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Aspects of E_8

in heterotic F-theory

duality

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Aspects of E_8 in heterotic F-theory duality:

In this thesis we study heterotic duality of the $SU(5) \times U(1) \times U(1)$ F-theory models introduced in [1] in compactifications to four and six dimensions. We investigate the relationship between GUT singlets in the F-theory geometry and singularities in the heterotic compactification. Moreover, we derive constraints on the models as to render the heterotic geometry smooth. In four dimensions, we find that two of the four models do not admit a smooth heterotic dual. We proceed to consider how these singularities are related to possible embeddings of the F-theory spectrum into a Higgsed E_8 and show that for the two models allowing for a smooth heterotic geometry, there are multiple embeddings such that all non-singular states are embeddable whereas the singular ones are not. In six dimensions all four models allow for a smooth heterotic dual, but the geometries are severely restricted by said constraints.

Aspekte von E_8 in heterotischer F-Theorie Dualität:

In dieser Arbeit untersuchen wir die Dualität zwischen heterotischer Stringtheorie und F-Theorie-Modellen mit Eichgruppe $SU(5) \times U(1) \times U(1)$, die in [1] eingeführt worden sind für Kompaktifizierungen in vier und sechs Dimensionen. Wir betrachten die Beziehung zwischen GUT Singlets in der F-Theorie-Geometrie und Singularitäten in der heterotischen Kompaktifizierung. Darüber hinaus leiten wir Einschränkungen her, welche glatte heterotische Geometrien sicherstellen. In vier Dimensionen zeigen wir, dass zwei der vier Modelle kein glattes heterotisches Dual besitzen. Wir fahren fort und untersuchen in welcher Beziehung Singularitäten und etwaige Einbettungen in eine gehiggste E_8 stehen. Im Speziellen zeigen wir, dass für die beiden Modelle, welche ein glattes heterotisches Dual besitzen, mehrere Einbettungen existieren, in denen alle nicht-singulären Zustände einbettbar sind, während alle singulären Zustände nicht einbettbar sind. In sechs Dimensionen besitzen alle vier Modelle ein glattes heterotisches Dual, aber die Geometrien sind durch die genannten Bedingungen stark eingeschränkt.

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Contents

1. Introduction

The analyses presented in this thesis deal with the duality of certain F-theory models for grand unified theories featuring additional abelian symmetries to the heterotic string.

To set the stage we start in section 2 by introducing the concept of grand unified theories and explain how additional abelian symmetries can be used to solve problems regarding proton decay and flavour hierarchies. Aside from the phenomenological description of such grand unified theories, it is desirable to explain them from first principle. Moreover, it is widely believed that they arise from a more fundamental theory, which should in particular unify quantum field theory and general relativity. Therefore, we sketch the necessity for such a theory and introduce string theory as the leading candidate for such a proposal. In doing so we shed light on dualities among the different types of superstring theories, which leads us to the introduction of an eleven dimensional theory called M-theory. Moreover, we introduce a non-pertubative formulation of the type IIB superstring, by considering this theory in the presence of D7-branes and O7-planes. This twelve-dimensional, non-pertubative description of the IIB string has been dubbed F-theory. Of course this theory features additional dualities, among which is the duality to M-theory. This latter duality carries much information regarding the resulting low-energy spectrum, as we will see. Lastly, we have a closer look at the objects F-theory encodes. In fact, we establish that the concepts of fundamental strings, D7-branes and O7-planes are replaced by more general notions away from the weakly-coupled IIB description.

Having established the basic foundations of F-theory, we turn our attention to the key mathematical aspects, which are related to the construction of so-called elliptic fibrations. We investigate, in particular, how non-abelian gauge groups arise in this description. Moreover, we consider the generalisation of intersecting D7-branes in F-theory and how these give rise to gauge enhancements along geometric loci of various codimensions in the compactification space. Furthermore, we discuss the appearance of elliptic fibrations with extra abelian symmetries and outline two different means of constructing such models. Additionally, we compute the charges of states under these $U(1)$s by introducing the so-called Shioda-map. Lastly, we have a more detailed look at the process of resolutions in singular manifolds.

After having outlined the mathematical methods involved in F-theory compactifications, we return to the concept of string dualities and investigate the duality between the heterotic string and F-theory. This duality is of particular interest as the heterotic string with its large structure groups and the IIB string (in its non-pertubative manifestation, F-theory) provides phenomenologically rich corners of the string landscape. We introduce the duality in its simplest form for compactifications to eight dimensions. From there we set out to extend the duality to the phenomenologically more interesting cases of six and four dimensional duality. In doing so we introduce the basic concepts for so-called Hirzebruch surfaces, as well as the construction of vector bundles over elliptic curves in terms of the so-called spectral cover construction. We then have a closer look at the matching of moduli between the two theories.

Having introduced the formalism to investigate such models we turn our attention

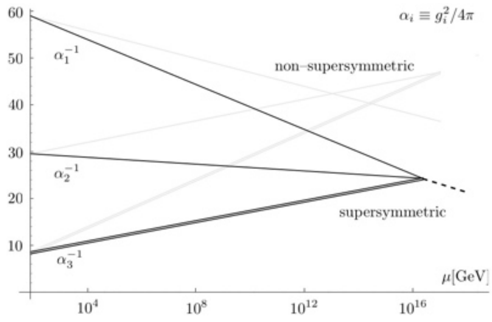

Figure 1: Running of the gauge couplings for non-supersymmetric and supersymmetric model [2]

to the $SU(5) \times U(1) \times U(1)$ F-theory models developed in [1]. More specifically, we constrain the compactification manifold to allow for a heterotic dual. Secondly, we impose two constraints, which can be understood in the F-theory description. This will further restrict the geometry. Lastly, we investigate the relation between these constraints and possible embeddings of the spectrum into a Higgsed E_8.

In appendix Appendix A we apply the same analysis to constructions with different gauge groups. Moreover, we consider duality in compactifications to six dimensions in appendix 6. This duality is more restrictive on the geometry, but does not allow for constraints associated to codimension two loci in the F-theory base.

2. GUTs, strings and branes

2.1. Grand unified theories

At low energies, interactions are described with great precision by the standard model of particle physics — a gauge theory of $SU(3) \times SU(2) \times U(1)$ with three a priori independent couplings. If one computes the running of the couplings, one notes that the three couplings intersect at almost the same energy scale (see fig. 1). This structure raises the questions whether the couplings are indeed independent or whether there is some underlying principle. A by now famous proposal of H. Georgi and S. H. Glashow [3] states that the standard model gauge groups are in fact part of some larger group which is spontaneously broken at low energies. The simplest such choice is $SU(5)$. Such a theory would relate the gauge couplings at low energies to that of $SU(5)$ at the breaking scale. A theory embedding the standard model in some larger group is therefore called *grand unified theory* (GUT). Indeed one may confirm that the quarks and leptons of the standard model can be organised in chiral, anomaly-free representations of $SU(5)$ [4]. As we noted above, while the couplings do approach each other, they do not in fact intersect at a common point, which would indicate two subsequent breaking processes. While such a theory provides a very simple explanation for the origin of the gauge couplings it also poses a problem: With the introduction of

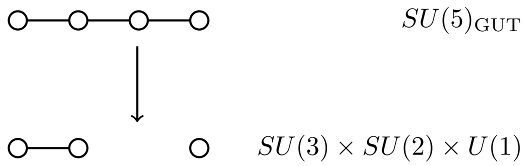

Figure 2: Breaking of an $SU(5)$ GUT group to $SU(3) \times SU(2) \times U(1)$

the breaking scale into the theory it seems very unnatural for the Higgs boson to obtain a mass of the observed order. This can be understood as follows: In the prescription given by renormalisation, the Higgs mass is determined by the difference of the bare mass and the counter terms. Both of these terms are of the order of the cutoff scale squared, which makes it technically highly unnatural for the Higgs boson to obtain a mass of the observed order ~ 100GeV. This open question is generally known as the *hierarchy problem*.

However, one does not necessarily expect the low energy physics to be described by the standard model alone, but rather by some supersymmetric extension thereof. Supersymmetry can indeed be shown to stabilise the Higgs mass. Repeating the analysis for the couplings of the *minimal supersymmetric standard model* (MSSM) also yields a common intersection point for the three gauge couplings at approximately 10^{16}GeV pointing at a single breaking process. It is important to stress that so far no supersymmetric particles have been detected in collider experiments. While this does not affect the possiblity of supersymmetry at high energy scales, it rules out many models for low scale supersymmetry, which would solve the hierarchy problem in total.

As we outlined, grand unified theories seem to be a promising direction for a better understanding of particle physics and therefore it is particularly interesting whether such models can be constructed from a more fundamental theory such as string theory in a natural fashion. We will come back to this question in the following sections repeatedly. Note, however, that GUT theories leave a number of questions unanswered. One such question concerns proton decay: While the experimental limits on the half-life of a proton are at least of the order 10^{34}years, it is non-trivial to account for such a high half-life in a grand unified theory. A second question is raised by the masses of the quarks and leptons: Albeit the different generations of quarks and leptons have identical quantum numbers their masses lie in the range of ~ 1MeV to $\sim 10^2$GeV. Of course such a large scale is very unnatural and necessitates an explanation. To date both questions remain unanswered although some proposals to their solution exist. Two very promising ideas to remedy these two shortcomings in GUTs are based on the introduction of additional abelian symmetries in the theory.

To see how to utilise such an extra $U(1)$ gauge factors to prohibit proton decay, note that the dangerous operators arise from couplings of the form $\mathbf{10}\,\bar{\mathbf{5}}\,\bar{\mathbf{5}}$ [5]. The idea to prohibit such operators is to assign charges under the additional $U(1)$ symmetry to these fields in such a way as to render the coupling non-neutral, as would be required for it to be realised. For a treatment of proton decay in this fashion within an F-theoretic framework, we refer the reader to [6, 7, 8, 9, 10].

The basic idea to tackle the second question — that is the high mass hierarchies — by using an extra abelian symmetry has been introduced by C. D. Froggatt and H. B. Nielsen in [11] and proceeds as follows [12]: One introduces an additional scalar field S into the theory, which is charged under the $U(1)$-symmetry and transforms as a singlet under the non-abelian symmetries. Furthermore, one assigns different $U(1)$-charges to the different fermion fields, such that the couplings have to be made neutral by different powers of S. If S obtains a VEV in such a setup, and breaks the abelian symmetry, this breaking will be mediated with different powers of $\epsilon := \frac{<S>}{M_*}$, where M_* denotes the scale of flavour dynamics. To make this idea more specific, consider a supersymmetry, two-family toy model in which we assign the $U(1)$-charges as follows [12]:

$$\{Q_3, u_3^{\rm c}\} : 0 \tag{2.1}$$

$$\{Q_2, u_2^{\rm c}\} : 2 \tag{2.2}$$

$$\{d_2, d_3^{\rm c}\} : 1 \tag{2.3}$$

$$\{H_u, H_d\} : 0 \tag{2.4}$$

One may now obtain the effective Lagrangian by integrating out the degrees of freedom above M_* — the so-called Froggatt-Nielsen fields $G_i, \bar{G}_i, F_i, \bar{F}_i$ ($i = 1, \ldots 4$), where the field G_i has the same gauge quantum numbers as u^c and F_i has those of d^c ($\bar{G}_i$ and $\bar{F}_i$ have the conjugate numbers). For brevity we do not give their $U(1)$-charges here, which may be read off from fig. 3 taken in this form from [12]. The masses of the quarks are induced by the effective couplings encoded in the diagrams. Consider for example figures b) and c), which encode the mass of b- and c-quark, respectively. One may read off that they correspond to effective couplings,

$$\mathcal{L}_b^{\rm eff} = Y_1 Y_2 (Q_3 d_3^{\rm c} H_{\rm d}) \left(\frac{S}{M_{F_1}} \right) \tag{2.5}$$

$$\mathcal{L}_c^{\rm eff} = \Pi_{i=1}^5 Y_i' (Q_2 u_3^{\rm c} H_{\rm u}) \left(\frac{S^4}{M_G^4} \right), \tag{2.6}$$

where Y_i and Y_i' denote order one Yukawa couplings and we assumed the same mass for all G_i. If S obtains a VEV $< S >$ in this scenario, the resulting mass matrices are given by

$$M_{\rm u} \sim \begin{pmatrix} \epsilon^4 & \epsilon^2 \\ \epsilon^2 & 1 \end{pmatrix} \qquad M_{\rm d} \sim \begin{pmatrix} \epsilon^3 & \epsilon^3 \\ \epsilon & \epsilon \end{pmatrix}, \tag{2.7}$$

from which one may deduce the mass ratios for the quarks as:

$$\frac{m_{u,2}}{m_{u,3}} \sim \epsilon^4 \tag{2.8}$$

$$\frac{m_{d,2}}{m_{d,3}} \sim \epsilon^2 \tag{2.9}$$

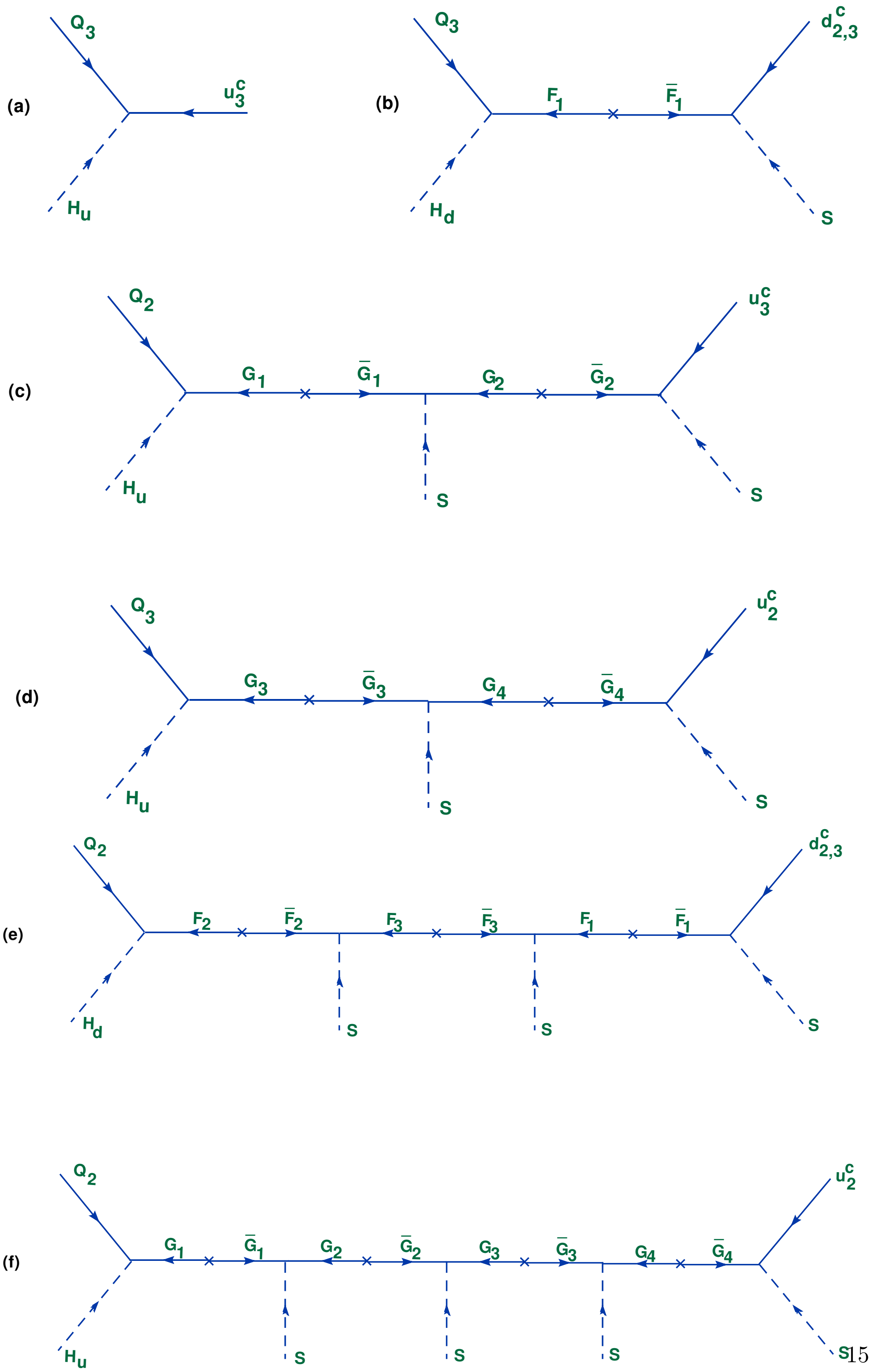

Figure 3: Interactions of the Froggatt-Nielsen fields, taken from [12].

As we indicated, the masses scale with different powers of ϵ allowing for realistic mass hierarchies for ϵ of the order one. This implies that realistic scenarios may be constructed if there is now large hierarchy between $< S >$ and M_*.

Note that more realistic models with three families can easily be obtained (see [12]). Of course this mechanism is a purely phenomenological approach and the remaining challenge lies in the construction of such models with the right field content, couplings and charges from a more fundamental theory. Constructing such models within F-theory has been addressed in [13, 14].

2.2. String theory

As we have underlined, the construction of GUT models is by itself purely phenomenological and should be justified from a more fundamental theory. A promising proposition for such a theory is *string theory* as we will try to argue in the following.

Modern physics rests on the foundation of two theories: General relativity on the one hand and quantum field theory — in its manifestation as the standard model of particle physics — on the other. Both theories are very successful in the description of many physical phenomena: General relativity in answering cosmological questions and the standard model in predicting particle interactions. The picture drawn by the two theories is, nevertheless, far from complete and moreover suffers from many problems. For quantum field theory those problems stem from UV divergences. While they can be treated in the framework of renormalisation to render the theory useful at low energies, this comes at the cost of predictive power and fails in the description of high energy physics. In general relativity problems arise from two origins: On the one hand spacetime singularities of infinite mass density may in particular develop from finite mass densities in formerly smooth geometries and might render the theory inconsistent. (Although it is conjectured that singularities without a surrounding event horizon are absent.) On the other hand general relativity lacks a quantum description in the sense that it is non-renormalisable in pertubation theory and to date also in non-pertubative descriptions. It is therefore desirable to find a unified description of the two theories which is also fundamental in the sense that it incorporates all physical aspects of the universe. Apart from this rather philosophical point of view, there is also a number of questions requiring a description incorporating quantum effects as well as relativistic effects. These are for example (small) black holes and questions of very early cosmology [15], such as cosmic inflation.

Today the only theory providing such a framework for a quantum description of all interactions is string theory; it is therefore widely considered to be the leading candidate for a fundamental theory. String theory incorporates the former two theories by demanding general covariance and a quantisation of excitations. These two axioms are supplemented by taking one dimensional objects — *strings* — as fundamental rather than point particles. Moreover, one introduces a further symmetry between fermionic and bosonic excitations in order to introduce fermions in the most basic way: *Supersymmetry.* Resting on these very basic assumptions, profound consequences can be drawn. In particular there is a unique way of describing string interactions which not only simplifies to a single process at every loop order (see fig. 4) but also remains finite

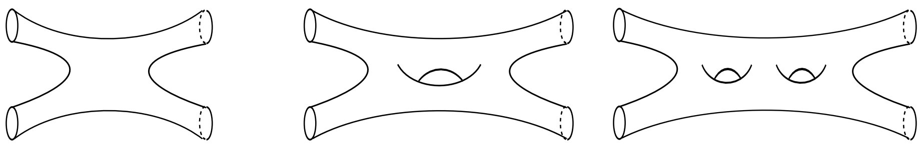

Figure 4: Closed string diagrams to second loop order

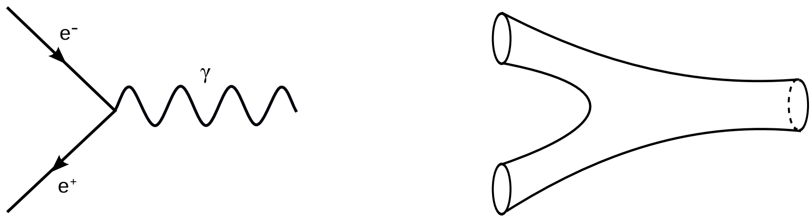

Figure 5: Tree level diagrams in quantum field theory [16] and string theory. Note that the interaction is localised to a vertex in the QFT diagram in contrast to the one in the string theory diagram

in the UV regime. This can be understood at an intuitive level as the consequence of the localisation of an interaction vertex in quantum field theory versus a non-localised interaction in string theory as depicted in fig. 5. This UV finiteness has been proven up to two-loop order and has been conjectured to hold for higher loop diagrams, as well.

Note, that string theory reproduces general relativity (i.e. the Einstein equations) as a corollary whereas standard model physics are not eminent immediately, as we will explain. Before we discuss this, we briefly digress and note that in fact there is no unique way of formulating a superstring theory, but rather five — at first seemingly unrelated — ones (type I, IIA, IIB and heterotic string theory with gauge group $E_8 \times E_8$ and $SO(32)$). We will come to relations among them in the next subsection. Note at this point that gauge groups come about quite differently in these theories. On the one hand the heterotic string theories allow for gauge groups to be embedded within their large structure groups and therefore provide a truly unified description of gravity and gauge theories. While on the other hand the type I, IIA and IIB superstring theories associate gauge degrees of freedoms to so called branes — hypersurfaces on which open strings may end — whereas gravity degrees of freedom are encoded in closed string dynamics. Coincident stacks of such branes are then described in terms of so called *Chan-Paton factors* which in turn give rise to gauge groups of types $U(n)$ and $Sp(n)$. Although considerable research has been done to reproduce standard model physics in terms of these theories, to date no exact description within string theory exists.

