`

- Disk amplitudes involving
both open and closed strings
share similar transcendentality properties

e.g.: $\langle AAAC \rangle$ or $\langle AACC \rangle$

Concluding remarks

- Results restrict (together with one–loop results) the ring of possible modular forms describing the perturbative and non–perturbative completion of the higher order terms in $D = 10$ type IIB superstring theory

- Results give possible constraints on counter terms in N=8 SUGRA
R. Kallosh: *"The fact of cancellation of the tree–level R^5 term in string theory is suggesting that N=8 SUGRA at the four 4–loop level will not have* a 5–point amplitude divergence."

Kallosh, `arXiv:0906.3495`

Open & closed vs. pure open string disk amplitude

$$\begin{aligned}
\mathcal{A}(1,2) &= A(1,2,3,4)\ , \\
\mathcal{A}(1,2;3) &= \sin(\pi t)\ A(1,2,3,4)\ , \\
\mathcal{A}(1,2,3;4) &= \sin(\pi t)\ A(1,5,2,4,3)\ , \\
\mathcal{A}(1,2;3,4) &= \sin\left(\frac{\pi s}{2}\right)\ \sin(\pi s)\ A(1,6,3,5,4,2) \\
&- \sin\left(\frac{\pi s}{2}\right)\ \sin(\pi t)\ A(1,3,5,4,2,6)\ , \\
\mathcal{A}(1,2,3,4;5) &= \sin(\pi s_4)\ A(1,6,4,5,3,2) \\
&- \sin\pi\left(\frac{s_1}{2}-\frac{s_3}{2}+s_5\right)\ A(1,4,3,5,2,6)\ , \\
&\ldots\ .
\end{aligned}$$

St.St. arXiv:0907.2211