Leaving aside the issue of these five theories for a moment, string theory produces unique results up to this point in our description. However, this does not hold if

Figure 6: Dualities among the different superstring theories.

one considers its resulting low energy physics: Namely, string theory predictions not only include interactions, but also the dimension of its ambient spacetime. Seemingly problematic, string theory is only consistent in ten dimensions (for the superstring). To derive a suitable theory in a four-dimensional spacetime, six dimensions therefore have to be put on a compact space. Although there are constraints acting on the compactification space, their choice is far from unique. As has been described before the advent of string theory by T. Kaluza and O. Klein [17, 18, 19] in quantum field theory, compactification produces a tower of fields with masses related to the curvature radii of the compact space. Since the non-zero modes in this tower acquire very high masses, the low-energy effective theory is governed by the massless excitations of the strings and is in particular strongly affected by the geometry of the compactification. Note that while the compactification procedure is not unique, it is constrained by the phenomenological requirement of $\mathcal{N} = 1$ supersymmetry in four dimensions and Ricci-flatness to reproduce Einstein's equations in vacuum. These two constraints fix the internal manifold to be a so called (conformal) *Calabi-Yau three-fold*. Such manifolds are hermitian with vanishing Kähler form and have $SU(n)$ holonomy. In particular they admit a complex structure.

2.3. String dualities and M-theory

Let us now come back to the five consistent formulations of superstring theories. After their discovery their interrelations have naturally been of great interest and it was subsequently discovered that they are linked by various dualities. That is, one may relate certain parameter regions of one theory to that of another. These dualities can be categorised in so called S- and T-dualities. The former relates a theory at strong coupling to a theory at weak coupling ($g \to \frac{1}{g}$), the latter relates a theory compactified on a circle of radius R to a theory compactified on a circle with radius $\frac{1}{R}$. Taking into account all five superstring theories, the dualities can be summarised as in fig. 6. From the figure we take that there are in a sense only two superstring theories: On the one hand the two heterotic strings and the type I superstring and on the other hand the two different type II superstrings. Furthermore it has been shown, that two superstring theories exhibit an eleventh dimension at strong coupling: The IIA string in the form of a circle and the heterotic $E_8 \times E_8$ string in the form of an interval. These phenomena led to the proposal of a unifying theory which not only incorporates the above theories as certain limits, but also has as its low-energy effective action the maximal eleven dimensional supergravity. This theory was dubbed *M-theory*. One hint at the duality between IIA superstring theory and M-theory can already be seen at the level of their low-energy effective actions: Dimensional reduction of the 11D SUGRA action associated to M-theory leads to the 10D IIA SUGRA action, which is indeed the low-energy effective action of the IIA superstring. However, the dualities among the two theories do not restrict to the low-energy regime.

Since M-theory is often approached in the framework of its low-energy effective action — the unique, maximal 11D supergravity action — let us briefly review its construction. One way to approach this theory is by considering the necessary field content: Firstly, we want to construct a theory including gravity; this means in particular that we have to include a graviton, i.e. a symmetric traceless tensor of $SO(D-2) = SO(9)$, we will henceforth denote it by $\hat{e}_M^A$ where M is a base space index and A a tangent space index. Note in particular, that such a field contributes 44 bosonic degrees of freedom. Secondly, the theory should exhibit local supersymmetry, which requires a gauge field ψ_M, that is a 32-component Majorana spinor for each M. To find out how many degrees of freedom it adds, note that the little group is the covering group of $SO(9)$ if we include spinors. This group, $Spin(9)$, has a real spinor representation of dimension 16. Thus, the gravitino transforms in 9×16 which can be decomposed into irreducible representations as $128 + 16$. However, the relevant part of the action $S_\psi \propto \int \bar{\psi_M} \Gamma^{MNP} \partial_N \psi_P \mathrm{d}^{11}x$ is invariant under transformations $\delta\psi_M = \partial_M \epsilon$, reducing the number to 128 fermionic degrees of freedom. One requirement of a supersymmetric theory is of course that the bosonic degrees of freedom should match the fermionic ones. So far, we introduced 44 bosonic and 128 fermionic degrees of freedom. This means, we are lacking 84 bosonic degrees of freedom. The solution to this problem is to introduce an additional three-form field C_3 together with the constraint that the theory should be invariant under gauge transformations $C_3 \to C_3 + \mathrm{d}\Lambda_2$ for some two-form Λ_2. This gives $\frac{9 \cdot 8 \cdot 7}{3!} = 84$ bosonic degrees of freedom, thus matching the fermionic ones. Now, taking into account invariance under C_3 gauge transformations, local Lorentz invariance and coordinate invariance, the bosonic part of the action is entirely fixed as

$$S_{\text{boson}} = \frac{1}{2} \int \mathrm{d}^{11} \sqrt{-\hat{g}} \left(\hat{R} - \frac{1}{2} \hat{F}_4 \wedge \star \hat{F}_4 - \frac{1}{6} \hat{C}_3 \wedge \hat{F}_4 \wedge \hat{F}_4 \right), \tag{2.10}$$

where η is a Majorana spinor and $\hat{F}_4 = \mathrm{d}\hat{C}_3$. This action is manifestly invariant under local Lorentz transformations [1]:

$$\delta \hat{e}_M^A = i\bar{\eta}\hat{\Gamma}^A \psi_M \tag{2.11}$$

$$\delta \hat{C}_{MNP} = 3i\bar{\eta}\hat{\Gamma}_{[MN}\psi_{P]} \tag{2.12}$$

$$\delta\psi_M = \hat{\nabla}_M \eta - \frac{1}{288} \left(\hat{\Gamma}_M^{PQRS} - 8\hat{\delta}_M^P \hat{\Gamma}^{QRS} \right) \hat{F}_{PQRS}\eta \tag{2.13}$$

[1] The following conventions have been used for the gamma matrices:

$$\hat{\Gamma}_M := \hat{e}_M^A \hat{\Gamma}^A$$

$$\hat{\Gamma}_{M_1 \ldots M_n} := \hat{\Gamma}_{[M_1} \ldots \hat{\Gamma}_{M_n]}$$

The covariant derivative is taken with respect to the spin connection ω: $\hat{\nabla}_M \eta = \partial_M \eta + \frac{1}{4}\omega_{MAB}\hat{\Gamma}^{AB}\eta$. Since a classical solution has vanishing fermionic fields, it is sufficient to concentrate on above bosonic action in this context. Moreover, in the following we will restrict the discussion to supersymmetric solutions as these have nice phenomenological properties:

$$0 = \delta\psi_M = \delta\hat{e}_M^A = \delta\hat{C}_{MNP} \tag{2.14}$$

Before we go on, note that there is a much deeper relationship between the IIA superstring and M-theory than just via their low-energy effective actions. The non-perturbative nature of this relationship can be probed using quantities protected by supersymmetry or more specifically so called *BPS-saturated states* — states which are tied to the central charge of the superconformal algebra and therefore transform in shorter supermultiplets. As long as these states are non-degenerate (and do not combine), they can be used to match states in different theories. In particular one may show, that the D0 branes in IIA superstring theory correspond to Kaluza-Klein excitations of the SUGRA multiplet of M-theory on a circle [20].

2.4. F-theory as non-pertubative IIB theory

After we have introduced M-theory as a non-pertubative description of both the IIA superstring and the heterotic string with $E_8 \times E_8$ and also noted that it is also related to the other string theories via the dualities among them, one might nevertheless wonder whether there are more direct non-pertubative formulations of other string theories. In particular the IIB string has proven to be very fruitful from a phenomenological point of view. Hence we will have a look at IIB superstring theory in the presence of 7-branes and from there set out to develop a deeper understanding of the theory along the lines of [5]. This will eventually lead us to a non-pertubative description of the theory. To do so, consider the backreaction of 7-branes on the fields in the low-energy effective action [5]

$$S_{\text{IIB}}^{(\text{S})} = \frac{2\pi}{l_{\text{s}}^8}\bigg(\int \mathrm{d}^{10}x e^{-2\phi}\left(\sqrt{-g}R + 4\partial_M\phi\partial^M\phi\right) - \frac{1}{2}e^{-2\phi}\int H_3 \wedge \star H_3 \tag{2.15}$$

$$-\frac{1}{4}\sum_{p=0}^{4}\int F_{2p+1}\wedge\star F_{2p+1} - \frac{1}{2}\int C_4 \wedge H_3 \wedge F_3\bigg). \tag{2.16}$$

As we know, the dilaton ϕ and the axion C_0 combine to the complex axio-dilaton $\tau = C_0 + ie^{-\phi}$.

If we consider a geometry including D7 branes we have to introduce a source of the C_8 field which satisfies a Poisson equation

$$\mathrm{d}\star F_9 = \delta^{(2)}(z - z_0), \tag{2.17}$$

where $z := x^8 + ix^9$ denote the normal coordinates to the brane and z_0 encodes the locus of a D7-brane. Integration yields

$$1 = \int_{\mathbb{C}} \mathrm{d} \star F_9 = \oint_{C_{z_0}} F_1 = \oint_{C_{z_0}} \mathrm{d}C_0, \tag{2.18}$$

where we used Stokes theorem in the second equality. This equation has the solution

$$\tau(z) = \tau_0 + \frac{1}{2\pi i} \log(z - z_0) + \text{regular} \tag{2.19}$$

In particular the string coupling is given by

$$\frac{1}{g_S} = e^{-\phi} = -\frac{1}{2\pi} \log(z - z_0), \tag{2.20}$$

which implies that the coupling strongly varies over the internal manifold and will in particular leave the non-perturbative regime. Another problematic property can be seen if we set for simplicity $z_0 \equiv 0$ and introduce $z \equiv re^{i\chi}$. Eq. (2.19) is now expressed as:

$$\tau(z) = \tau_0 + \frac{1}{2\pi i} \log(r) + \frac{\chi}{2\pi} \tag{2.21}$$

This shows that encircling the brane — i.e. $\chi \to \chi + 2\pi$ — transforms the axio-dilation as $\tau \to \tau + 1$. So at a given point in the compactification $t(z)$ is not uniquely defined. Mathematically this is expressed as the presence of a monodromy in the theory. At this point it seems thus questionable whether the IIB string is well defined in the presence of D7-branes at all.

To answer this, consider the IIB action using the following field redefinitions:

$$\tau = C_0 + ie^{-\phi} \tag{2.22}$$

$$G_3 = F_3 - \tau H_3 \tag{2.23}$$

$$\tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3 \tag{2.24}$$

It is then given by [5]:

$$S_{IIB,E} = \frac{2\pi}{l_S^8} \Big(\int \mathrm{d}^{10}x R - \frac{1}{2} \int \frac{1}{(\Im\tau)^2} \mathrm{d}\tau \wedge \star \mathrm{d}\bar{\tau} + \int \frac{1}{\Im\tau} G_3 \wedge \star G_3 + \frac{1}{2} \int \tilde{F}_5 \wedge \star \tilde{F}_5 + \int C_4 \wedge H_3 \wedge F_3 \Big)$$

supplemented with the self duality constraint $\star\tilde{F}_5 = \tilde{F}_5$, which is imposed at the level of the equations of motion. Introducing these new fields makes manifest that the IIB superstring theory is invariant under $SL(2, \mathbb{Z})$ transformations:

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{2.25}$$

$$\begin{pmatrix} H \\ F \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} H \\ F \end{pmatrix} \tag{2.26}$$

$$\tilde{F}_5 \to \tilde{F}_5 \tag{2.27}$$

$$g_{MN} \to g_{MN} \tag{2.28}$$

In this light, the monodromy we encountered above can be expressed as an $SL(2,\mathbb{Z})$ transformation:

$$M_{1,0} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{2.29}$$

This underlines that the monodromy we observed, should instead be understood as a symmetry of the theory. Note, moreover, that S-duality of the IIB superstring to itself is also incorporated as a special case of such a gauge transformation by:

$$M_{\mathrm{S-dual.}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.30}$$

$$\Rightarrow \tau \to \frac{1}{\tau} \tag{2.31}$$

At this point you might also have noticed that the newly encountered symmetry matches the structure group of a torus. Indeed IIB superstring theory resembles a theory obtained from compactification of some twelve dimensional theory on a torus with complex structure given by the axio-dilaton τ. In this interpretation F_3 and H_3 would be components of a twelve dimensional four form $\hat{F}_4$ and the $SL(2,\mathbb{Z})$ symmetry would originate in the reparametrisation invariance of the torus. In view of the resemblance of the underlying concepts to the ones we encountered between the IIA string and M-theory this twelve dimensional theory has been dubbed *F-theory*. In the context of such a theory we can also develop a geometric understanding of the divergence of the complex structure $\tau(z)$ at the position of the brane. In terms of the two additional toroidal dimensions, branes would encode loci along which one circle of the torus shrinks to zero size, see fig. 7. However, there are a few problems in the interpretation of IIB superstring theory as a theory obtained from some higher twelve dimensional theory, which show that M- and F-theory are on quite different standings despite their conceptual similarities. Among others, there is no twelve dimensional supergravity which could function as a low-energy effective action of such a theory. Secondly, it seems unnatural that there is no dependence on the area of the compactification torus in the theory.

2.5. M-/F-theory duality

To shed more light on these questions we approach F-theory from the perspective of M-theory by employing the dualities we introduced above based on the discussion in [21]. More specifically we investigate how T-duality between the IIA and IIB string carries over to their non-pertubative extensions M- and F-theory. We start by considering M-theory compactified on $M_9 \times T^2 =: M_9 \times S^1_A \times S^1_B$ for a torus of complex structure τ and area v. The metric of the resulting space is given by:

$$\mathrm{d}s^2_M = \frac{v}{\tau_2}\left((\mathrm{d}x + \tau_1 \mathrm{d}y)^2 + \tau_2^2 \mathrm{d}y^2\right) + \mathrm{d}s^2_9, \tag{2.32}$$

where x and y are coordinates of the circles S^1_A and S^1_B respectively. Moreover τ_1 and τ_2 are real and imaginary part of the complex structure. We can now employ

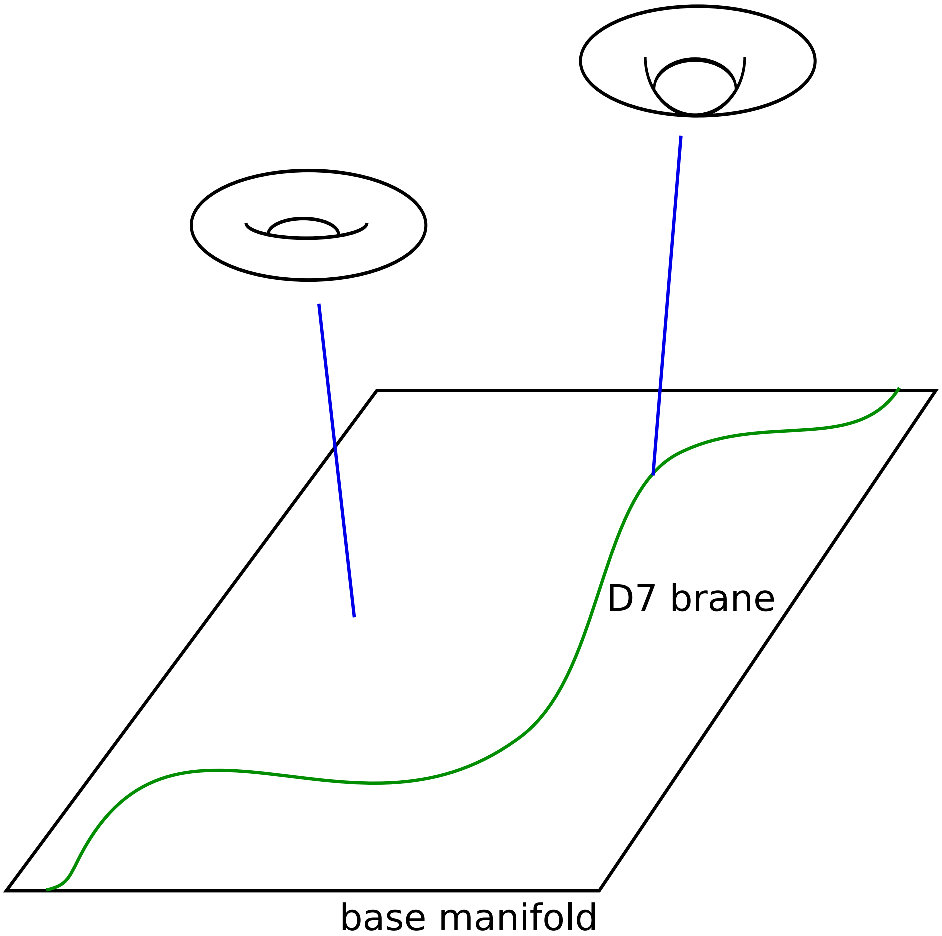

Figure 7: Schematic appearance of the fibre at different loci in the base manifold. At a generic point we encounter a torus, whereas at the loci of D7 branes the torus becomes singular.

the duality between M-theory and the IIA string by compactifying the former on the A-circle to obtain the latter on $M_9 \times S^1_B$. The metrics are related by

$$\mathrm{d}s^2_M = L^2 e^{\frac{4\chi}{3}} (\mathrm{d}x + C_1)^2 + e^{\frac{-2\chi}{3}} \mathrm{d}s^2_{IIA}, \tag{2.33}$$

where we introduced $L^2 e^{\frac{4\chi}{3}} := \frac{v}{\tau_2}$ and χ is only defined up to a constant shift, such that L can be chosen conveniently. Moreover, $C_1 := \tau_1 \mathrm{d}y$ is the IIA one-form field and the IIA metric is given by:

$$\mathrm{d}s^2_{IIA} = \sqrt{\frac{v}{\tau_2}} \frac{1}{L} (v\tau_2 \mathrm{d}y^2 + \mathrm{d}s^2_9) \tag{2.34}$$

Recall now that the IIA superstring on a circle S^1_B with radius R_B is dual to a IIB superstring on a circle $\tilde{S}^1_B$ with a radius $\sim \frac{1}{R_B}$. The couplings of the two theories are related by $g_{IIB} = \frac{l_S}{R_B} g_{IIA}$ and the IIB axion is given by $C_0 = (C_1)_y$. Having used these two dualities now, we can express the quantities of IIB superstring theory in this compactification in terms of M-theory quantities, using $l_S^2 = l_M^3$ and $g_{IIA} l_S^2 = e^{\chi} l_M^3$. This yields an expression for the metric in Einstein-frame as:

$$\mathrm{d}s^2_{IIB,E} = \frac{\sqrt{v}}{L} \left(\frac{L^2 l_S^4}{v^2} \mathrm{d}y^2 + \mathrm{d}s^2_9 \right) \tag{2.35}$$

Moreover, we can relate the complex structure to the IIB axion and string coupling:

$$\tau = C_0 + \frac{i}{g_{IIB}} \tag{2.36}$$

In summary this means M-theory on a torus is dual to IIB superstring theory on a circle and in particular the complex structure of the M-theory torus is given by the IIB axio-dilaton.

Now let us specialise this analysis to $M_9 = \mathbb{R}^{1,2} \times B_6$ where B_6 is some 6 dimensional manifold. As we argued before the manifold we compactify a theory on has to be Calabi-Yau in order for supersymmetric solutions to exist. In this case therefore $B_6 \times T^2$ has to be a Calabi-Yau manifold. In this specialised compactification the metric for the dual IIB superstring is given as:

$$\mathrm{d}s^2_{\mathrm{IIB,E}} = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + \frac{l_S^4}{v} \mathrm{d}y^2 + \mathrm{d}s^2_{B_6} \tag{2.37}$$

Crucially, if we take the limit of vanishing torus area $v \to 0$ the IIB-circle decompactifies and we get four-dimensional Minkowski spacetime. So in particular one of the internal M-theory dimensions now becomes part of the IIB spacetime.

Since we are interested in supersymmetric solutions it is essential to know how the supersymmetry generators of M-theory and IIB superstring theory relate; to see this, recall that a Dirac spinor in a $2k$ (or $2k+1$) dimensional space has 2^k degrees of freedom. For phenomenological reasons we try to construct $\mathcal{N} = 1$ supersymmetry in the IIB theory and since we have 4d Minkowski spacetime this means $k = 2$ and we

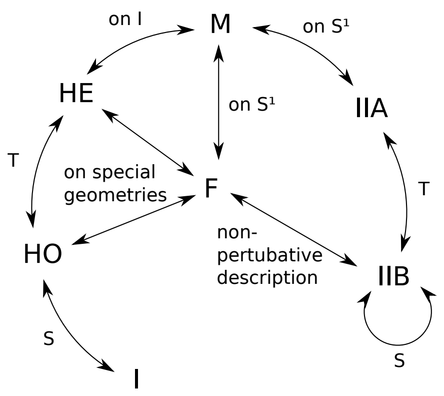

Figure 8: Dualities among the different superstring theories, M-theory and F-theory.

have 4 degrees of freedom. However, in three dimensions $k = 2$ and therefore a spinor has only two degrees of freedom. This means in order to match the degrees of freedom of the SUSY-generators in both theories we need two spinors in the dual M-theory compactification.

Putting all this together, we can summarise: M-theory compactified on $\mathbb{R}^{1,2} \times B_6 \times T^2$ with $N = 2$ supersymmetry is dual to the IIB superstring on $\mathbb{R}^{1,3} \times B_6$ with $\mathcal{N} = 1$ supersymmetry. The compactification procedure we just outlined therefore gives us a recipe to construct phenomenologically interesting IIB theories.

We can now further generalise the former analysis to compactifications in which the internal M-theory manifold is not given by a direct product of $B_6 \times T^2$ rather than a varying torus over the remaining six dimensions. Mathematically, this can be achieved by considering a fibration structure

$$T^2 \hookrightarrow Z_8 \tag{2.38}$$
$$\downarrow \tag{2.39}$$
$$B_6 \tag{2.40}$$

Physically, this realises a varying axio-dilation over the base manifold B_6. In order to arrive at a suitable, four-dimensional spacetime, we have to take the limit of vanishing fibre size $v \to 0$. In this context we can understand why there is no dependence on the area of the torus in IIB superstring theory. In summary this analysis therefore tells us that really we should think of F-theory as the limit of vanishing fibre size of the M-theory geometries we discussed above.

Coming back to our original net of dualities, we can now supplement it by M- and F-theory and their dualities as shown in fig. 8. Note that we will elaborate the dualities between F-theory and the heterotic string in section 4.

2.6. D7 branes and O7 planes in the light of F-theory

In pertubative IIB string theory a fundamental string is charged electrically under the Kalb-Ramond field B_2. As we have outlined in section 2.4, $SL(2,\mathbb{Z})$ transformations mix B_2 with the two-form field C_2. Therefore invariance under these transformations implies the existence of objects charged electrically under C_2. We know that such an object is provided by a D1-brane. In fact one may combine the fundamental string and the D1-brane into an $SL(2,\mathbb{Z})$-doublet, which underlines that we have to account for more generic objects carrying p units of charge under B_2 and q units of charge under C_2, so called $[p,q]$-strings. A consequence of this is of course that one also has to introduce more generic objects than D7-branes in the form off $[p,q]$-7-branes which encode hypersurfaces on which $[p,q]$-strings can end. The monodromy introduced by such a $[p,q]$-brane can be expressed as [5]:

$$M_{p,q} = g_{p,q} M_{1,0} g_{p,q}^{-1} = \begin{pmatrix} 1-pq & p^2 \\ -q^2 & 1+pq \end{pmatrix} \tag{2.41}$$

For $g_{p,q} \in SL(2,\mathbb{Z})$ such that a fundamental string is transformed into a $[p,q]$-string under $g_{p,q}$. and $M_{1,0}$ is given in eq. (2.29). This shows in particular that every brane can be mapped to a D7-brane by an $SL(2,\mathbb{Z})$ transformation. However, in a system of multiple such branes not all of them can be transformed to a D7-brane simultaneously. Physically this translates to the statement that a $[p,q]$-brane locally resembles a D7-brane, whereas the global description is more complicated in general.

Apart from D7-branes one very important ingredient in IIB compactifications are so called orientifolds. This stems from the fact that the IIB string compactified on such geometries gives the type I string. Let us, therefore, briefly recall the construction of orientifolds: Given a smooth manifold $\mathcal{M}$, two discrete groups G_1, G_2 and the worldsheet parity operator:

$$\Omega_P : \quad \sigma \to 2\pi - \sigma \tag{2.42}$$

An orientifold is given by:

$$\mathcal{M} \Big/ (G_1 \cup \Omega G_2) \tag{2.43}$$

Given the significance of such compactifications for the relation between the type IIB string and the type I string, one might therefore wonder how such geometries lift to F-theory. Consider for instance IIB on a torus T^2 with the orientifold action $\Omega(-1)^{\mathrm{F_L}}\sigma$. One may confirm that this action acts on the complex coordinate of the torus as $z \to -z$ generically, while it admits four hyperplanes — so called *O7-planes* — encoding the fixpoints of the orientifold action. Tadpole cancellation then links the number of these O7-planes to that of the D7-branes and constrains them such that there are four D7-branes and their respective images on top of every O7-plane. This results in a gauge group of $SO(8)^4$. A famous result by Sen [22] states that each such set of D7-branes and O7-plane corresponds to a system of three different types of $[p,q]$-branes

$$A : [1,0] \quad B : [3,-1] \quad C : [1,-1], \tag{2.44}$$

where the BC-system originates in the O7-plane as can be seen by noting [5]:

$$M_{BC}\begin{pmatrix}1\\0\end{pmatrix} = M_{3,-1}M_{1,-1}\begin{pmatrix}1\\0\end{pmatrix} = -\begin{pmatrix}1\\0\end{pmatrix} \tag{2.45}$$

So in summary, F-theory unifies the treatment of D1 branes and fundamental strings as well of D7-branes and O7-planes. Moreover, we have established that F-theory provides a truly non-pertubative formulation in the string coupling g_S. Its weakly-coupled limit is related to the IIB string in the presence of D7-branes and O7-planes.

3. Elliptic fibrations, gauge groups and singularities

In the previous sections we concluded that the geometry of F-theory compactifications is described by a torus fibred over a threefold base such that the resulting space is a Calabi-Yau fourfold. In order to have a well-defined notion of the complex three dimensional physical compactification space, we additionally need an embedding of the base within the fourfold. The most common way to realise this is to consider geometries admitting a global section — so called *elliptic fibrations*. Since the understanding of elliptic fibration is at the heart of F-theory, we will devote the following section to describe the properties of such geometries along the lines of [5]. Before we proceed to discuss elliptic fibrations in more detail, note that also more general genus one fibrations without a global section have been considered recently [23, 24, 25].

In section 3.1 we start by first deriving an expression of a single elliptic curve, that is a complex, algebraic curve of genus one with a rational point. In the following we will then fibre this expression over some base manifold. Having obtained an expression for these geometries we can then proceed to investigate the interplay of fibration structure and resulting field theory. More specifically we will investigate how non-abelian gauge algebras arise in F-theory in section 3.2 as well as how the geometry is related to further gauge enhancements in section 3.3. Moreover, as has been outlined in the last section, further abelian symmetries are very desirable for phenomenological applications and therefore we will consider how to construct fibrations admitting extra $U(1)$ symmetries in section 3.4. Lastly, the arising of gauge algebras in F-theory is profoundly related to singularities in the geometry and therefore we will discuss how to resolve singular spaces and how the resolution is related to physics in section 3.5.

3.1. Elliptic fibrations

As we just outlined we start by deriving a representation of a single elliptic curve: One possibility of describing an elliptic curve is in terms of a hypersurface within a complex two dimensional weighted projective space $\mathbb{P}_{[k,l,m]} = \mathbb{C}^3/_{\sim}$ for an equivalence relation $(x, y, z) \sim (\lambda^k x, \lambda^l y, \lambda^m z)$ and $\lambda \in \mathbb{C}^*$. That is, within this projective space, we define an elliptic curve as the vanishing locus of a homogeneous polynomial P in x, y, z. An elliptic curve defined in a such a way will be Ricci-flat if the degree of the polynomial equals $k + l + m$. A proof of this can be found in [26]. To understand the general idea we will argue for the non-weighted projective space (i.e. $\mathbb{P}_{[1,1,1]} = \mathbb{P}^2$) why this holds

[27]: We start by computing the Chern class of $\mathbb{P}^2$. To do so we consider instead the bundle $T\mathbb{P}^2 \oplus \mathcal{O}$ (i.e. the Whitney sum of $\mathbb{P}^2$ and the trivial bundle). This bundle has an identical Chern class, but is a vector bundle, which means we can apply the *splitting principle* and assert that the curvature two-form is diagonalisable to $\mathrm{diag}(J, J, J)$ for the Kähler form J on $\mathbb{P}^2$. This implies in particular that the Chern class of $\mathbb{P}^2$ is given by:

$$c(\mathbb{P}^2) = (1 + J)^3 \tag{3.1}$$

As $\mathbb{P}^2$ is complex two dimensional, J^3 vanishes. Specialising this result to the locus of the hypersurface given by $X = \{P = 0\}$, we use the splitting into tangent and normal bundle inside $\mathbb{P}^2$:

$$T\mathbb{P}^2 = TX \oplus NX \tag{3.2}$$

Which implies for the Chern class

$$c(T\mathbb{P}^2|_{P=0}) = c(TX) \wedge c(NX) \tag{3.3}$$

$$c(TX) = \frac{(1 + J)^3}{1 + tJ}, \tag{3.4}$$

where t is the degree of the polynomial P. From above equation we compute the first Chern class by expanding above expression in J:

$$c_1(TX) = (3 - t)J \tag{3.5}$$

In order to have vanishing first Chern class we therefore have $t = 3$. So for the non-weighted projective space indeed, the hypersurface will be Ricci flat, if its defining polynomial is of the same degree as the sum of the weights $1 + 1 + 1$ of its ambient space $\mathbb{P}_{[1,1,1]}$.

A well known result from the mathematics literature (see e.g. [28]), is that every elliptic curve can in particular be described within the space $\mathbb{P}_{[2,3,1]}$ in so called *Weierstraß form*.

$$P_{\mathrm{W}} = y^2 - x^3 - fxz^4 - gz^6 \tag{3.6}$$

Although the Weierstraß form is the most generic, it will turn out below that for some applications other representations prove to be more useful and therefore we will also deal with curves defined as a quartic in $\mathbb{P}_{[1,1,2]}$ and as a cubic in $\mathbb{P}_{[1,1,1]} = \mathbb{P}^2$, in this thesis. Both of these different representations can however be taken to the standard Weierstraß representation. For an example how to transform the former to Weierstraß form, see e.g. [29]. The latter is related to Weierstraß form by a so called Nagell transformation (see e.g. [30])

In section 2.4 we noted that the elliptic fibre degenerates along the loci of 7-branes, which translates to singularities of the hypersurface. In terms of the Weierstraß form this translates to loci along which $P_{\mathrm{W}} = 0 = \mathrm{d}P_{\mathrm{W}}$, that is the tangent space degenerates. In affine coordinates ($z = 1$) this amounts to

$$
\begin{aligned}
0 = \mathrm{d}P_{\mathrm{W}} &= \frac{\partial}{\partial x}\left(-x^3 - fx - g\right)\mathrm{d}x + 2y\mathrm{d}y && (3.7)\\
\Rightarrow 0 &= y && (3.8)\\
\Rightarrow 0 &= (x-b_2)(x-b_3) + (x-b_1)(x-b_3) + (x-b_1)(x-b_2), && (3.9)
\end{aligned}
$$

where the b_i label the roots of $-x^3 - fx - g$. Accordingly:

$$0 = P_{\mathrm{W}} = (x-b_1)(x-b_2)(x-b_3) \tag{3.10}$$

In order for the two equations (3.9) and (3.10) to be satisfied at the same time, at least two of the roots b_i have to coincide. This is of course encoded in the vanishing of the discriminant. For above cubic in x, it is given by

$$\Delta = 27g^2 + 4f^3 \tag{3.11}$$

As we have seen in section 2.4, the axio-dilation τ is not invariant under $SL(2,\mathbb{Z})$ transformations. One can, however, derive from it the *j-function*. The j-function is the unique function from the fundamental region of τ to $\mathbb{C}$ which is invariant under $SL(2,\mathbb{Z})$ and holomorphic away from a pole at the origin. Omitting a more rigorous definition for brevity, we define it as the expansion [5]:

$$j(\tau) = \exp(-2\pi i\tau) + 744 + \exp(2\pi i\tau) + \ldots \tag{3.12}$$

For an elliptic curve it can be computed as (see e.g. [30]):

$$j(\tau) = \frac{4(24f)^3}{\Delta} \tag{3.13}$$

Note in particular that two elliptic curves are isomorphic if and only if their j-functions are identical [31].

Now that we have obtained an expression for a single elliptic curve and also discussed how to detect singularities, we want to generalise this to an elliptically fibred $n+1$-fold Y_{n+1} over some base B_n. At this point we leave the dimensions of the fibration arbitrary as we will work with compactifications to various dimensions, when dealing with heterotic duality in section 4. Generalisation of the above description to a fibration of course implies in particular that f and g depend on the base coordinates u_i. These dependencies can be further constrained by considering the Calabi-Yau property of the total space. In particular we can relate the first Chern class of Y_{n+1} to that of B_n by [32]:

$$c_1(T_{Y_{n+1}}) = \pi * \left(c_1(T_{B_n}) - \sum_i \frac{\delta_i}{12}[\Gamma_i]\right) + (\text{higher codim. degen.}) \tag{3.14}$$

The higher codimension degenerations are irrelevant for the following [5]. Where $\delta_i = \mathcal{O}(\Delta)|_{\Gamma_i}$ counts the vanishing order of the discriminant along the divisors Γ_i and

$\pi : Y_{n+1} \to B_n$ is a fibre bundle with pullback $\pi^* : H^2(B_n, \mathbb{Z}) \to H^2(Y_{n+1}, \mathbb{Z})$. Restricting this to Calabi-Yau spaces, the left hand side vanishes due to the total space being Ricci-flat. Above formula therefore implies:

$$12c_1(T_{B_n}) = \sum_i \delta_i[\Gamma_i] \tag{3.15}$$

This shows in particular that in the presence of 7-branes B_n is itself not Ricci-flat and therefore not Calabi-Yau. Moreover, as the divisors are the vanishing loci of the discriminant, we can compute its class as:

$$[\Delta] = \sum_i \delta_i[\Gamma_i] \tag{3.16}$$

Thus, taking the expression for Δ from eq. (3.11), we can deduce how g and f transform

$$2[g] = 12c_1(T_{B_n}) = 12c_1(\bar{\mathcal{K}}) \tag{3.17}$$
$$\Rightarrow g \in H^2(B_n, \bar{\mathcal{K}}^6) \tag{3.18}$$
$$3[f] = 12c_1(\bar{\mathcal{K}}) \tag{3.19}$$
$$\Rightarrow f \in H^2(B_n, \bar{\mathcal{K}}^4) \tag{3.20}$$

for the anticanonical bundle $\bar{\mathcal{K}}$ of B_n. Going back to the Weierstraß equation (3.6), we can now deduce how x and y transform, by noting that every term in P_{W} has to be in the same class:

$$x \in H^2(B_n, \bar{\mathcal{K}}^2) \tag{3.21}$$
$$y \in H^2(B_n, \bar{\mathcal{K}}^3) \tag{3.22}$$

Note moreover that the fibre degenerates along a codimension one locus given by the vanishing of the discriminant $\Delta = 0$. From the IIB perspective, we expect this to encode the degeneracy of τ along a locus of coincident 7-branes. To make this correspondence more precise, label the coordinate normal to the degeneracy locus by ω and suppose $\Delta \sim \omega^N$ while f is generically non-zero along $\omega = 0$. Computing to leading order from eq. (3.13) gives:

$$\tau \sim \frac{N \ln(\omega)}{2\pi i} \tag{3.23}$$

Which is exactly what we would expect from a stack of N coincident D7-branes at $\omega = 0$. So indeed the vanishing order of the discriminant Δ seems to encode the number of branes, at least in this simple case.

3.2. Non-abelian gauge groups

We have seen above that the vanishing order of the discriminant corresponds to the number of coincident D7-branes. From the IIB string, we know that stacks of D7

ord(f)	ord(g)	ord(Δ)	fiber type	singularity type
≥ 0	≥ 0	0	smooth	none
0	0	n	I_n	A_{n-1}
≥ 1	1	2	II	none
1	≥ 2	3	III	A_1
≥ 2	2	4	IV	A_2
2	≥ 3	$n+6$	I_n^*	D_{n+4}
≥ 2	3	$n+6$	I_n^*	D_{n+4}
≥ 3	4	8	IV^*	E_6
3	≥ 5	9	III^*	E_7
≥ 4	5	10	II^*	E_8

Table 1: Kodaira classification of singularities [32]. Taken in this form from [33]

branes give rise to gauge algebras via Chan-Paton factors. This indicates a relation between the appearance of particular non-abelian gauge algebras and the singularity structure of the elliptic fibration. The different degeneration types of the elliptic fibre have been classified according to their vanishing order in Δ, f and g by Kodaira for the specific case of compactifications to six dimensions on K3 surfaces [32] (see tab. 1), but have since been generalised to other compactifications [33].

Before we proceed, note that not all degenerations of the elliptic fibre yield singularities which need a resolution. Consider for example a locus along which Δ vanishes to order one whereas f and g are non-zero. According to Kodaira's classification this corresponds to an I_1 singularity over which the elliptic curve degenerates to a single $\mathbb{P}^1$ with one self-intersection point. This degeneration in the fibre does not give rise to a singularity in the total space. In terms of the j-function we can confirm that such a singularity yields a logarithmic behaviour for the complex axio-dilaton τ matching what we observed for a single 7-brane in our motivation of F-theory from the IIB string. Physically it makes sense therefore that this does not give rise to a non-abelian gauge algebra. For higher order singularities we will, however, need a resolution process, which gives rise to gauge algebras as we will shortly see. In the specific compactifications considered by Kodaira these correspond to the simply laced Dynkin diagrams, yielding the *ADE classification* of singularities. More generic compactifications allow for more general gauge algebras by incorporating monodromies along the singular locus.

To demonstrate the relation between resolution process and gauge algebra, consider a singularity along a divisor $S \subset B_n$ and restrict for simplicity to cases where a *split simultaneous resolution* is applicable. That is we can replace the singular fibre over S by n projective lines $\mathbb{P}^1$, such that the resulting space $\tilde{Y}_{n+1}$ is a non-singular Calabi-Yau. We will see how this is done in more detail in 3.5. Denoting the $\mathbb{P}^1$s by Γ_i, the resolution process naturally yields n different $\mathbb{P}^1$ fibrations $\hat{D}_i : \Gamma_i \to S$ over the singular locus. Define furthermore $\hat{D}_0 = \hat{S} - \sum_i a_i \hat{D}_i$, where $\hat{S}$ is the elliptic fibre over S and a_i are the Dynkin labels of some Lie algebra G of rank n. We denote the Poincaré dual two forms by $[\hat{D}_i] \in H^2(\bar{Y}_4)$. One can then show that for some $\tilde{\omega} \in H^4(B_n)$, the resolution divisors intersect like the Cartan C_{ij} of G [5]:

$$\int_{\bar{Y}_4} [\hat{D}_i] \wedge [\hat{D}_j] \wedge \tilde{\omega} = -C_{ij} \int_S \tilde{\omega} \tag{3.24}$$

This equation implies that singularities not only give rise to non-abelian gauge algebras, but moreover that the fibre over these singularities behaves exactly as the extended Dynkin diagram of the corresponding algebra. Taking the limit of zero volume for the $\mathbb{P}^1$ takes us back to the singular F-theory geometry Y_4.

Having described the relation between the resolution process and the appearance of the Lie algebra G, we still lack a physical understanding of the situation. This can be seen via M-/F-theory duality: Namely, reducing the M-theory three-form C_3 along a non-singular fibre gives the IIB fields B_2, C_2. Contrastingly over degenerations extra harmonic two-cycles appear. Reducing C_3 along these additional cycles produces extra massless vector states, accordingly. More specifically, consider for instance an A_{n-1} singularity which is resolved by introducing $n-1$ $\mathbb{P}^1$s in the fibre. Reducing C_3 along these Γ_i gives one-forms $A_i = \int_{\Gamma_i} C_3$ acting like the gauge potentials associated to the Cartan subalgebra of $SU(n)$ [5]. Secondly, the M2 brane wrapping chains of the resolution spaces $S_{ij} = \Gamma_i \cup \Gamma_{i+1} \cup \cdots \cup \Gamma_j$ for $i \leq j$, yields extra massless states propagating along the brane. Taking into account both orientations yields $n^2 - n$ states which become massless in the limit of vanishing fibre size and give rise to the W bosons of $SU(n)$ [5]. These two additional sources for massless vector fields give in total $n^2 - 1$ generators indeed matching those of the adjoint representation of $SU(n)$.

Locally one can always bring a Weierstraß form to *Tate form*:

$$P_{\mathrm{T}} = x^3 - y^2 + a_1 xyz + a_2 x^2 z^2 + a_3 yz^3 + xz^4 + a_6 z^6 = 0 \tag{3.25}$$

The exact procedure to map a Weierstraß form to (3.25) locally, has been developed in [33] and is generally referred to as Tate's algorithm. Given a Tate form we can obtain the standard Weierstraß form of an elliptic curve by completing the square in y and the cube in x:

$$\beta_2 \equiv a_1^2 + 4a_2 \tag{3.26}$$

$$\beta_4 \equiv a_1 a_3 + 2a_4 \tag{3.27}$$

$$\beta_6 \equiv a_3^2 + 4a_6 \tag{3.28}$$

$$f = -\frac{1}{48}\left(\beta_2^2 - 24\beta_4\right) \tag{3.29}$$

$$g = -\frac{1}{864}\left(-\beta_2^2 + 36\beta_2\beta_4 - 216\beta_6\right) \tag{3.30}$$

$$\Rightarrow \Delta = -\frac{1}{4}\left(\beta_2\beta_6 - \beta_4^2\right)\beta_2^2 - 8\beta_4^3 - 27\beta_6^2 + 9\beta_2\beta_4\beta_6 \tag{3.31}$$

Note that, while a Tate form can always be mapped to Weierstraß form, the converse does not hold in general.

In the Tate form the gauge algebra associated to a singularity can be determined from the vanishing orders of the a_i and Δ along the codimension one locus in B_n defined by the vanishing of the discriminant, analogously to the Kodaira classification for f, g and Δ. Note that the discriminant may further factorise which corresponds to multiple intersecting stacks of 7-branes. This can be understood as follows: A factorisation of the discriminant $\Delta = \Pi_i^N P_i^{N_i}$ corresponds in homology to $[\Delta] = \sum_i^n N_i[P_i]$ meaning that the vanishing locus is comprised of n independent divisors. The precise correspondence between different vanishing orders of a_i and Δ has been worked out in [33]. And can be found in table 2, taken in this form from [34].

Having obtained the knowledge to adjust the a_i such as to induce a particular gauge symmetry, it is of course phenomenologically interesting to construct an $SU(5)$ gauge group along some divisor in order to construct GUT theories in the spirit of [3]. Note that recently also direct constructions of the standard model gauge groups within F-theory have been investigated in [35, 36, 37]. From table 2, we see that an $SU(5)$-GUT group along a divisor $S : \omega = 0$ is realised for an fibre degeneration of type I_5^s, which requires leading vanishing order of the a_i as:

$$a_1 \quad a_2 = a_{2,1}\omega \quad a_3 = a_{3,2}\omega^2 \quad a_4 = a_{4,3}\omega^3 \quad a_6 = a_{6,5}\omega^5 \tag{3.32}$$

As can be readily confirmed:

$$a_i \in H^0(B, \bar{\mathcal{K}}^i) \tag{3.33}$$

$$\Rightarrow a_{i,j} \in H^0(B, \bar{\mathcal{K}}^i\omega^{-j}) \tag{3.34}$$

Note that this specifies only the leading vanishing order in the perpendicular coordinate ω. That is, the $a_{i,j}$ might depend on ω, but do not contain additional overall factors of it. Plugging this specific model into our expression for the discriminant yields [5]:

$$\Delta \sim \omega^5 \left(a_1^4 P + \omega a_1^2(8a_{2,1}P + a_1 R)\omega^2(16a_{2,1}^2 a_{3,2}^2 + a_1 Q) + \mathcal{O}(\omega^3)\right) \tag{3.35}$$

where we define:

$$P = (a_{2,1}a_{3,2}^2 - a_1 a_{3,2}a_{4,3} + a_1^2 a_{6,5}) \tag{3.36}$$

$$R = (-a_{3,2}^3 - a_1 a_{4,3}^2 + 4a_1 a_{2,1} a_{6,5}) \tag{3.37}$$

$$Q = 2(-18a_{2,1}a_{3,2}^3 - 8a_{2,1}^2 a_{3,2}a_{4,3} + 15a_1 a_{3,2}^2 a_{4,3} \tag{3.38}$$

$$- 4a_1 a_{2,1} a_{4,3}^2 + 24a_1 a_{2,1}^2 a_{6,5} - 18a_1^2 a_{3,2} a_{6,5}) \tag{3.39}$$

A generic elliptic fibration has to be brought to Tate form locally to determine the gauge enhancement. However, some elliptic fibrations can be brought to Tate form not only locally, but also globally. This makes it particularly simple to read off the gauge group along a divisor. Such models are, however, not as generic as Weierstraß models and are related to E_8 in the sense that a non-abelian gauge group G over some divisor can always be embedded in E_8 [10]. Contrastingly, a generic Weierstraß model may encode enhancements beyond E_8.

sing. type	discr. $\deg(\Delta)$	gauge enhancement type	group	coefficient vanishing degrees a_1	a_2	a_3	a_4	a_6
I_0	0		—	0	0	0	0	0
I_1	1		—	0	0	1	1	1
I_2	2	A_1	$SU(2)$	0	0	1	1	2
$\mathrm{I}_{2k}^{\mathrm{ns}}$	$2k$	C_{2k}	$SP(2k)$	0	0	k	k	$2k$
$\mathrm{I}_{2k}^{\mathrm{s}}$	$2k$	A_{2k-1}	$SU(2k)$	0	1	k	k	$2k$
$\mathrm{I}_{2k+1}^{\mathrm{ns}}$	$2k+1$		[unconv.]	0	0	$k+1$	$k+1$	$2k+1$
$\mathrm{I}_{2k+1}^{\mathrm{s}}$	$2k+1$	A_{2k}	$SU(2k+1)$	0	1	k	$k+1$	$2k+1$
II	2		—	1	1	1	1	1
III	3	A_1	$SU(2)$	1	1	1	1	2
$\mathrm{IV}^{\mathrm{ns}}$	4		[unconv.]	1	1	1	2	2
IV^{s}	4	A_2	$SU(3)$	1	1	1	2	3
$\mathrm{I}_0^{*\,\mathrm{ns}}$	6	G_2	G_2	1	1	2	2	3
$\mathrm{I}_0^{*\,\mathrm{ss}}$	6	B_3	$SO(7)$	1	1	2	2	4
$\mathrm{I}_0^{*\,\mathrm{s}}$	6	D_4	$SO(8)$	1	1	2	2	4
$\mathrm{I}_1^{*\,\mathrm{ns}}$	7	B_4	$SO(9)$	1	1	2	3	4
$\mathrm{I}_1^{*\,\mathrm{s}}$	7	D_5	$SO(10)$	1	1	2	3	5
$\mathrm{I}_2^{*\,\mathrm{ns}}$	8	B_5	$SO(11)$	1	1	3	3	5
$\mathrm{I}_2^{*\,\mathrm{s}}$	8	D_6	$SO(12)$	1	1	3	3	5
$\mathrm{I}_{2k-3}^{*\,\mathrm{ns}}$	$2k+3$	B_{2k}	$SO(4k+1)$	1	1	k	$k+1$	$2k$
$\mathrm{I}_{2k-3}^{*\,\mathrm{s}}$	$2k+3$	D_{2k+1}	$SO(4k+2)$	1	1	k	$k+1$	$2k+1$
$\mathrm{I}_{2k-2}^{*\,\mathrm{ns}}$	$2k+4$	B_{2k+1}	$SO(4k+3)$	1	1	$k+1$	$k+1$	$2k+1$
$\mathrm{I}_{2k-2}^{*\,\mathrm{s}}$	$2k+4$	D_{2k+2}	$SO(4k+4)$	1	1	$k+1$	$k+1$	$2k+1$
$\mathrm{IV}^{*\,\mathrm{ns}}$	8	F_4	F_4	1	2	2	3	4
$\mathrm{IV}^{*\,\mathrm{s}}$	8	E_6	E_6	1	2	2	3	5
III^*	9	E_7	E_7	1	2	3	3	5
II^*	10	E_8	E_8	1	2	3	4	5
non-min	12		—	1	2	3	4	6

Table 2: Classification of singularities in elliptic fibres according to Kodaira and refined to curves given in Tate form in [33]

3.3. Matter curves and Yukawa couplings

As we have seen, non-abelian gauge algebras are located along divisors D_a in the base B_n of the elliptic fibration. Each D_a carries a gauge theory G_a. Apart from the vector multiplet transforming in the adjoint representation of G_a, extra massless charged matter states can appear in two ways: First in terms of so called *bulk matter*, whose gauge degrees of freedom are located on the entire divisor D_a and secondly in terms of states appearing along codimension two singularities. Both of these extra states can be motivated from a IIB perspective. As we will only deal with localised matter in the subsequent chapters, we skip a discussion of bulk matter and refer the reader to [5], for a derivation from IIB perspective.

The appearance of localised matter along codimension two loci is in accordance with the expectation from the IIB string where vector multiplets associated to open strings stretching between stacks of intersecting D7 branes become massless along the intersection loci. From the F-theory point of view, such intersection loci correspond to further gauge enhancements along intersection curves of two divisors:

$$C_{ab} = D_a \cap D_b \tag{3.40}$$

In terms of the resolution process of such a singularity one observes that the $\mathbb{P}^1$s $\Gamma_{a,i}$ and $\Gamma_{b,i}$ involved in the resolution of the individual divisors are both part of the fibre along along C_{ab} and intersect like the affine Dynkin diagram of a new gauge algebra G_{ab} with $\mathrm{rk}(G_{ab}) = \mathrm{rk}(G_a) + \mathrm{rk}(G_b)$. Note that there are multiple possibilities of such gauge enhancements for two such divisors, as we will see below. Along such curves there are extra two-cycles along which the M-theory three-form can be reduced on. Additionally M2 branes wrapping chains of resolution $\mathbb{P}^1$s yield extra massless states in the limit of vanishing fibre where the volume of the resolution chain is zero. The massless states along C_{ab} form the adjoint of G_{ab} composed of the massless matter from G_a, G_b and the extra vector multiplets appearing. To determine the representation under which the extra massless states transform one considers the decomposition

$$G_{ab} \to G_a \times G_b \tag{3.41}$$

$$\mathrm{ad}_{G_{ab}} \to (\mathrm{ad}_{G_a}, 1) \oplus (1, \mathrm{ad}_{G_b}) \oplus \sum (R_x, U, x), \tag{3.42}$$

where the first two terms correspond to the states propagating along D_a and D_b and the last term corresponds to extra localised matter along their intersection. To determine the specific type of gauge enhancement one considers Tate's algorithm. Take for instance an $SU(5)$ singularity along a divisor S. Localised matter arises at the intersection locus with the I_1 singularity along the divisor S_1 which carries no additional non-abelian gauge algebra; the rank of the gauge algebra thus enhances by one, along the intersection curve. This leaves us with two gauge enhancements [5]: Firstly, an enhancement $A_4 \to D_5$ which corresponds to the decomposition

$$SO(10) \to SU(5) \times U(1) \tag{3.43}$$

$$\mathbf{45} \to (\mathbf{24})_0 + (\mathbf{1})_0 + \mathbf{10}_2 + \bar{\mathbf{10}}_{-2}, \tag{3.44}$$

where the subscripts denote the charges under the extra $U(1)$ appearing in the breaking. One can check that such an enhancement occurs along

$$P_{10} : \{\omega = 0\} \cap \{a_1 = 0\}, \tag{3.45}$$

where the vanishing order of the discriminant increases to $\Delta|_{P_{10}} \sim \omega^7$, as one can confirm from eq. (3.35). Secondly, the gauge algebra enhances as $A_4 \to A_5$ yielding extra states as:

$$SU(6) \to SU(5) \times U(1) \tag{3.46}$$

$$\mathbf{35} \to (\mathbf{24})_0 + (\mathbf{1})_0 + \mathbf{5}_1 + \bar{\mathbf{5}}_{-1} \tag{3.47}$$

This is localised along:

$$P_5 : \{\omega = 0\} \cap \{P \equiv a_{3,2}^2 a_{2,1} - a_{4,3}a_{3,2}a_1 + a_{6,5}a_1^2 = 0\} \tag{3.48}$$

See fig. 9 for an illustration of the various gauge enhancements.

From the Kodaira classification (tab. 2), we see that certain singularities in the fibre give rise to exceptional gauge algebras - something which is unfamiliar from the IIB perspective of intersecting stacks of D7-branes. Recall, that in this context gauge algebras arise via Chan-Paton factors, labelling the branes on which open strings end. F-theory therefore accounts for more generic gauge enhancements as known from the weak-coupling description. How can this be understood? The idea is that in the non-pertubative description, the distinction between strings and D1-branes vanishes (see section 2.6) and therefore D1-branes on which strings end become dynamical themselves. This results in objects behaving like multi-pronged strings. The endpoints of such a n-pronged string are intuitively labelled by n indices as opposed to the 2 indices of the Chan-Paton factors, thus generalising the construction to more general gauge algebras [38]. Note in particular that this allows for the construction of exceptional gauge algebras, which have been a great source of innovations for phenomenological constructions in heterotic compactifications.

As we will shortly see in more detail, additional GUT singlets can arise at loci away from the divisor S, which can, however, intersect S. They occur at self intersections of S_1, along which the gauge algebra enhances to A_1. In compactifications to four dimensions the gauge algebra may further enhance at loci of codimension 2 in the base along which two or more such matter curves intersect. The specific gauge enhancement can be determined via Tate's algorithm in the same way as for the matter curves. Consider for instance the intersection of three curves $C_{ab} \cap C_{bc} \cap C_{ac}$ along which the gauge algebra enhances to G_{abc}. Such a gauge enhancement involves a cubic interaction term in the algebras adjoint representation which yields Yukawa couplings in the effective field theory. Restricting this analysis to $SU(5)$ models, we note that the minimal enhancement at a Yukawa point is therefore from the rank four A_4 to rank six. There exist three distinct such enhancements: Firstly $A_4 \to E_6$, giving rise to a $\mathbf{10 10 5}$ coupling at $\{a_1 = a_{2,1} = 0\}$. Secondly $A_4 \to D_6$, yielding a coupling $\mathbf{10}\bar{\mathbf{5}}\bar{\mathbf{5}}$ at $\{a_1 = a_{3,2} = 0\}$ and lastly an enhancement $A_4 \to A_6$ realising a $\mathbf{5}\bar{\mathbf{5}}\mathbf{1}$ coupling at $\{P = R = 0\} \cap (a_1, a_{2,1}) \neq (0,0)$. Note that the first gauge enhancements is truly non-pertubative in the sense that it cannot be engineered in pertubative IIB string theory. There are of course other enhancements of higher rank, depending on the ranks of the gauge algebras along the divisors. In particular the rank might increase such that the fibre becomes too singular to be resolved and the geometry does not give rise to a well defined theory anymore. Compactifications involving these so called *non-flat points* have to be restricted further by imposing extra conditions to turn the points in question off in homology, that is to set their class to zero. See [1] for details.

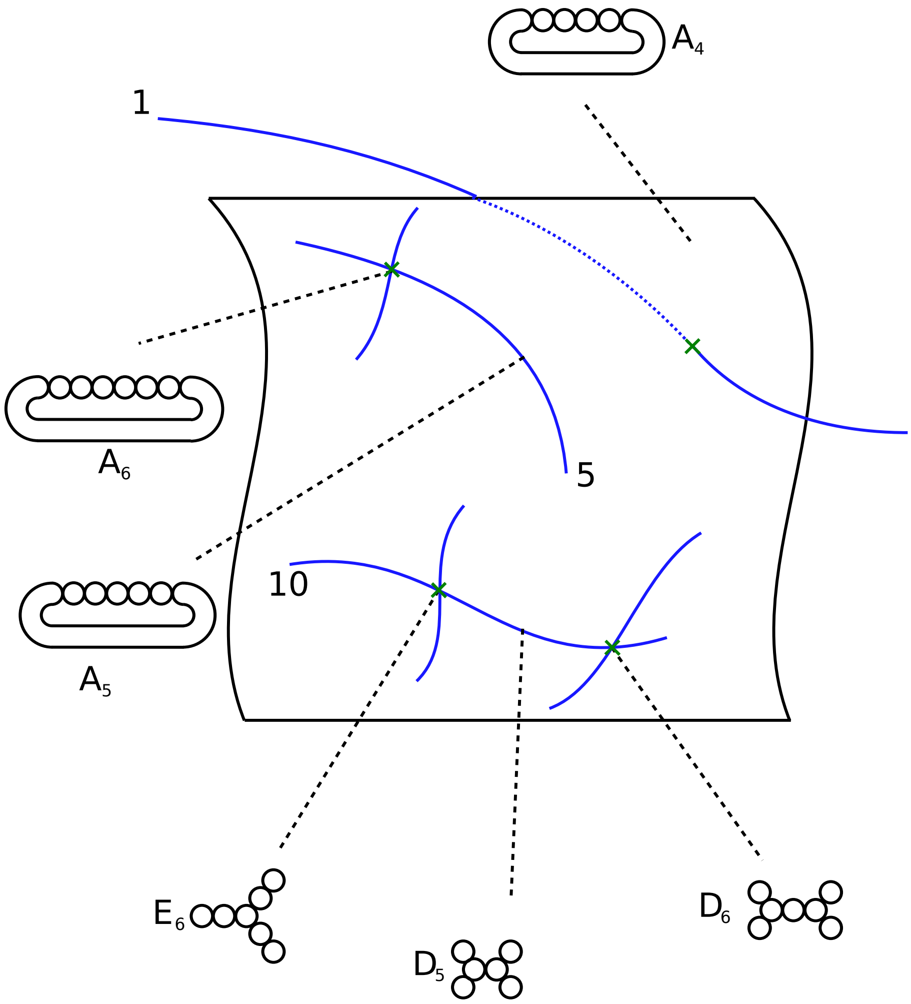

Figure 9: Schematic illustration of the gauge enhancements, discussed in this section. The framed area depicts the GUT divisor, while the blue lines represent gauge enhancements along curves. The curves associated to the **10** and **5** representations lie within the GUT divisor, whereas possible curves associated to GUT-singlets **1** do not lie in the GUT-divisor, but may intersect it.

3.4. Extra $U(1)$s

Having explained the construction of compactifications with non-abelian gauge algebras, the second ingredient needed for our analysis are additional $U(1)$ symmetries, which are vital for many phenomenological mechanisms as we indicated in section 2.1. Moreover, $U(1)$ symmetries may allow for the construction of gauge flux, thereby inducing a chiral massless spectrum. The construction of abelian symmetries was first approached in the description of a local limit in vicinity of a divisor [8, 39, 40] by considering so called *split spectral-covers*. Within this approach $U(1)$s are reasonably well understood. However, further research revealed that in F-theory compactifications $U(1)$ symmetries exhibit non-trivial behaviour away from this divisor necessitating a description away from the brane as well. This behaviour in the global geometry may for instance include breaking of the symmetry. Additionally models have been constructed for which the spectrum cannot be embedded in a single representation of E_8 (see e.g. [1]) underlining that a split spectral cover is not sufficient. In order to keep the analysis of compactifications exhibiting extra $U(1)$s as general as possible, it is desirable to consider restrictions of the fibration without specifying a particular base geometry. From the IIB perspective this resembles analyses of intersecting D7-branes without specifying their ambient space.

In F-theory compactifications, additional $U(1)$s correspond to extra rational sections different from the holomorphic zero section. This correspondence has been made precise in [41] for compactifications to six dimensions. It can be seen as follows: Given an elliptically fibred Calabi-Yau three-fold X over a base B, we consider the tensor- and vector-multiplets. The scalars of the tensor multiplets are in one-to-one correspondence to the Kähler classes of B with the exception of the overall volume form, which corresponds to a hyper-multiplet. Therefore, the number of tensor-multiplets is given by:

$$T = h^{1,1}(B) - 1 \tag{3.49}$$

By employing the duality to the IIA string — sketched in section 2.5 — one can infer that the rank of the vector multiplet is determined as:

$$\begin{aligned} r(V) &= h^{1,1}(X) - T - 2 \\ &= h^{1,1}(X) - h^{1,1}(B) - 1 \end{aligned} \tag{3.50}$$

We noted in section 3.2, that one may determine the non-abelian gauge group factors V' in the total gauge group V via the singularity type of 7-branes, according to Kodairas classification in tab. 2. This implies we can compute the rank of the abelian factors by considering the difference of the total rank $r(V)$ and the rank of the non-abelian part $r(V')$.

Geometrically, the elements of $H^{1,1}(X)$ can be categorised into three groups: First, they may correspond to divisors in the base (i.e. elements of $H^{1,1}(B)$) resulting in tensor multiplets. Secondly they may be associated to extra components of reducible fibres responsible for the non-abelian contributions. Lastly, they may arise from sections of the fibration. This means the number of $U(1)$ factors is given by rank of the group of sections. Intuitively, this corresponds to the different base embeddings in terms of these sections. A section defines a divisor class with a dual two-form w. Considering the dual M-theory picture by expanding the three-form as

$$C_3 = A \wedge w + \dots \tag{3.51}$$

results in $U(1)$ gauge potentials in the low-energy effective theory. Note that the introduction of extra rational sections leads to additional singularities in the elliptic fibration [29]. To see this, consider a Weierstraß form with the section

$$[x : y : z] = [A : B : 1]. \tag{3.52}$$

This implies in particular, that we can write in affine coordinates for some C:

$$P_{\mathrm{W}} = (y - B)(y + B) - (x - A)(x^2 + Ax + C) = 0 \tag{3.53}$$

Expansion and comparison with the standard Weierstraß form yield:

$$f = C - A^2 \tag{3.54}$$

$$g = AC - B^2 \tag{3.55}$$

This means the discriminant can be expressed in terms of the coordinates of the section:

$$\Delta = B^2(27B^2 - 54AC) + (C + 2A^2)(4C - A^2) \tag{3.56}$$

One can readily verify that extra singularities arise along:

$$\{y = 0 \cap x = A \cap B = 0 \cap C + 2A^2 = 0\} \tag{3.57}$$

This encodes loci of codimension two in the base at $\{0 = B = f + 3A^2\}$. As we will see in section 3.5, these codimension two singularities may be resolved by performing a blow-up in the ambient space. In doing so one also identifies the fibre to be of type I_2 [29]. After performing the resolution process, one can show that such sections may wrap entire fibre components along curves in the base B_3 (see e.g. [1]). A well known fact [30] from the geometry of elliptic curves states that the rational points on such a curve form the so called *Mordell-Weil group*. This extends naturally to rational sections in elliptic fibrations. The rank of the Mordell-Weil group is then equal to the number of additional $U(1)$s [41].

Having established that additional abelian gauge groups correspond to extra sections in the fibration, we now consider means of constructing such models. Given a generic elliptic curve in standard Weierstraß form, no extra sections are present. Moreover, constructing compactifications exhibiting such features is generally a less understood procedure in the Weierstraß form. Recall, however, that at the beginning of the section we noted that there are multiple ways of constructing elliptic curves apart from this form. As we will see, it is possible to use these different representations in order to construct fibrations with extra sections, which can then be mapped to Weierstraß forms of highly non-generic form, if necessary.

One way class of models exhibiting extra $U(1)$s has been dubbed *factorised Tate models*. In this formalism it is particularly simple to introduce not only additional $U(1)$ symmetries, but also an $SU(5)$ GUT divisor by choosing the leading vanishing order of the a_i in a coordinate ω perpendicular to said divisor, appropriately. Although we will not use this approach in the later sections, we nevertheless discuss it here for completeness as well as to provide an example for the resolution process in the next section. The procedure introduced in [42] works as follows: As outlined, we start with a Tate form adjusted such as to exhibit an $SU(5)$ singularity at $\{\omega = 0\}$. We are looking for sections — apart from the universal zero section $z = 0$ — along which the Tate form reduces to $y^2 = x^3$. Introducing such sections will impose further constraints on the coefficients of the Tate model. In order to bring the Tate form to a more practical form, we introduce the coordinate $t \equiv \frac{y}{x}$ and eliminate y from the Tate form:

$$P_{\mathrm{T}} = x^2(x - t^2) + x^2tza_1 + x^2z^2a_{2,1}\omega + txz^3a_{3,2}\omega^2 + xz^4a_{4,3}\omega^3 + z^6a_{6,5}\omega^5 \quad (3.58)$$

Restricting this to the form of our desired sections $0 = x - t^2$ gives:

$$P_{\mathrm{T}}|_{0=x-t^2} = t^5za_1 + t^4z^2a_{2,1}\omega + t^3z^3a_{3,2}\omega^2 + t^2z^4a_{4,3}\omega^3 + z^6a_{6,5}\omega^5 \quad (3.59)$$

Requiring extra sections is now equivalent to a factorisation as

$$P_{\mathrm{T}}|_{0=x-t^2} = -z\Pi_{i=1}^{n}Y_i, \quad i = 1, \ldots, n \quad (3.60)$$

for some holomorphic polynomials Y_i. This implies, that away from the section $0 = x - t^2$, the Tate form is given by:

$$P_{\mathrm{T}} = XQ - z\Pi_{i=1}^{n}Y_i \quad (3.61)$$

For some holomorphic polynomial Q. Given such a factorised Tate model, there is one further constraint we have to impose: Observe, namely, that the Tate form does not contain a term proportional to z^5. Imposing this extra constraint, reduces the number of extra sections to $n - 1$. Note moreover, that the splitting of the Tate form is not unique for a given number of extra $U(1)$s: As the degree of $P_{\mathrm{T}}|_{0=x-t^2}$ in the coordinate t is 5, there exist multiple splittings into polynomials X_i of degree n_i such that $\sum_i n_i = 5$. Considering in particular fibrations of Mordell-Weil rank 1 and 2, they are:

$$(n_1, n_2) = (1, 4) \text{ or } (2, 3) \quad (3.62)$$

$$(n_1, n_2, n_3) = (1, 1, 3) \text{ or } (1, 2, 2) \quad (3.63)$$

An alternative procedure to construct compactifications exhibiting extra sections has been developed in [29] for fibrations of Mordell-Weil rank one: Since such a fibration has two rational sections, there are two generically distinct points P, Q over every base point. Then $M \equiv \mathcal{O}(P + Q)$ is the line bundle vanishing at these two points. A degree two line bundle such as this has two sections which we henceforth denote by u and v. Accordingly, the line bundle $2M$ has four sections, three of which can be formed from the monomials u^2, uv and v^2 and an additional independent one w. Proceeding with this computation for the degree three line bundle $3M$ has six sections corresponding to the six monomials one can form: $u^3, u^2v, uv^2, v^3, uw, vw$. However, when considering $4M$, we can form nine monomials, whereas there are only eight independent sections. This implies that there is a relation among them. Introducing coefficients b_i and c_i, we express it as:

$$w^2 + b_0u^2w + b_1uvw + b_2v^2w = c_0u^4 + c_1u^3v + c_2u^2v^2 + c_3uv^3 + c_4v^4 \tag{3.64}$$

Which defines a hypersurface in $\mathbb{P}_{[1,1,2]}$. The elliptic fibration defined in this way features a rational section by construction. Without loss of generality we can set u to be the sections vanishing at P and Q, which means the two points are encoded in the solutions of

$$w^2 + b_2v^2w - c_4v^4 = 0 \tag{3.65}$$

By an appropriate birational transformation we can set $c_4 = b_0 = b_1 = 0$ leaving us with

$$w^2 + bv^2w = u(c_0u^3 + c_1u^2v + c_2uv^2 + c_3v^3) \tag{3.66}$$

And moreover P and Q are given by:

$$P : [u : v : w] = [0, 1, 0] \tag{3.67}$$

$$Q : [u : v : w] = [0, 1, -b] \tag{3.68}$$

as one can readily confirm. At $b = 0$ the two rational zeros of the hypersurface defined in eq. come together. At this point the discriminant vanishes and the singularity in the fibre can be identified to be of type I_2. Note that the models considered in [10] are special examples of such fibrations.

The same procedure we just described can be applied to the case of two $U(1)$s as follows [1]: Let P,Q and R denote the three distinct points and $L = \mathcal{O}(P+Q+R)$ the degree three line bundle vanishing at all three points, with sections u,v,w. Repeating above computations we end up with ten monomials for the degree nine line bundle $3L$, yielding a hypersurface equation as they cannot be independent. Once again we set u to be the section vanishing at all distinct points. This implies

$$\tilde{c}_0w^3 + \tilde{c}_1w^2v + \tilde{c}_2wv^2 + \tilde{c}_3v^3 = 0 \tag{3.69}$$

has to factor into three distinct roots:

$$(\alpha_1w + \beta_1v)(\alpha_2w + \beta_2v)(\alpha_3w + \beta_3v) = 0 \tag{3.70}$$

One can then relabel $(\alpha_1w + \beta_1) \to w$ and $(\alpha_2w + \beta_2v) \to v$ so that:

$$vw(c_1w + c_2v) = 0 \tag{3.71}$$

The two sections c_1 and c_2 can be determined from α_i and β_i, but are not needed in the following. Using these redefinitions yield a hypersurface equation:

$$P_{\mathrm{T}} = vw(c_1w + c_2v) + u(b_0v^2 + b_1vw + b_2w^2) + u^2(d_0v + d_1w + d_2u) \tag{3.72}$$

b_0	b_1	b_2	c_1	c_2	d_0	d_1	d_2
$\alpha-\beta+\bar{\mathcal{K}}$	$\bar{\mathcal{K}}$	$-\alpha+\beta+\bar{\mathcal{K}}$	$-\alpha+\bar{\mathcal{K}}$	$-\beta+\bar{\mathcal{K}}$	$\alpha+\bar{\mathcal{K}}$	$\beta+\bar{\mathcal{K}}$	$\alpha+\beta+\bar{\mathcal{K}}$

Table 3: The classes of the sections corresponding to eq. (3.72) as taken from [1]

Which vanishes for:

$$P : [u : v : w] = [0 : 0 : 1] \tag{3.73}$$

$$Q : [u : v : w] = [0 : 1 : 0] \tag{3.74}$$

$$R : [u : v : w] = [0, -c_1, c_2] \tag{3.75}$$

Once again, two of the rational points coincide if $c_1 = 0$ or $c_2 = 0$.

For future reference we also discuss how to compute the classes of the sections b_i, c_i and d_i. To do so, we can set w and v to be sections of $\alpha \otimes \mathcal{L}$ and $\beta \otimes \mathcal{L}$ for two arbitrary line bundles α, β over the base and $\mathcal{L}$ the line bundle corresponding to the hyperplane class of the $\mathbb{P}^2$-fibre [1]:

$$[w] \equiv \alpha \tag{3.76}$$

$$[v] \equiv \beta \tag{3.77}$$

The Calabi-Yau condition on the hypersurface $[P_{\mathrm{T}}] = \alpha + \beta + \bar{\mathcal{K}}$ then sets:

$$[u] = 0 \tag{3.78}$$

It is easy now to compute the transformation properties of all coefficient sections [1] as in tab. 3

In order to investigate possible embeddings of the spectrum of such a compactification featuring extra $U(1)$s, as well as to utilise their presence for phenomenological applications such as Froggatt-Nielsen mechanism, we need to compute the charges of states under these additional symmetries. Before we can proceed to the actual computation however, we need to define the so called *Shioda-map* (see [29]): Consider a fibration with Mordell-Weil rank V_A, that is a fibration possessing not only the universal zero section defined by the vanishing of the fibre coordinate z but also some rational sections defined by the vanishing of $\{s_1, \ldots, s_{V_A}\}$. In the four-fold the sections define divisors to which we can associate two-forms $Z, S_1, \ldots, S_{V_A} \in H^2(\hat{Y}_4)$ by de Rham- and Poincaré-duality. We denote possible fibral divisors introduced in the resolution process as $E_{\kappa,J}$. Their intersection structure determined by a non-abelian gauge algebra G_κ with simple roots $\alpha_{\kappa,I}$ and Cartan matrix $(C_\kappa)_{IJ} \equiv \frac{2<\alpha_I,\alpha_J>}{<\alpha_I,\alpha_I>}$. We then define the Shioda map as:

$$\sigma(s_i) = S_i - Z - S_i \cdot Z \cdot \omega_4 - \bar{\mathcal{K}} + \sum_{I,J,\kappa} (S_i \cdot \alpha_{\kappa,I})(C_\kappa^{-1})_{IJ} E_{\kappa,J} \tag{3.79}$$

For the anticanonical class of the base $\bar{\mathcal{K}}$ and any four-form ω_4 of the base. The point is now that there is a one-to-one correspondence between the abelian vector fields $\{A_1, \ldots, A_{V_A}\}$ and the six-cycles $\{\sigma(s_1), \ldots, \sigma(s_{V_A})\}$ and in particular the $\sigma(s_i)$ serve as generators for the additional $U(1)$ symmetries associated to s_i. Applied to the scenario of section 2 in [1] we have a Mordell-Weil group of rank two and no extra non-abelian singularities (they are subsequently introduced). That means the summation term vanishes and the generators are shown to be [43]:

$$w_1 := 5\sigma(s_1) = 5(S_1 - Z - \bar{\mathcal{K}}) \tag{3.80}$$
$$w_2 := 5\sigma(s_2) = 5(S_2 - Z - \bar{\mathcal{K}} - [c_1]) \tag{3.81}$$

Where the normalisation has been set for later convenience.

As discussed M2 branes wrapping $\mathbb{P}^1$ fibre components over the singular codimension two loci $C_{\mathbf{1}^{(i)}}$, induce extra massless vector multiplets charged under the additional $U(1)$s. Their charges are determined by the integral of the generators w_i over the $\mathbb{P}^1$s. Recall, that at a point p in the base, a section intersects the fibre F_p once, by definition:

$$\int_{F_p} S_i = \int_{F_p} Z = 1 \tag{3.82}$$

From the relations in eq. (3.80) – (3.81) we, therefore, take that the integral of w_i over the entire fibre vanishes ($\bar{\mathcal{K}}$ and $[c_1]$ arise from classes of the base). Moreover, if the fibre splits into two curves along some singularity — as is the case over the loci $C_{\mathbf{1}^{(i)}}$ — the vector multiplets originating from these curves are oppositely charged and can therefore be understood as charge conjugate to each other.

In order to compute the charges now we need to investigate the intersections of the divisors Z, S_1, S_2 with the fibre components along the singular loci. We do so in terms of a particular example taken from above reference: As can be derived from above considerations and also verified directly within the discriminant, the elliptic fibration is singular along the locus $C_{\mathbf{1}^{(1)}} = \{b_0\} \cap \{c_2\}$, along which the Tate model emits an s_1 factor as:

$$P_{\mathrm{T}}|_{C_{\mathbf{1}^{(1)}}} = s_1 p_1 \tag{3.83}$$

For some polynomial p_1. This factorisation underlines that the fibre splits into two $\mathbb{P}^1$s given by:

$$\mathbb{P}^1_{A_1} = \{s_1\} \cap C_{\mathbf{1}^{(1)}} \cap D \subset \hat{X}_5 \tag{3.84}$$
$$\mathbb{P}^1_{B_1} = \{p_1\} \cap C_{\mathbf{1}^{(1)}} \cap D \subset \hat{X}_5, \tag{3.85}$$

where D denotes a divisor on the base space intersecting $C_{\mathbf{1}^{(1)}}$ in a point to isolate the fibre. From these equations we can confirm that along the singularity S_1 wraps the entire $\mathbb{P}_{A_1}$. Secondly we note that zs_1 is in the SR-ideal and therefore the two coordinates cannot vanish simultaneously. This implies in particular that Z does not intersect this fibre component. As the intersection of this section with the total fibre has to satisfy $Z \cdot (\mathbb{P}^1_{A_1} + \mathbb{P}^1_{B_1}) = 1$, we deduce that Z intersects $\mathbb{P}^1_{B_1}$) in a point. Lastly

S_2 intersects $\mathbb{P}^1_{A_1}$ in a point determined by $\{u\} \cap C_{\mathbf{1}^{(1)}}$. We summarise the arguments we just gave as:

$$\begin{array}{ll} \mathbb{P}_{A_1} \cdot Z = 0 & \mathbb{P}_{B_1} \cdot Z = 1 \\ \mathbb{P}_{A_1} \cdot S_1 = 1 & \mathbb{P}_{B_1} \cdot S_1 = 0 \\ \mathbb{P}_{A_1} \cdot S_2 = 1 & \mathbb{P}_{B_1} \cdot S_2 = 1 \end{array} \tag{3.86}$$

These considerations now allow us to compute the charges of the vector multiplets arising along the singular locus:

$$\int_{\mathbb{P}_{B_1}} w_1 = 5 \int_{\mathbb{P}_{B_1}} (S_1 - Z - \bar{\mathcal{K}}) = -5 \tag{3.87}$$

$$\int_{\mathbb{P}_{A_1}} w_1 = 5 \tag{3.88}$$

$$\int_{\mathbb{P}_{A_1}} w_2 = 5 \int_{\mathbb{P}_{A_1}} (S_2 - Z - \bar{\mathcal{K}} - [c_1]) = 5 \tag{3.89}$$

$$\int_{\mathbb{P}_{B_1}} w_2 = -5 \tag{3.90}$$

We therefore encounter states $\mathbf{1}_{(5,-5)} + c.c.$ along the singularity.

3.5. Resolution of singularities

In section 2.5 we discussed that F-theory should be understood via its duality to M-theory in the limit of vanishing fibre, but M-theory is only well-defined on smooth geometries, while we discussed how singularities in the elliptic fibre give rise to gauge enhancements in F-theory. The way to understand this apparent discrepancy is in terms of a resolution process: Mathematically, a resolution provides us with a smooth description of the hypersurface $\{P_{\mathrm{W}} = 0\}$ in some ambient space. In the limit of vanishing fibre this smooth geometry will converge to the singular F-theory geometry. This means we can dualise the resolved Calabi-Yau fourfold $\bar{Y}_4$ to M-theory were we can understand the appearance of additional states along loci of gauge enhancements in terms of three-from C_3 and M_2 branes. The masses of these states are proportional to the volume of the corresponding resolution divisors. This means they are massless in the limit of vanishing fibre.

Paraphrasing the discussion in [44] and also [1] we will first demonstrate the resolution procedure for an elliptic fibration in Tate form with one $U(1)$ but without non-abelian symmetries. Secondly we will apply the same procedure to a fibration within a different projective space and exhibiting two extra $U(1)$s. Lastly we will extend this analysis to the case of a Tate model with an $SU(5)$ singularity as well an additional $U(1)$. Moreover we will investigate the intersection structure of the resolution divisors. In doing so we will shed more light on the interrelation between gauge algebras and singularities in the elliptic fibre.

	x	y	z	s
z	2	3	1	$\cdot$
s	-1	-1	$\cdot$	1

Table 4: Rescaling behaviour of the blown-up space

To make the procedure clear, we start with the most simple example: We consider a generic Tate model, but set $a_6 \equiv 0$ which introduces an extra $U(1)$ [10]:

$$P_{\mathrm{T}} = y^2 + a_1 xyz + a_3 yz^3 = x^3 + a_2 x^2 z^2 + a_4 xz^4 \tag{3.91}$$

The discriminant can then be obtained from eq. (3.31). One may verify that we encounter a singularity at $\{a_3 = a_4 = 0 = x = y\}$. According to the Kodaira classification (tab. 2) it is of type I_2. In order to resolve this singularity we perform a so called *blow-up* in the ambient space by introducing an additional homogeneous coordinate s:

$$x = \tilde{x}s \tag{3.92}$$

$$y = \tilde{y}s \tag{3.93}$$

Furthermore we require that the original coordinates remain invariant under rescaling of s:

$$(x, y) = (\tilde{x}s, \tilde{y}s) \sim (\tilde{x}'\lambda s, \tilde{y}'\lambda s) \tag{3.94}$$

$$\Rightarrow (\tilde{x}, \tilde{y}) \sim (\lambda^{-1}\tilde{x}, \lambda^{-1}\tilde{x}) \tag{3.95}$$

Dropping the tildes, we can therefore summarise the degrees of x, y, z, s under the new rescaling as well as the rescaling of the $\mathbb{P}_{[2,3,1]}$ as in tab. 4. Introducing such a blow-up changes the so called *Stanley-Reisner ideal* which indicates the coordinates not allowed to vanish simultaneously. Before applying the resolution, it contained only the case of all x, y, z vanishing at the same time, $\{xyz\}$, while one can confirm that is adjusted to $\{xy, zs\}$ after the procedure. The resulting ambient space is denoted as $\mathrm{Bl}_1\mathbb{P}_{[2,3,1]}$. Note that the resolved space now has two apparent sections. Namely the universal zero section $\{z = 0\}$, but also an additional section given by $S : \{s = 0\} \cap \{P_{\mathrm{T}} = 0\}$. It is in particular interesting to investigate the fibre structure over the intersection of the Tate form P_{T} with this new section S, it is given by:

$$P_{\mathrm{T}} \cap S = \{a_3 yz^3 = a_4 xz^4\} \tag{3.96}$$

Since zs is in the Stanley-Reisner ideal, we can set $z \equiv 1$. For generic a_3, a_4 this provides an embedding of the base B_3 within the complex five dimensional ambient space. However at the curve C_{34} both sides vanish such that the relation between x and y is missing and correspondingly this defines a $\mathbb{P}^1$ space parametrised by $[x : y]$. Intuitively speaking this behaviour can be understood as follows: While generically the Tate form and the divisor S intersect in a point, they do not on the singularity locus C_{34}. This means essentially we pasted in a $\mathbb{P}^1$ fibre at the singularity. More formally, the intersection defined by

$$\{s=0\} \cap \mathrm{Bl}_1\mathbb{P}_{[2,3,1]} \tag{3.97}$$

is a linear subspace of the unresolved ambient space $\mathbb{P}_{2,3,1}$, which is isomorphic to $\mathbb{P}^1$. Lastly, note that the resolution enlarges $h^{1,1}$ by the class of S. Which we know corresponds to extra massless matter.

For future reference we now sketch the application of the same procedures to the $\mathbb{P}^2$ fibrations exhibiting two additional $U(1)$s as considered in [1]. They are given by the hypersurface equation:

$$P_\mathrm{T} = vw(c_1w + c_2v) + u(b_0v^2 + b_1vw + b_2w^2) + u^2(d_0v + d_1w + d_2u) \tag{3.98}$$

Together with the Stanley-Reisner ideal $\{uvw\}$. One can confirm that this fibration is singular along the two loci:

$$\{u = v = c_1 = b_2 = 0\} \tag{3.99}$$
$$\{u = w = c_2 = b_0 = 0\} \tag{3.100}$$

Hence we will subsequently turn both singularities off by applying a blow-up. To do so, we introduce a new homogeneous coordinate s_0:

$$u \to us_0 \tag{3.101}$$
$$v \to vs_0 \tag{3.102}$$

In order to leave the original coordinates invariant under rescaling of s_1, we additionally introduce the scaling relation $(u, v, w, s_0) \to (\lambda_0^{-1}u, \lambda_0^{-1}v, w, \lambda_0 s_0)$. To resolve the second singularity, we introduce another coordinate by:

$$u \to us_1 \tag{3.103}$$
$$w \to ws_1 \tag{3.104}$$

Supplemented by the scaling relation $(u, v, w, s_0, s_1) \to (\lambda_1^{-1}u, v, \lambda_1^{-1}w, s_0, \lambda_1 s_1)$. The resolution procedure can therefore be summarised as $(u, v, w) \to (us_0, vs_0, w) \to (us_0s_1, vs_0, ws_1)$. This leaves us with a hypersurface equation given by:

$$P_\mathrm{T} = vw(c_2s_0v + c_1s_1w) + u(b_0s_0^2v^2 + b_1s_0s_1vw + b_2s_1^2w^2) + s_0s_1u^2(d_2s_0s_1u + d_0s_0v + d_1s_1w) \tag{3.105}$$

One can check that the resolution procedure changes the Stanley-Reisner ideal to:

$$\{uv, uw, vs_1, ws_0, s_0s_1\} \tag{3.106}$$

Up to this point we only considered the resolution process for elliptic fibrations with extra $U(1)$ factors. After having outlined the general idea of the resolution process, we can now consider a Tate model containing a divisor yielding a non-abelian gauge

singularity	Δ	group
$\{a_1\}$	7	$SO(10)$
$\{a_{3,2}\}$	6	$SU(6)$
$\{a_1 a_{4,3} - a_{2,1} a_{3,2}\}$	6	$SU(6)$
$\{a_1 \cap a_{2,1}\}$	8	E_8
$\{a_1 \cap a_{3,2}\}$	8	$SO(12)$
$\{a_{3,2} \cap a_{4,3}\}$	7	$SU(7)$

Table 5: Codimension two and three singularities of $U(1)$ restricted Tate model with $SU(5)$ singularity [44]. The two codim. 3 enhancements of vanishing order 7 in Δ can be categorised when considering the coefficients $a_{i,j}$.

group. As an example we consider the case of a divisor $\{\omega = 0\}$ admitting an $SU(5)$ singularity. Additionally we restrict it further as in above case by setting $a_6 = 0$ in order to possess an additional $U(1)$ symmetry — that is we consider an elliptic fibration with Mordell-Weil rank two:

$$P_{\mathrm{T}} = -x^3 + y^2 - a_1 xyz - a_{2,1} x^2 z^2 \omega - a_{3,2} y z^3 \omega^2 - a_{4,3} x z^4 \omega^3 \tag{3.107}$$

The discriminant can be obtained from eq. (3.35) by setting $a_{6,5} = 0$. As one can readily verify this Tate model encounters the singularities provided in tab. 5. Given this data we now proceed to first resolve the $SU(5)$ singularity by subsequently introducing new blow-up coordinates. From the technical perspective we go about this by first considering above Tate form $P_{\mathrm{T}}(x_0, y_0, z, \omega)$. We then repeat the following three steps, until we arrive at a smooth manifold: Introduce ambient space coordinates e_i by

$$(x_k, y_l, \omega) \to (x_k e_i, y_l e_i, \omega e_i) \tag{3.108}$$

and consider the leading order of P_{T} in e_i, if it contains an overall factor of x_k or y_l, we define $x_k \to x_{k+1}$ or $y_l \to y_{l+1}$, respectively. Otherwise the procedure terminates and all singularities associated to the $SU(5)$ divisor are resolved. Lastly, we resolve the singularity arising in the $U(1)$ restriction as we did above by $(x, y) \to (xs, ys)$. Performing above procedure step by step results in:

$$\begin{array}{rr} & (x_0, y_0, \omega) \to (x_0 e_1, y_0 e_1, \omega e_1) \\ y_0 \to y_1 \omega & (x_0, y_1, \omega) \to (x_0 e_2, y_1 e_2, \omega e_2) \\ x_0 \to x_1 \omega & (x_1, y_1, \omega) \to (x_1 e_3, y_1 e_3, \omega e_3) \\ y_1 \to y_2 \omega & (x_1, y_2, \omega) \to (x_1 e_3, y_2 e_3, \omega e_3) \\ & (x_1, y_2) \to (x_1 s, y_2 s) \end{array} \tag{3.109}$$

Which can be summarised in (we drop the subscripts from now on):

$$(x, y, \omega) \to (x s e_1 e_2 e_3^2 e_4^2, y s e_1 e_2^2 e_3^2 e_4^3, \omega e_1 e_2 e_3 e_4) \tag{3.110}$$

F-theory		het. $E_8 \times E_8$ string
$T^2 \hookrightarrow$	$Y_2 = K3$	
	$\downarrow$	T^2
	$B_1 = \mathbb{P}^1$	

Table 6: Fibration structure for heterotic/F-theory duality in eight dimensions.

In particular we arrive at a Tate form given by:

$$P_{\mathrm{T}} = e_1 e_3^2 e_4 s^2 x^3 - e_2 e_4 s y^2 + a_1 sxyz + a_{2,1} e_1 e_3 s x^2 z^2 \omega \qquad (3.111)$$

$$+ a_{3,2} e_1 e_2 y z^3 \omega^2 + a_{4,3} e_1^2 e_2 e_3 x z^4 \omega^3 \qquad (3.112)$$

Where the e_i correspond to the additional $\mathbb{P}^1$s appearing over the singular locus. As in the two simpler resolution procedures above, the introduction of every divisor changes the SR-ideal and gives an additional scaling relation acting on the coordinates. Note that the resolution is not unique but different choices of Stanley-Reisner ideals are possible, which nevertheless give the same scaling relations.

4. Heterotic/F-theory duality

Apart from the origins of F-theory as a non-pertubative description of the IIB superstring and its resulting duality to M-theory, also the duality to the heterotic string has been of great interest in the past. This is to a large part due to the role of the heterotic groups $E_8 \times E_8$ and $SO(32)$ in GUT-model building. The advantage of dualising such heterotic compactifications to F-theory lies in the geometrisation of gauge groups along singular loci in this context. As we will see in the following, this duality implies in particular that the vector bundle data from the heterotic geometry is mapped to the moduli of the elliptically fibred Calabi-Yau four-fold in F-theory. For brevity we will limit the following discussion to the heterotic string with structure group in $E_8 \times E_8$ introduced in [45] rather than the $SO(32)$ string. Note also, that most of this discussion follows [46], to which one should also refer for recent insights into the relation of geometric constraints in the two theories.

4.1. Duality in eight dimensions

Duality between heterotic string theory and F-theory is deduced [47] from the conjectured duality between the former and M-theory [20]. In its most simple form it states that the heterotic string compactified on a torus is dual to F-theory on an elliptically fibred $K3$ surface (see tab. 6).

In the so called *stable degeneration limit* the moduli spaces of the two compactifications can be matched exactly. This limit translates into the splitting up of the $K3$

F-theory	het. $E_8 \times E_8$ string
$\begin{array}{ccc} K3 & \hookrightarrow & Y_{n+2} \\ & & \downarrow \\ & & B_n \end{array}$	$\begin{array}{ccc} T^2 & \hookrightarrow & X_3 \\ & & \downarrow \\ & & B_n \end{array}$
$\begin{array}{ccc} T^2 & \hookrightarrow & Y_{n+2} \\ & & \downarrow \\ \mathbb{P}^1 & \hookrightarrow & B_{n+1} \\ & & \downarrow \\ & & B_n \end{array}$	

Table 7: Fibration structure for heterotic/F-theory duality in generic dimensions.

surface into a fibre product of two dP_9 surfaces along an elliptic curve E [48, 49, 41]. In particular the heterotic string coupling relates to the F-theory geometry as

$$e^{2\phi} = \text{vol}(B_1), \tag{4.1}$$

where we denoted the $\mathbb{P}^1$-base by B_1. Moreover, gauge groups arising in heterotic string theory as the commutants $G_1 \times G_2$ of the vector bundle structure groups $H_1 \times H_2$ within $E_8 \times E_8$ — can be matched to the ADE degenerations of the elliptic fibration Y_2.

Of course, duality in eight dimensions is limited in the sense that it only allows for codimension one gauge enhancements in the F-theory base B_1. To generalise said duality, one therefore considers fibrewise extension of above geometries, i.e. compactifications of the heterotic string on an elliptically fibred $n+1$ fold X_{n+1} over some base B_n as dual to F-theory on a K3 fibration Y_{n+2} over the same base. To define a suitable F-theory geometry, Y_{n+2} also has to admit a second fibration structure in terms of an elliptic fibration over some $n+1$ fold B_{n+1}, which is itself a $\mathbb{P}^1$-fibration over the common base B_n (see tab. 7). Reversing this argument implies that only F-theory compactifications $T^2 \hookrightarrow B_{n+1}$ allowing for such a $\mathbb{P}^1$-fibration structure have a heterotic dual in terms of above geometries.

Before we consider duality in six and four dimensions, note that singularities in the heterotic compactification space may yield additional gauge group factors [5] - something we will not consider in the following discussions.

4.2. Duality in six dimensions

A first generalisation of the duality between the two theories are compactifications to six dimensions which allow for further gauge enhancements along loci of codimension two in the base. Following above introduction to the duality, the heterotic compactification is given by an elliptically fibred Calabi-Yau two-fold. The only such spaces admitting such a structure in two dimensions are $K3$ surfaces, which means the common base is given by $B_1 = \mathbb{P}^1$. The dual F-theory geometry is then given as an elliptic fibration over a two-fold B_2, which is itself a $\mathbb{P}^1$ fibration over a $\mathbb{P}^1$ base. That is to say, the base is a so called *Hirzebruch surface* $\mathbb{F}_n$.

As we will often encounter these complex surfaces in the following sections, we briefly digress into their properties: As should be clear from the definition, we can construct such a manifold by considering some rank two vector bundle V over $\mathbb{P}^1$ and take its projectivisation, i.e. the fibrewise extension of a vector space projectivisation. In particular we may take the vector bundle as $V = \mathcal{O} \oplus \mathcal{O}(n)$, $n \in \mathbb{N}$ and define $\mathbb{F}_n := \mathbb{P}(V) = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$ (see e.g. [50]). One can show that indeed all Hirzebruch surfaces may be constructed in this way and are therefore uniquely characterised by the single integer n. We denote the fibre class of $\mathbb{F}_n$ by $\mathcal{E}$ and the classes corresponding to the two sections induced by the sub-bundles $\mathcal{O}$ and $\mathcal{O}(n)$ by $\mathcal{S}_\infty$ and $\mathcal{S}_0$, respectively. Then, as argued in [50], their intersections are given by:

$$\mathcal{E} \cdot \mathcal{E} = 0 \qquad \mathcal{S}_\infty \cdot \mathcal{S}_\infty = -n \qquad \mathcal{S}_0 \cdot \mathcal{S}_0 = n \tag{4.2}$$

$$\mathcal{E} \cdot \mathcal{S}_\infty = 1 \qquad \mathcal{E} \cdot \mathcal{S}_0 = 1 \qquad \mathcal{S}_\infty \cdot \mathcal{S}_0 = 0 \tag{4.3}$$

And furthermore:

$$\mathcal{S}_0 = \mathcal{S}_\infty + n\mathcal{E} \tag{4.4}$$

such that $\mathcal{S}_\infty$ and $\mathcal{E}$ provide a basis of $\mathrm{H}_2(\mathbb{F}_n, \mathbb{Z})$ and are moreover both effective. [2] Lastly, [50] compute the first and second Chern classes as:

$$c_1(\mathbb{F}_n) = 2\mathcal{S}_\infty + (n+2)\mathcal{E} \tag{4.5}$$

$$c_2(\mathbb{F}_n) = 4 \tag{4.6}$$

Having outlined these basic facts about Hirzebruch surfaces, we return to the duality. Where we note that apart from the restriction of the base space by the heterotic Calabi-Yau condition, there are more constraints imposed by the $\mathcal{N} = 1$ SUSY condition in six dimensions: On the F-theory side this condition translates to the requirement that the resolved compactification space $\hat{Y}_3$ is Calabi-Yau, as well. On the heterotic side it additionally constrains the gauge bundles in $V_1 \oplus V_2$ to satisfy the Hermitian Yang-Mills equations [51, 52]. In addition to this, their first Chern class has to satisfy:

$$c_1(V_i) = 0 \bmod 2 \tag{4.7}$$

Given a pair fo dual F-theory and heterotic compactifications, it can be characterised by a single integer $n \in \mathbb{N}$ encoding distinct properties on each side of the duality. While we encountered it already on the F-theory side as determining the base geometry $\mathbb{F}_n$,

[2] To define the notion of effectiveness, a few definitions are in order (see e.g. [50]): A curve is called *irreducible*, if it is not the union of two curves. Correspondingly a class in H_2 is called irreducible, if it is the class of an irreducible curve. Furthermore a class is called *algebraic*, if it is a linear combination of irreducible classes with integer cooeficients. If all coefficients are non-negative, it is called *effective*.

it restricts the second Chern class of the heterotic gauge as $c_2(V_{1,2}) = 12 \pm n$ on the heterotic side [46]. The integer n is not entirely unconstrained, however: As has been shown in [49], the heterotic anomaly cancellation condition as well as Hermitian Yang-Mills equations restrict to $n \leq 12$ from the heterotic point of view. Correspondingly, for $n > 12$ also the F-theory geometry does not yield a physical setup: This can be derived from the existence of an effective divisor in $\mathbb{F}_n$ with self-intersection $-n$ (in our notation $\mathcal{S}_\infty$). A divisor with a self-intersection number smaller than -12 renders the geometry too singular as to allow for a resolution to a smooth Calabi-Yau space [46]. The realisation of this very same constraint as a consequence of two very distinct arguments in heterotic string theory and F-theory is an example for the non-triviality of this duality, already in six dimensions.

We will now have a closer look at how the moduli are mapped between the two theories: First of all, the heterotic string coupling is determined by the fibre- and base-volume of $\mathbb{F}_n$ as [5]:

$$e^{2\phi} = \frac{\mathrm{vol}_f}{\mathrm{vol}_b} \tag{4.8}$$

Consider, moreover, the elliptic fibration on the F-theory side given in Weierstraß from:

$$0 = P_{\mathrm{W}} = y^2 - x^3 - f(z_1, z_2)x - g(z_1, z_2) \tag{4.9}$$

For z_1, z_2 coordinates on $\mathbb{F}_n$. Since f and g are sections of $\mathrm{H}^2(B_2, \bar{\mathcal{K}}^4)$ and $\mathrm{H}^2(B_2, \bar{\mathcal{K}}^6)$, respectively, we can expand them locally as:

$$f(z_1, z_2) = \sum_{i=0}^{I} f_{8+n(4-i)}(z_2) \qquad I \leq 8 \qquad 8 + n(4 - I) \geq 0 \tag{4.10}$$

$$g(z_1, z_2) = \sum_{j=0}^{J} g_{12+n(6-j)}(z_2) \qquad J \leq 12 \qquad 12 + n(6 - I) \geq 0 \tag{4.11}$$

(4.12)

The subscript denotes the degree in z_2. So, in particular one may compute the degrees of freedom encoded in the two functions f and g for a given n.

As we have stressed many times now, in F-theory gauge groups arise the geometry of the elliptic fibration, whereas they are encoded in the vector bundle $V_1 \oplus V_2$ on the heterotic side. From the point of view we took above, therefore, some of the f_i and g_i should map to the moduli of the vector bundle. Indeed this correspondence is intuitively easy to understand: Without loss of generality pick z_1 to be the coordinate normal to some gauge divisor. From tab. 1 we take that all possible gauge enhancements are encoded in $f_0, \ldots f_3$ and $g_0, \ldots g_5$. This points at the fact that these functions are mapped to the moduli of the first vector bundle V_1. The correspondence we just hinted at, has been made precise in [48, 49, 41] and additionally assures that f_4 and g_6 parametrise the elliptic fibration in the heterotic compactification, while $f_5, \ldots f_8$ and $g_7, \ldots g_{12}$ correspond to the V_2-moduli.

As we have outlined before, the vector bundles V_i have structure group $H_i \subseteq E_8$. The commutant G_i of H_i in E_8 then corresponds to the resulting gauge group. In F-theory this is encoded in a singularity along $\{z_1 = 0\}$ in the base B_1 giving rise to a gauge group G_1 in terms of the ADE classification. Physically speaking, therefore, the heterotic gauge groups are associated to 7-branes wrapping a divisor in B_1. One can employ the same analysis for G_2 at $z_1 = \infty$. We can make this more precise in the stable degeneration limit $Y_3 \to Y_1 \cup_{\mathrm{K3}} Y_2$ where Y_i are two dP_9 fibred three-folds. The infinitesimal deformation spaces can then be shown to satisfy [49, 41, 33, 53, 54]:

$$\mathrm{Def}(Y_i) \simeq \mathrm{Def}(V_i) \simeq \mathrm{Def}(\mathrm{K3}) \tag{4.13}$$

More specifically, one can show that the two sets of f_i and g_i parametrise two so-called *spectral covers* describing the vector bundles. As the understanding of spectral covers is paramount ot the construction of vector bundles on elliptic curves, we briefly outline their properties.

4.3. Spectral cover construction

The aim of the spectral cover construction on an elliptically fibred manifold is to construct semi-stable $U(n)$ and $SU(n)$ subbundles of E_8 over the fibration. In this context *semi-stable* means these bundles have a reducible Yang-Mills field strength or, put differently, the holonomy commutes with more than just the center of the group. The following discussion proceeds largely along the lines of [50]. And is intended only as a reminder rather than a complete introduction.

Consider first a single elliptic curve E. A theorem by Looijenga assures in particular that the moduli space of a semi-stable $SU(n)$-bundle on E is given by the complex projective space $\mathbb{P}^{n-1}$. In order to construct such a bundle, one starts with a $U(n)$ bundle V over the elliptic curve E. For the bundle to be well-defined we additionally need to specify how it twists (mathematically speaking this amounts to specifying the holonomy). It can be argued that the bundle can be diagonalised as:

$$V = \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_n \tag{4.14}$$

Note that the bundle is only fixed up to interchange of the line bundles $\mathcal{N}_i$, the permutations of which form the Weyl-group. The additional constraint on the determinant in restricting to $SU(n)$ is implemented as:

$$\mathcal{N}_1 \otimes \cdots \otimes \mathcal{N}_n = \mathcal{O} \tag{4.15}$$

It can be shown that semi-stability implies for the $\mathcal{N}_i$ to have the same degree which is henceforth set to zero. On an elliptic curve there is a unique point $Q_i \in E$ such that $\mathcal{N}_i$ has a section which is allowed to have a pole at Q_i and which is zero at the origin p:

$$\mathcal{N}_i = \mathcal{O}(Q_i) \otimes \mathcal{O}(p)^{-1} \tag{4.16}$$

Using above equations, the points satisfy (for an $SU(n)$ bundle):

$$\sum_{i=1}^{n}(Q_i - p) = 0 \tag{4.17}$$

In affine coordinates ($z = 1$) of the Weierstraß equation this translates to [50]:

$$s = a_0 + a_2 x + a_3 y + a_4 x^2 y + \cdots + \begin{cases} a_n x^{\frac{n}{2}} & n \text{ even} \\ a_n x^{\frac{n-3}{2}} y & n \text{ odd} \end{cases} \tag{4.18}$$

The points Q_i are encoded as the roots $s \overset{!}{=} 0$. Note that they are only determined up to an overall scaling factor, underlining that the moduli space of the Q_i is indeed $\mathbb{P}^{n-1}$ with homogeneous coordinates a_i.

The next step is now to generalise this procedure from a single elliptic curve to an elliptic fibration. The basic idea in doing so is to specify how the points Q_i vary over the base. This information is encoded in the so-called *spectral cover bundle*:

$$\pi_C : C \to B \tag{4.19}$$

Since for every $b \in B$ there lie n points in the fibration, C is called and *n-fold cover* of the base B. For a general fibre E_b we can proceed as for an elliptic curve E:

$$V|_{E_b} = \mathcal{N}_{1b} \oplus \cdots \oplus \mathcal{N}_{nb} \tag{4.20}$$

Since there is a section $\sigma(b) \in E_b$ for $b \in B$ we can also give $V|_B$ embedded via σ. These two restrictions determine a unique line bundle $\mathcal{N}$ on C such that $\pi_{c*}\mathcal{N} = V|_B$ is a vector bundle on B with fibre give by $\bigoplus_i \mathcal{N}|_{Q_i}$. Since C is an n-fold cover over the base B, its cohomology class in $H^2(X, \mathbb{Z})$ is given by $[C] = n + \sigma + \eta$ where η is a class in $H^2(B, \mathbb{Z})$ or equivalently, the line bundle on X determined by C is given by

$$\mathcal{O}_X(C) = \mathcal{O}_X(n\sigma) \otimes \mathcal{M}, \tag{4.21}$$

with $\mathcal{M}$ is a line bundle on X satisfying $\eta = c_1(\mathcal{M})$.

Before we proceed and give the entire bundle V, we need to define the so called *Poincaré bundle* $\mathcal{P}$ which is determined by two constraints. First, for $x \in E'_b$: $\mathcal{P}|_{E_b \times x}$ is the line bundle on E_b determined by x and second, $\mathcal{P}_{\sigma \times_B X'}$ is the trivial bundle. (X' denotes the dual fibration to X. Since X has a section $X \cong X'$. In the above expression $X \times_B X'$ is the fibre product with base B and fibre $E_b \times E'_b$ at $b \in B$.) This means $\mathcal{P}$ is the bundle whose sections are meromorphic functions on $X \times_B X'$ with first order poles on $\mathcal{D}$ (the divisor representing the graph of the isomorphism $X \to X'$) and which vanish on $\sigma \times_B X'$ and $X \times_B \sigma'$:

$$\mathcal{P} = \mathcal{O}_{X \times_B X'}\left(\mathcal{D} - \sigma \times_B X' - X \times_B \sigma'\right) \otimes K_B \tag{4.22}$$

Taking all these ingredients one can give the $U(n)$-bundle as:

$$V = \pi_{1*}\left(\pi_2^* \mathcal{N} \otimes \mathcal{P}\right) \tag{4.23}$$

$$\pi_1 : X \times_B C \to X \tag{4.24}$$

$$\pi_2 : X \times_B C \to C \tag{4.25}$$

Constraining this further to $SU(n)$ amounts to setting $\mathcal{M}|_{E_b}$ to the trivial bundle and requiring $V|_B$ to be an $SU(n)$ bundle itself.

4.4. Duality in four dimensions

When generalising the duality further to four dimensions, once again the common base is strongly restricted by the heterotic Calabi-Yau condition. This is due to the fact that only very few classes of Calabi-Yau three-folds allow for an elliptic fibration. One can show that the only possible such bases B_2 of such compactifications are (see e.g. [5]) the Enriques surface $\mathrm{K3}/\mathbb{Z}_2$, a Hirzebruch surface $\mathbb{F}_n$, a complex projective space $\mathbb{P}^2$ or blow-ups of these tree. The dual F-theory picture has been investigated in [48]: As the base B_3 of the elliptic fibration is a $\mathbb{P}^1$ fibration over B_2, it can be constructed as $B_3 = \mathbb{P}(\mathcal{O} \oplus \mathcal{L})$ for some line bundle $\mathcal{L}$ over B_2 as well as the canonical line bundle over the same base and their first Chern classes:

$$R := c_1(\mathcal{O}) \qquad\qquad T := c_1(\mathcal{L}) \tag{4.26}$$

There are then two sections Σ_- and $\Sigma_+ = \Sigma_- + T$ of B_3 with intersection $\Sigma_- \cdot \Sigma_+ = 0$. In cohomology this is expressed as $R(R+T) = 0$. On the heterotic side the curvature of the two vector bundles in $V_1 \oplus V_2$ splits as [46]

$$\frac{1}{30}\mathrm{Tr}F_i^2 = \eta_i \wedge \omega_0 + \zeta_i, \tag{4.27}$$

where ω_0 is the dual of the section an η_i, ζ_i are the pullbacks of some two- and four-form, respectively. It can be shown [48] that duality is possible, only if:

$$\eta_{1,2} = 6c_1(B_2) \pm T \tag{4.28}$$

Moreover, the Bianchi identity yields $\eta_1 + \eta_2 = 12c_1(B_2)$.

Note, that more sections in the heterotic fibration might allow for more general F-theory duals. Lastly, a smooth heterotic dual may only exist if there are only two non-abelian gauge group factors in the F-theory geometry, associated to the divisors Σ_- and Σ_+. The mapping of the moduli between these singularities and the heterotic vector bundles can be generalised from the procedure in six dimensions.

4.5. Identification of moduli

In section 3, we noted that some elliptic fibrations can be brought to Tate form not only locally around some divisor, but also globally. One might wonder if the moduli of such a model can be mapped to heterotic string theory directly and also how this approach relates to the one via the Weierstraß model. The following discussion proceeds closely along the lines of [55]. We start with a Tate form P_T and a gauge group G localised at the divisor $\{\omega = 0\}$, i.e.:

$$P_\mathrm{T} = y^2 - x^3 - b_{1,j_1}\omega^{j_1}xy - b_{2,j_2}\omega^{j_2}x^2 - b_{3,j_3}\omega^{j_3}y - b_{4,j_4}\omega^{j_4}x - b_{6,j_6}\omega^{j_6} \tag{4.29}$$

Note that the b_{i,j_i} can still depend on ω, but may not contain an overall factor of ω. This is in contrast to the approach relating f and g to the heterotic moduli. We therefore expand:

$$b_{i,j_i} = a_{i,j_i} + a_{i,j_i+1}\omega + a_{i,j_i+2}\omega^2 + \dots \tag{4.30}$$

Consider for a moment only the information encoded in $b_{i,j_i}|_{\omega=0} = a_{i,j_i}$. As has been argued in [33], on Hirzebruch surfaces $\mathbb{F}_n$ these five functions have exactly the right number of degrees of freedom $(36+5n)$ to match the bundle moduli of the heterotic $E_8 \times E_8$ string. On the other hand one might consider which higher order terms play a role for gauge enhancements by computing the discriminant [55]:

$$\begin{aligned}\Delta = -\omega^5 \Big(& P_{10}^4 P_5 + \omega P_{10}^2 \big(8a_{2,1}P_5 + P_{10}\left(R_0 + R_1\right)\big) \\ & +2\omega^2 \left(-8a_{3,2}^2 a_{2,1}^3 + \mathcal{O}\left(P_{10}\right)\right) + \mathcal{O}\left(\omega^3\right) \Big)\end{aligned} \tag{4.31}$$

For:

$$P_{10} = a_{1,0} \tag{4.32}$$
$$P_5 = a_{3,2}^2 a_{2,1} - a_{4,3}a_{3,2}a_{1,0} + a_{6,5}a_{1,0}^2 \tag{4.33}$$
$$R_0 = -a_{3,2}^3 - a_{4,3}^2 a_{1,0} + 4a_{6,5}a_{2,1}a_{1,0} \tag{4.34}$$
$$\begin{aligned}R_1 = {} & 4a_{1,1}a_{2,1}a_{3,2}^2 + a_{1,0}a_{2,2}a_{3,2}^2 + 2a_{1,0}a_{2,1}a_{3,2}a_{3,3} - 5a_{1,0}a_{1,1}a_{3,2}a_{4,3} \\ & - a_{1,0}^2 a_{3,3}a_{4,3} - a_{1,0}^2 a_{3,2}a_{4,4} + 6a_{1,0}^2 a_{1,1}a_{6,5} + a_{1,0}^3 a_{6,6}\end{aligned} \tag{4.35}$$

Note that all higher order terms are contained in R_1. We are therefore interested in the question whether there are any gauge enhancements depending on R_1. From eq. (4.31) we read off the following codimension one and two enhancements:

$$\{\omega = 0\} \tag{4.36}$$
$$\{\omega = 0\} \cap \{P_{10} = 0\} \tag{4.37}$$
$$\{\omega = 0\} \cap \{P_5 = 0\} \tag{4.38}$$

This shows that no gauge divisors or matter curves arise as a consequence of higher order terms in b_{i,j_i}. For compactifications to six dimensions this is the end of the story. Contrastingly, in four dimensions, localised Yukawa couplings at codimension three singularities have to be included. This occurs at:

$$\{\omega = 0\} \cap \{P_{10} = 0\} \cap \{P_5 = 0\} \tag{4.39}$$
$$\{\omega = 0\} \cap \{P_5 = 0\} \cap \{R_0 + R_1 = 0\} \tag{4.40}$$

The second singularity indeed depends on the higher order terms encoded in R_1 and therefore shows that in compactifications to four dimensions, the leading order terms of b_{i,j_i} in ω are not sufficient to describe all gauge enhancements.

4.6. Matter in heterotic compactifications to four dimensions

In the following we will be concerned with relations between the matter spectrum arising in F-theory models and that of heterotic models in four dimensions. We, therefore, very briefly recapitulate the aspects of matter states in such heterotic compactifications. As has been shown, there are two ten-dimensional supersymmetric heterotic string theories, one with gauge group $SO(32)$ and one with $E_8 \times E_8$ [45]. We restrict our discussions to the latter one. The massless matter spectrum of this theory transforms under $SO(8)_{\rm spin} \times E_8 \times E_8$ as

$$\begin{aligned}&(\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{28},\mathbf{1},\mathbf{1}) + (\mathbf{35},\mathbf{1},\mathbf{1}) + (\mathbf{56},\mathbf{1},\mathbf{1}) + (\mathbf{8}',\mathbf{1},\mathbf{1})\\ &+(\mathbf{8}_v,\mathbf{248},\mathbf{1}) + (\mathbf{8},\mathbf{248},\mathbf{1}) + (\mathbf{8}_v,\mathbf{1},\mathbf{248}) + (\mathbf{8},\mathbf{1},\mathbf{248}),\end{aligned}$$

where we refer the reader to [56] for a pedagogical derivation. When considering the phenomenologically interesting case of compactifications to four dimensions $\mathcal{M}_4 \times K$, one may derive strong constraints on both manifolds $\mathcal{M}_4$ and K. Namely, as has been derived in [57], requiring a maximally symmetric four-dimensional spacetime, as well as unbroken $N = 1$ supersymmetry in four dimensions, implies that $\mathcal{M}_4$ is given by the flat Minkowski spacetime. Moreover, K is shown to have $SU(3)$ holonomy as well as vanishing curvature in such situations. That is, K is given by a Calabi-Yau three-fold. Furthermore, it is shown that the gauge field has to satisfy:

$$\frac{1}{30}\mathrm{Tr}F \wedge F = \mathrm{Tr}R \wedge R \tag{4.41}$$

Compactifications satisfying these constraints generically break the gauge group and thereby change the properties of the matter spectrum, as can be seen in the following example: Namely, one solution to the condition in eq. (4.41), resulting in realistic models, is to identify the gauge field with the spin connection [57]. In doing so, the spin connection — an $SU(3)$ field in this case — has to be embedded in $E_8 \times E_8$. Indeed, one may confirm that $SU(3) \times E_6$ is a maximal subgroup of E_8, which shows that one E_8 factor is broken in such a scenario, whereas the other one remains unbroken. [3]

5. Four dimensional heterotic duality on $SU(5) \times U(1) \times U(1)$ F-theory models

In the following we will investigate aspects of duality between the F-theory GUTs constructed in [1] to the heterotic $E_8 \times E_8$ string. To do so, we will first derive constraints from putting the four top models on a base admitting this duality. Secondly, we investigate the behaviour of the elliptic fibre on the heterotic side of the duality. In doing so we identify singular loci in the heterotic compactification present along particular loci of gauge enhancement in the dual F-theoretic picture. Lastly, we will approach the top constructions from a purely group theoretic point of view by computing all possible embeddings of the four top models into a Higgsed E_8. The question underlying this

[3] Note that analogous compactifications for the $SO(32)$ do not give realistic models [57].

b_0	b_1	b_2	c_1	c_2	d_0	d_1	d_2
$\alpha-\beta+\bar{\mathcal{K}}$	$\bar{\mathcal{K}}$	$-\alpha+\beta+\bar{\mathcal{K}}$	$-\alpha+\bar{\mathcal{K}}$	$-\beta+\bar{\mathcal{K}}$	$\alpha+\bar{\mathcal{K}}$	$\beta+\bar{\mathcal{K}}$	$\alpha+\beta+\bar{\mathcal{K}}$

Table 8: Classes of the sections present in the top models [1].

analysis is, whether there is a relation between fields which are not embeddable in this sense and loci yielding singularities on the heterotic side.

5.1. Effective sections and non-flat points

As we have outlined in section 4, the heterotic E_8 string is dual to F-theory on certain geometries. Namely a heterotic compactification on an elliptically fibred three fold X_3 over a base B_2 is dual to F-theory on an elliptically fibred four fold Y_4 over a base B_3 which is itself a $\mathbb{P}^1$ fibration over the common base B_2 (see tab. 7). The base of the elliptic fibration on the F-theory side can be expressed as $B_3 = \mathbb{P}(\mathcal{O}\oplus\mathcal{L})$ for two line bundles $\mathcal{O}$ and $\mathcal{L}$ over the base B_2. We can choose them such that:

$$c_1(\mathcal{O}) \equiv 0 \tag{5.1}$$
$$c_1(\mathcal{L}) =: t \tag{5.2}$$

Defining $\omega := c_1(\mathcal{O}_{B_3}(1))$ the anticanonical divisor $\bar{\mathcal{K}}$ of the fibration B_3 is given by [48]:

$$\bar{\mathcal{K}} = c_1(B_3) = c_1(B_2) + 2\omega + t \tag{5.3}$$

In the top models, ω corresponds to the class of the divisor supporting the non-abelian singularity.

Recall from [1] that the top models contained points of non-flat fibre, that is points on which the fibre becomes too singular to allow for a resolution in terms of a smooth Calabi-Yau four-fold. Such geometries therefore do not yield physical compactifications in general. To construct a well-defined theory, we have to turn off these points in cohomology. Secondly, we have to ensure that the classes corresponding to the sections $k_{i,j}$ (see 3.4) defining the top models are effective [4]: $[k_{i,j}] \geq [k_i] - j\omega$ (j is the leading vanishing order of this section in a particular top model). As you can see from tab. 8, both constraints impose restrictions on the line bundles α, β.

Before we consider a particular top model, let us outline the general procedure we are going to follow: First we expand the line bundles α and β into a piece depending on the base B_2 and one depending on ω:

$$\alpha = \alpha_{B_2} + a_\omega\omega \tag{5.4}$$
$$\beta = \beta_{B_2} + b_\omega\omega \tag{5.5}$$
$$a_\omega, b_\omega \in \mathbb{Z} \tag{5.6}$$

[4] This ensures that the sections are holomorphic, as required.

$b_{0,3}$	b_1	$b_{2,1}$
$\alpha-\beta+c_1(B_2)+t-\omega$	$c_1(B_2)+2\omega+t$	$-\alpha+\beta+c_1(B_2)+\omega+t$
c_1	c_2	
$-\alpha+c_1(B_2)+2\omega+t$	$-\beta+c_1(B_2)+2\omega+t$	
$d_{0,2}$	d_1	$d_{2,1}$
$\alpha+c_1(B_2)+t$	$\beta+c_1(B_2)+2\omega+t$	$\alpha+\beta+c_1(B_2)+\omega+t$

Table 9: Classes of the sections in top 2.

For a specific base B_3 (i.e. a Hirzebruch fibration $\mathbb{F}_{k,m,n}$), we could further expand the first piece into the generators of the Mori cone of B_2. Since we want to consider generic $\mathbb{P}_1$ fibrations here, we cannot do this, however. We can nevertheless derive restrictions from the two constraints described above: As ω is independent from the classes coming from the B_2-piece, we can consider it separately. Effectiveness will then give a set of inequalities acting on the ω-coefficients a_ω and b_ω. Turning off the non-flat point by setting its dual class trivial in cohomology will give an additional equality to be satisfied. [5] Of course such an analysis only gives a necessary constraint rather than a sufficient one, as we do not consider the effectiveness in the expansion of generators of B_2.

As an example consider now top 2 from [1]. In this model the leading vanishing orders of the sections in the coordinate ω normal to the divisor are given by:

$$b_0 \to b_{0,3}\omega^3 \tag{5.7}$$
$$c_2 \to b_{2,1}\omega \tag{5.8}$$
$$d_0 \to d_{0,2}\omega^2 \tag{5.9}$$
$$d_2 \to d_{2,1}\omega \tag{5.10}$$

Taking the classes fron tab. 8, we can now plug in eq. (5.3) as well as above vanishing orders, yielding the classes given in tab. 9. Top 2 has a non-flat point at $\{b_1 = c_{2,1} = 0\}$, which we turn off by setting:

$$0 \overset{!}{=} [c_{2,1}] = \bar{\mathcal{K}} - \omega - \beta \tag{5.11}$$
$$\Rightarrow \beta = \bar{\mathcal{K}} - \omega \tag{5.12}$$

Using this expression and also the expansion from eq. (5.5), the ω-coefficients yield

[5]The class corresponding to the non-flat point is determined by the wedge product of the classes corresponding to the curves intersecting at that point. For a generic base we may only turn off the non-flat point by setting one of the factors to zero, whereas for a specific base there might exist non-zero choices of orthogonal classes.

	top 1	top 2	top 3	top 4
non-flat point	$\{b_1 = d_1 = 0\}$ $\Rightarrow \beta = -\bar{\mathcal{K}}$	$\{b_1 = c_{2,1} = 0\}$ $\Rightarrow \beta = \bar{\mathcal{K}} - \omega$	$\{b_1 = c_1 = 0\}$ $\Rightarrow \alpha = \bar{\mathcal{K}}$	$\{b_1 = b_{0,1} = 0\}$ $\Rightarrow \beta = \alpha + \bar{\mathcal{K}} - \omega$
restriction on a_ω, b_ω	cannot be satisfied	$a_\omega = 2$ $b_\omega = 1$	$a_\omega = 2$ $b_\omega = 0$	$a_\omega = -1$ $b_\omega = 0$

Table 10: Constraints on a_ω and b_ω from the effectiveness of the $k_{i,j}$ and turning off the non-flat points.

the following set of inequalities for the $k_{i,j}$ to be effective:

$$0 \leq a_\omega - 2 \tag{5.13}$$
$$0 \leq a_\omega + 3 \tag{5.14}$$
$$0 \leq -a_\omega + 2 \tag{5.15}$$
$$0 \leq a_\omega \tag{5.16}$$
$$0 \leq 3 \tag{5.17}$$
$$0 \leq a_\omega + 2 \tag{5.18}$$
$$\tag{5.19}$$

Combined, they yield:

$$a_\omega = 2 \tag{5.20}$$

The results of applying the same analysis to the other top models can be found in tab. 10. In summary, for the models top 2–4 there is a unique choice of the ω-coefficients of the line bundles α, β such that all sections are effective and the non-flat point is turned off. In contrast, top 1 does not yield a well-defined theory when put on $\mathbb{P}_1$ fibred bases B_3, that is it does not allow for a smooth heterotic dual in terms of the geometries considered.

5.2. Singular loci

The top models considered here are given as cubics in dP_2, which can be mapped to Weierstraß form

$$P_{\mathrm{W}} = y^2 - x^3 - fxz^4 - gz^6 \tag{5.21}$$

using a Nagell transformation that has been computed in [1]:

$$d = b_1^2 + 8b_0b_2 - 4c_1d_0 - 4c_2d_1 \tag{5.22}$$

$$c = -\frac{4}{c_1}(b_0b_2^2b_2c_1d_0 + c_1^2d_2) \tag{5.23}$$

$$e = \frac{2c_1(b_0(b_1c_1d_1 - b_1^2b_2 + 2b_2c_1d_0 + 2b_2c_2d_1 - 2c_1^2d_2))}{b_0b_2^2 + c_1(c_1d_2 - b_2d_0)} \tag{5.24}$$

$$+ \frac{2c_1(-2b_0^2b_2^2 + c_2(b_1b_2d_0 + b_1c_1d_2 - 2b_2c_2d_2 - 2c_1d_0d_1))}{b_0b_2^2 + c_1(c_1d_2 - b_2d_0)} \tag{5.25}$$

$$k = \frac{c_1^2(b_0b_1b_2 - b_0c_1d_1 - b_2c_2d_0 + c_1c_2d_2)^2}{(b_0b_2^2 + c_1(c_1d_2 - b_2d_0))^2} \tag{5.26}$$

$$f = -\frac{d^2}{3} + ce \tag{5.27}$$

$$g = -\frac{df}{3} - \left(\frac{d}{3}\right)^3 + c^2k \tag{5.28}$$

The information about the geometry and also the two E_8 bundles on the heterotic side can be obtained by plugging in the leading order vanishing in the GUT divisor ω for all sections k_i. The procedure from [41] then states that the F-theory Weierstraß model splits according to

$$P_{\mathrm{W}} = \tilde{P}_{\mathrm{W}} + \tilde{P}_{\mathrm{T},1} + \tilde{P}_{\mathrm{T},2} = 0 \tag{5.29}$$

$$\tag{5.30}$$

That is, a Weierstraß model $\tilde{P}_{\mathrm{W}}$ and two Tate models $\tilde{P}_{\mathrm{T},i}$, the splitting into which can be read of the order in ω as follows:

$$\tilde{P}_{\mathrm{W}} = -y^2 + x^3 + \tilde{f}x\omega^4 + \tilde{g}\omega^6 \tag{5.31}$$

$$\tilde{P}_{\mathrm{T},1} = \text{all terms of order } \omega^i, i < 4 \tag{5.32}$$

$$\tilde{P}_{\mathrm{T},2} = \text{all terms of order } \omega^i, i > 6 \tag{5.33}$$

$$\tag{5.34}$$

As stated before, on the heterotic side of the duality one considers elliptically fibred manifolds with two E_8 gauge bundles. The information about these three parts is encoded as follows: The geometry of the elliptic fibre over a point in the base is determined by $\tilde{P}_{\mathrm{W}}$ whereas the two $\tilde{P}_{\mathrm{T},i}$ parametrise the bundle data for each vector bundle V_i in terms of a spectral covers.

In the following analysis we will focus on the geometry of the heterotic compactification and therefore we only consider the Weierstraß form and leave specifics of the gauge bundles to future analysis. To exemplify this procedure we return once again to top 2. We restrict to this model by plugging the leading vanishing order in ω from eq. (5.7) – (5.10) into the expressions given in eq. (5.22) – (5.28) to determine the sections f and g of the F-theory Weierstraß form, which is not given here for brevity. To read off the heterotic fibration part, we collect the terms of fourth and sixth order in the GUT-divisor ω. These correspond to the sections $\tilde{f}$ and $\tilde{g}$ of the heterotic Weierstraß model $\tilde{P}_{\mathrm{W}}$:

$$\tilde{f} = -\frac{16}{3}(c_1^2 d_{0,2}^2 - b_{0,3} b_2 c_{2,1} d_1 - 3 b_{0,3} c_1^2 d_{2,1}) \tag{5.35}$$

$$\tilde{g} = \frac{16}{27}(6 b_{0,3}^2 b_1^2 b_2^2 - 18 b_{0,3} b_1 b_2^2 c_{2,1} d_{0,2} + 27 b_2^2 c_{2,1}^2 d_{0,2}^2 - 8 c_1^3 d_{0,2}^3 - 18 b_{0,3}^2 b_1 b_2 c_1 d_1 \tag{5.36}$$

$$+ 6 b_{0,3} b_2 c_1 c_{2,1} d_{0,2} d_1 + 27 b_{0,3}^2 c_1^2 d_1^2 - 72 b_{0,3} b_2^2 c_{2,1}^2 d_{2,1} + 36 b_{0,3} c_1^3 d_{0,2} d_{2,1}) \tag{5.37}$$

$$\tag{5.38}$$

On the F-theory side singular loci over the base encode enhancements of the gauge group associated to the GUT divisor, as explained in 3.2. The vanishing order of the discriminant Δ, f and g along these curves determine the precise enhancement according to Kodaira's classification (see tab. 1). Contrastingly, on the heterotic side the situation is far less clear: While such singularities may encode non-pertubative effects in the form of further gauge enhancements beyond E_8 they can also render the theory inconsistent. Therefore it is interesting to compute which loci in the geometry yield singularities on the heterotic side. So assuming a smooth base manifold B_2 we are looking for loci in X_3 along which the elliptic fibre develops singularities. This is encoded in the vanishing of the heterotic discriminant along those loci. For a general Weierstraß model the discriminant is computed as (see e.g. [5]):

$$\Delta = 27\tilde{g}^2 + 4\tilde{f}^3 \tag{5.39}$$

For brevity we will not give the entire expression for the specific top models.

The duality to the heterotic string translates the physics on a F-theory divisor (here ω) to geometry and bundle data on the heterotic side (see section 4). As we argued in section 3.4, further loci of gauge enhancement are present also away from the divisor in the form of GUT singlets, but may intersect the divisor in points. It is therefore particularly interesting to consider how these global aspects of the fibration translate to the heterotic dual. As these loci correspond to the intersection of the singlet curves with the GUT divisor, one should therefore consider the heterotic discriminant at these points. Take for instance the singlet $\mathbf{1}_3$ given by the equations $\{b_2 = c_1 = 0\}$ and consider top 2, where we turned off the non-flat point by $c_{2,2} \equiv 1$. We can now compute $\Delta_{\text{top 2}}|_{b_2=c_1=0, c_{2,2}\equiv 1}$, which indeed vanishes. The same computations can be performed for all singlets in top 2–4 (top 1 does not admit a $\mathbb{P}_1$ as a base, see section 5.1). The results are shown in tab.11. An analogous computation shows that there are no singular loci along any of the matter couplings within the GUT divisor in the top models.

At this point it is interesting to compute how turning off the singular singlets in cohomology constrains the line bundles α and β and to check whether this agrees with the constraints derived in section 5.1.[6] Let us now go through the computations in

[6] Note that one wants to turn off the singular couplings rather than the singular singlets. However, since the class of the coupling is given by $[C_{\mathbf{1}_{(i)}}] \wedge \omega$ and we do not want to turn off the GUT divisor ω, we proceed by turning of the singlet locus.

	singular singlets	f_4	g_6	Δ_{het}	type	group
top 2	$\mathbf{1}_3$	1	2	3	III	$SU(2)$
top 3	$\mathbf{1}_1$	2	2	4	IV	$SU(3)$
top 4	$\mathbf{1}_3$	0	0	2	I_2	$SU(2)$
	$\mathbf{1}_5$	2	2	4	IV	$SU(3)$

Table 11: The singlets listed are those over which the heterotic discriminant vanishes, i.e. those which yield singular loci on the heterotic side.

detail for top 2, before we give the results for the other models. Turning off $C_{\mathbf{1}_{(3)}}$ amounts to:

$$0 \overset{!}{=} [C_{\mathbf{1}_{(3)}}] = [b_2] \wedge [c_1] \tag{5.40}$$

$$= (-\alpha + \beta + \bar{\mathcal{K}}) \wedge (-\alpha + \bar{\mathcal{K}}) \tag{5.41}$$

$$\overset{\text{n.f.p}}{=} (\alpha + 2\bar{\mathcal{K}} - \omega) \wedge (-\alpha + \bar{\mathcal{K}}) \tag{5.42}$$

$$\tag{5.43}$$

From this we have two possibilities to turn off the singular locus:

$$\text{(i) } \alpha = 2\bar{\mathcal{K}} - \omega = 2c_1(B_2) + 2t + \omega \tag{5.44}$$

$$\text{(ii) } \alpha = \bar{\mathcal{K}} = c_1(B_2) + t + 2\omega \tag{5.45}$$

Lastly we can compare these results with the constraints derived in section 5.1, where we computed $a_\omega = 2$. It is apparent for (i) that they can only be satisfied if ω is trivial in cohomology. This means there cannot be a smooth heterotic dual and an $SU(5)$ singularity at the same time. Contrastingly, choice (ii) agrees with the constraint derived before, without turning off the GUT divisor. In summary for top 2 there is a unique choice of the line bundles α and β (or more precisely their ω-coefficients) such that there is an $SU(5)$ singularity, but no singular loci in the heterotic dual. The results of the analysis for top 3 and 4 can be found in tab. 12. One can see that most choices to turn off the singular loci do not admit for an $SU(5)$ singularity. In summary for top 2 and 3 there is a unique choice to turn off the singular singlets and satisfy the constraints from turning off the non-flat point and having effective sections $k_{i,j}$. For top 4 no such choice exists.

5.3. Embedding into a Higgsed E_8

In section 4.6 we outlined how the heterotic matter arises from an $E_8 \times E_8$ gauge symmetry in ten dimensions. It therefore, seems promising to check for an embedding of the F-theory spectrum into a Higgsed E_8 and compare the results with the considerations from sections 5.1 and 5.2. To make the embedding process more specific, note that we decompose E_8 to the GUT group and its commutant, i.e.:

$$E_8 \to SU(5)_{\text{GUT}} \times SU(5)_\perp \tag{5.46}$$

		constr. on coupling	turned off curves
top 2	(↯)	$\alpha = 2c_1(B_2) + 2t + 3w$	
	(✓)	$\alpha = c_1(B_2) + t + 2w$	$\mathbf{1}_{(1)}$,$\mathbf{1}_{(3)}$,$\mathbf{1}_{(5)}$,$\mathbf{5}_{(1)}$,$\mathbf{5}_{(2)}$
top 3	(↯)	$\beta = 2c_1(B_2) + 2t + 2w$	
	(✓)	$\beta = c_1(B_2) + t$	$\mathbf{1}_{(1)}$,$\mathbf{1}_{(3)}$,$\mathbf{1}_{(5)}$,$\mathbf{5}_{(2)}$
top 4	(↯)	$\alpha = c_1(B_2) + t + w$	
	(↯)	$2c_1(B_2) + 2t + 3w = 0$	
		$\alpha = -w$	

Table 12: Different choices to turn off the singular singlets. The ✓ and ↯ symbol indicate whether a particular choice agrees with the constraints derived from turning off the non-flat point and having effective sections $k_{i,j}$. Note that there is a unique choice to do so in top 2 and 3 and no such choice in top 4. The last column which curves are turned off by these restrictions.

The additional abelian gauge group factors are contained in $SU(5)_\perp$ and we have to provide an embedding. In general, many such embeddings exist, but the most generic one is in terms of its Cartan subgroup $G_\perp = U(1)^4$. To parametrise the embedding in $S\left[U(1)^4\right]$, we need five parameters a_i satisfying a tracelessness constraint $\sum_{i=1}^{5} a_i = 0$ to ensure a trivial determinant. In this description the embedding of the additional $U(1)$ factors is therefore given by:

$$U(1)_A = \sum_{i=1}^{5} = a_i^A t^i \tag{5.47}$$

A state is represented by a linear combination of t_j obeying $t_j t^i = \delta_j^i$ such that its charge can be obtained by contraction with $U(1)_A$. Decomposing adj E_8 under $S\left[U(1)^5\right]$ yields three different representations:

$$\mathbf{10}_i : t_i \tag{5.48}$$

$$\bar{\mathbf{5}}_{ij} : t_i + t_j \qquad i \neq j \tag{5.49}$$

$$\mathbf{1}_{ij} : t_i - t_j \qquad i \neq j \tag{5.50}$$

$$\tag{5.51}$$

Given such an embedding one may therefore compute the charges of all possible fields and compare them with those obtained from the geometry. If the charges of a field from the spectrum can be matched to one of the combinations provided by eq. (5.48) – (5.50), it is embeddable for this choice of a_i.

In order to determine such an embedding for the case of two additional $U(1)$ factors, we therefore need ten (5×2) equations determining the value of the a_i^A ($i = 1, \ldots, 5$, $A = 1, 2$). The first equation is provided by the tracelessness constraint. For the second one we can set $\mathbf{10} : t_1$ and take the respective charge of the top model in consideration. For the next three equations we can take three $\bar{\mathbf{5}}$ representations and their charges. One can then work ones way through all combinations of $t_i + t_j$ for the three types of representations. This means for each such combination we get a system of linear equations as:

$$M_{ij} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ \vdots & & \vdots & & \vdots \end{pmatrix} \tag{5.52}$$

$$M_{ij} a_j^A = q_i^A \tag{5.53}$$

$$\tag{5.54}$$

Where the first line is the tracelessness condition (therefore $q_1 = (0, 0)$) and the second corresponds to the **10** representation (i.e. q_2: $U(1)_1$ and $U(1)_2$ charge of **10**). The following three lines correspond to the $\bar{\mathbf{5}}s$. Having solved this equation one compute the charges of all combinations $t_i + t_j$ and match them to the remaining $\bar{\mathbf{5}}s$. If all of them can be matched, one has found an embedding for the matter spectrum (this does not imply that also the couplings can be embedded into a Higgsed E_8, in general). In the same way one can check which $\mathbf{1}s$ can be embedded by computing the charges of all $t_i - t_j$.

Next we have to find out whether the couplings are embeddable in the sense that they are gauge invariant under $SU(5)_{\text{GUT}}$. The way to go about this is first to compute the $U(1)$ charges of all possible couplings. We know that all couplings are present which have charge 0 under both $U(1)$s. However, this does not necessarily imply gauge invariance straight away: To see this, consider for example top 2 with the embedding ($\bar{\mathbf{5}}_{ij} = t^i + t^j$):

$$U(1)_1 = -t^1 + 4t^2 - t^3 - t^4 - t^5 \tag{5.55}$$

$$U(1)_1 = -2t^1 + 3t^2 + 3t^3 - 2t^4 - 2t^5 \tag{5.56}$$

$$\text{matter} : \mathbf{10}_1, \bar{\mathbf{5}}_{23}, \bar{\mathbf{5}}_{12}, \bar{\mathbf{5}}_{14}, \bar{\mathbf{5}}_{13} \tag{5.57}$$

As can be verified the following couplings have charge 0 under the two $U(1)$s:

$$\mathbf{10}_1 \bar{\mathbf{5}}_{23} \bar{\mathbf{5}}_{14} : 2t_1 + t_2 + t_3 + t_4 \tag{5.58}$$

$$\mathbf{10}_1 \bar{\mathbf{5}}_{12} \bar{\mathbf{5}}_{13} : 3t_1 + t_2 + t_3 \tag{5.59}$$

$$\bar{\mathbf{10}}_1 \bar{\mathbf{10}}_1 \bar{\mathbf{5}}_{14} : -t_1 + t_4 \tag{5.60}$$

The fact that the t_i do not add up to 0 in the couplings means, they are not gauge invariant under the full group. However, since we are interested in embeddings into a Higgsed E_8 to $SU(5) \times U(1) \times U(1)$ and the full Cartan is $S[U(1)^4]$, two singlets have obtained a vev. This corresponds to identifying $t_i = t_j$ and $t_k = t_l$ for some $i \neq j$ and $k \neq l$. Looking at above example, we see that the generators indeed add up to 0, if we identify $t_1 = t_4$ and $t_1 = t_5$ showing that the couplings in above model can be embedded into a Higgsed E_8. A comprehensive study of all embeddings of the matter sector into a E_8 has been performed using a script based on the above algorithm and shows that there are ten such embeddings which allow for the matter couplings to be embedded via a Higgsing of the theory.

If we take into account that singlets $\mathbf{1}^{(1)}$ and $\mathbf{1}^{(5)}$ are turned off when setting the point of non-flat fibre trivial in cohomology and that singlet $\mathbf{1}^{(3)}$ corresponds to the singular coupling, it is interesting to see, whether there exists an embedding such that the remaining singlet spectrum (i.e. $\mathbf{1}^{(2)}$, $\mathbf{1}^{(4)}$ and $\mathbf{1}^{(6)}$) is embeddable and moreover how these embeddings restrict to the singular coupling. It is easy to confirm that of the ten above mentioned embeddings four also contain singlets with the correct charges as to embed the non turned-off singlets. One of these four embeddings does in particular not contain a singlet representation to match the charges of the singular singlet$\mathbf{1}^{(3)}$, whereas the remaining three do. However, when considering the singular coupling (rather than the singlet alone), which is of type $\mathbf{15\bar{5}}$ one notes that all three of them lead to a coupling which cannot be embedded via Higgsing of two E_8 singlets.

For top 3 the situation looks similar at first: There are also ten embeddings of the matter sector (including the couplings) into a Higgsed E_8. Five of these embeddings once again allow for an embedding for the relevant singlet curves. However, when considering the embeddability of the singular singlet couplings, three of these five models allow for an embedding not only of the singlet itself, but also of the coupling into a Higgsed E_8. The remaining two embeddings do not embed the singular singlet at all. So it is important to stress that although the fate of the singular coupling is not as uniquely determined as for top 2, there are nevertheless two embeddings which have the property that they embed the matter spectrum, the matter couplings and also the non-singular couplings while leaving the singular singlet unembeddable.

Lastly, concerning top 4 there is only one embedding such that the matter spectrum is embeddable and this embedding does not allow for all matter couplings to be embedded via Higgsing. It is interesting to note that top 4 did not allow for a smooth heterotic dual at the same time as having an $SU(5)$ singularity.

6. Six dimensional heterotic duality on $SU(5) \times U(1) \times U(1)$ F-theory models

It is also possible to consider duality to the heterotic string in six dimensions. In this case the base is uniquely determined to be a Hirzebruch surface $\mathbb{F}_k = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k))$ [50]. As we outlined in 4.2, for such manifolds the first Chern class can be expressed in terms of the section $\mathcal{S}$ and the fibre class $\mathcal{E}$ as:

$$\bar{\mathcal{K}} = c_1(\mathbb{F}_k) = 2\mathcal{S} + (k+2)\mathcal{E} \tag{6.1}$$

Since the Mori cone is generated by these two independent classes, also the divisor class associated to the vanishing locus of ω is not independent, but instead given as:

$$\omega = \mathcal{S}_0 = \mathcal{S} + k\mathcal{E} \tag{6.2}$$

With this information it is possible to repeat the analysis we performed in four dimensions on $\mathbb{F}_k$. Once again the sections $k_{i,j}$ (given in tab. 8) are required to be effective. We impose this by first expanding the line bundles α and β in terms of the Mori cone

	singular singlets	f_4	g_6	Δ_{het}	type	group
top 1	$\mathbf{1}_1$	0	0	2	I_2	$SU(2)$
	$\mathbf{1}_3$	1	2	3	III	$SU(2)$
	$\mathbf{1}_5$	0	0	2	I_2	$SU(2)$
top 2	$\mathbf{1}_1$	0	0	2	I_2	$SU(2)$
	$\mathbf{1}_3$	1	2	3	III	$SU(2)$
top 3	$\mathbf{1}_1$	2	2	4	IV	$SU(3)$
	$\mathbf{1}_3$	0	0	2	I_2	SU(2)
	$\mathbf{1}_5$	0	0	2	I_2	SU(2)
top 4	$\mathbf{1}_1$	0	0	2	I_2	$SU(2)$
	$\mathbf{1}_3$	0	0	2	I_2	$SU(2)$
	$\mathbf{1}_5$	2	2	4	IV	$SU(3)$

Table 13: The singlets listed are those over which the heterotic discriminant vanishes, i.e. those which yield singular loci on the heterotic side.

generators:

$$\alpha = a_{\mathcal{S}}\mathcal{S} + a_{\mathcal{E}}\mathcal{E} \tag{6.3}$$

$$\beta = b_{\mathcal{S}}\mathcal{S} + b_{\mathcal{E}}\mathcal{E} \tag{6.4}$$

$$a_i, b_i \in \mathbb{Z} \tag{6.5}$$

Now the effectiveness constraint translates into two sets of inequalities for each top model. The first one relates the $\mathcal{S}$ coefficients $a_{\mathcal{S}}$ and $b_{\mathcal{S}}$ and constrains them to a parameter region. This is depicted in fig. 10. The second set of inequalities for the coefficients $a_{\mathcal{E}}$ and $b_{\mathcal{E}}$ depends on the integer k and is more complicated in general. However it allows us to restrict the integer k further to a subset of $k = 0, \ldots, 12$. For the four top models it is given by:

$$\text{top 1:} \quad k = 1, \ldots, 12 \tag{6.6}$$

$$\text{top 2:} \quad k = 1, \ldots, 4 \tag{6.7}$$

$$\text{top 3:} \quad k = 1, \ldots, 8 \tag{6.8}$$

$$\text{top 4:} \quad k = 1, \ldots, 6 \tag{6.9}$$

(6.10)

Next we determine the singlet couplings giving rise to a singularity in the heterotic elliptic fibration. These are in general different to the four dimensional case, as the non-flat points do not have to be set trivial in homology to render the fibration physical. This also implies that the vanishing order of f, g and Δ may decrease in comparison with the four dimensional case. The results of this computation can be found in tab. 13. Let us briefly compare the geometrical constraints from six and four dimensional duality. Namely, one the one hand we do not have to impose any requirement to turn off non-flat points in homology in six dimensions, as no codimension two singularities

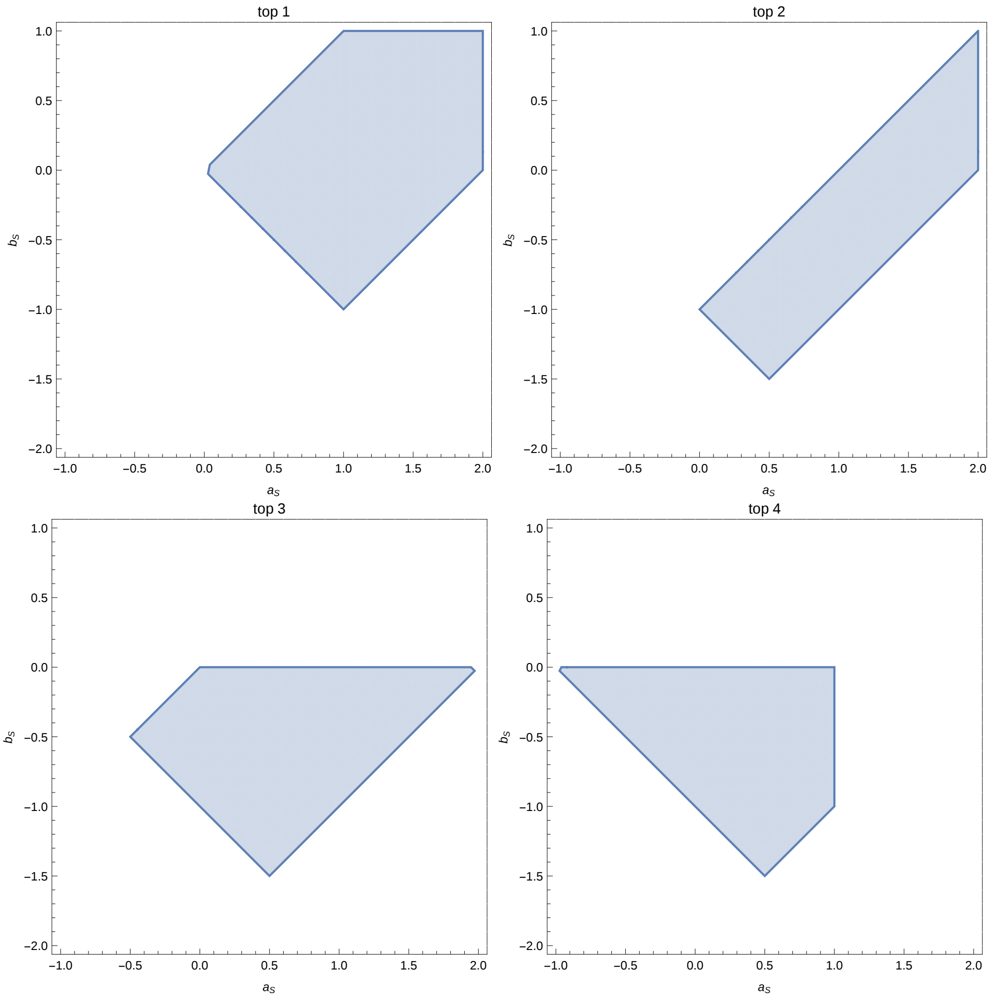

Figure 10: Allowed parameter region for $a_{\mathcal{S}}$ and $b_{\mathcal{S}}$ for the four top models derived from the effectiveness constraint.

exist. On the other hand the base geometry is far more restricted in six dimensions and furthermore the heterotic fibration is singular along more singlets. So in the latter sense, requiring a smooth geometry is more restrictive in six dimensions.

As indicated we now derive the conditions on a smooth heterotic dual, which means that we have to turn off those singlets in homology which lead to singularities in the heterotic elliptic fibration. The dual classes of the relevant singlets are given as

$$[\mathbf{1}^{(1)}] = [b_{0,j_1}] \wedge [c_{2,j_5}] \tag{6.11}$$
$$[\mathbf{1}^{(3)}] = [b_{2,j_3}] \wedge [c_{1,j_4}] \tag{6.12}$$
$$[\mathbf{1}^{(5)}] = [c_{1,j_4}] \wedge [c_{2,j_5}], \tag{6.13}$$

where the j_i are determined by the individual top model in question. These classes can now be expressed in terms of $\mathcal{S}$ and $\mathcal{E}$ by first consulting tab. 3 and using eq. (6.1) – (6.5), which yields:

$$\begin{aligned}
[\mathbf{1}^{(1)}] &= ((2 + a_{\mathcal{E}} - b_{\mathcal{E}} + k - j_0 k)\mathcal{E} + (2 + a_{\mathcal{S}} - b_{\mathcal{S}} - j_0)\mathcal{S})\wedge \\
&\quad ((2 - b_{\mathcal{E}} + k - j_4 k)\mathcal{E} - (-2 + b_{\mathcal{S}} + j_4)\mathcal{S}) \\
[\mathbf{1}^{(3)}] &= ((2 - a_{\mathcal{E}} + b_{\mathcal{E}} + k - j_2 k)\mathcal{E} - (-2 + a_{\mathcal{S}} - b_{\mathcal{S}} + j_2)\mathcal{S})\wedge \\
&\quad ((2 - a_{\mathcal{E}} + k - j_3 k)\mathcal{E} - (-2 + a_{\mathcal{S}} + j_3)\mathcal{S}) \\
[\mathbf{1}^{(5)}] &= ((2 - a_{\mathcal{E}} + k - j_3 k)\mathcal{E} - (-2 + a_{\mathcal{S}} + j_3)\mathcal{S})\wedge \\
&\quad ((2 - b_{\mathcal{E}} + k - j_4 k)\mathcal{E} - (-2 + b_{\mathcal{S}} + j_4)\mathcal{S})
\end{aligned}$$

To turn off the singular singlets in homology, one imposes that all relevant coefficients equal zero (taking into account $\mathcal{E}\wedge\mathcal{S} = -\mathcal{S}\wedge\mathcal{E}$). For brevity we do not list the resulting constraints here.

In summary we have to sets of constraints: First of all the requirement of effective $[k_{i,j}]$ and secondly that of a smooth heterotic geometry, which translates to turning off singular singlets. Combining these results severely constraints the base geometry by restricting k and secondly the line bundles α and β. The results of these constraints can be found in tab. 14. As is apparent, the possible geometries are quite restricted. Moreover, given a Hirzebruch surface $\mathbb{F}_k$, the choice of line bundles α, β satisfying all constraints is unique.

top 1	
$k > 0$	$\alpha = 2\mathcal{S} + (2+k)\mathcal{E}$ $\beta = \mathcal{S} + 2\mathcal{E}$
top 2	
$k = 1$	$\alpha = 2\mathcal{S} + (2+k)\mathcal{E}$ $\beta = \mathcal{S} + 2\mathcal{E}$
$k = 2, \ldots, 4$	$\alpha = 2\mathcal{S} + (2+k)\mathcal{E}$ $\beta = \mathcal{S} + (4-k)\mathcal{E}$
top 3	
$k = 1$	$\alpha = 2\mathcal{S} + 3\mathcal{E}$ $\beta = \mathcal{E}$
$k = 2, \ldots, 8$	$\alpha = 2\mathcal{S} + 4\mathcal{E}$ $\beta = (2-k)\mathcal{E}$
top 4	
$k = 1, \ldots, 6$	$\alpha = \mathcal{S} + 2\mathcal{E}$ $\beta = (2-k)\mathcal{E}$

Table 14: Possible choices that allow for a smooth heterotic dual as well as ensure effective $[k_{i,j}]$ The possible geometries are more restricted than in the case of general duality (where $k = 0, \ldots, 12$). Moreover, the choice of line bundles α and β is unique for a given Hirzebruch surface.

7. Conclusions

In this thesis we studied heterotic duality to four and six dimensions of the $SU(5) \times U(1) \times U(1)$ F-theory models, introduced in [1]. In particular we investigated the relationship between smoothness of the heterotic compactification space and embeddability of the F-theory spectrum. In doing so we paid special attention to singlet states under the GUT group. More specifically, we compactified said F-theory models on K3-fibred Calabi-Yau $(n+2)$-folds over some n-fold base which is dual to heterotic string theory on an elliptic fibration over the same base. To be well-defined, the F-theory geometry also has to admit an elliptic fibration over some $(n+1)$-fold base, which is itself a $\mathbb{P}^1$-fibration over the same n-fold base, where $n = 2$ for compactifications to four dimensions and $n = 1$ for compactifications to six dimensions, respectively. To render the F-theory elliptic fibration well-defined, we had to impose an effectiveness constraint on the classes of the defining coefficients $k_{i,j}$ of the elliptic fibration.

We first considered the four dimensional case, in which so-called non-flat points in the three-fold base are too singular to result in a physical theory and had to be avoided by setting their corresponding class trivial. Implementing these constraints severely restricts the geometry and shows in particular that one of the four models (top 1) is inconsistent on these geometries. After we obtained these basic F-theory constraints, we considered the heterotic compactification and investigated whether the elliptic fibration develops singularities along points associated to couplings. Indeed, we showed that some of the singlet couplings correspond to loci of singular fibre, whereas the matter couplings correspond to loci of smooth fibre. Furthermore, we investigated whether the classes of the $k_{i,j}$ can be chosen in such a way as to render the heterotic geometry smooth. That is whether one can consistently set the singular loci associated to the GUT singlets trivial in homology. We showed that for two of the models (top 2 & 3) there is a unique such choice, whereas for the last model (top 4) no such choice exists. Lastly, we computed possible embeddings of the F-theory spectrum into a Higgsed E_8. We have been particularly interested in the question whether there is an embedding in which all non-singular states are embeddable and the singular states are either not embeddable or their couplings cannot be rendered gauge invariant. Indeed, for two models (top 2 & 3) multiple such embeddings exist, whereas such an embedding does not exist for the last model (top 4). We noted in particular that for this model no smooth heterotic dual exists.

In the six dimensional case, the common two-fold base is uniquely determined to be a Hirzebruch surface $\mathbb{F}_k$. While the geometry is therefore more restricted than in the four-dimensional case, there are less constraints to impose. Namely, since no loci of codimension two may exist in the base, there are no points on which the fibre is too singular to be resolved. This also yields more singlet states, which are set trivial in the four dimensional case. Imposing the effectiveness constraints strongly restricts the geometry. We noted that more singlets give rise to singularities in the heterotic elliptic fibration and in fact we showed that given a specific Hirzebruch surface there is a unique choice of the $k_{i,j}$ such that the heterotic geometry is smooth.

As we outlined, the former analyses solely considered the heterotic geometry. In the future it would, therefore, be interesting to include data on the heterotic vector bundles into the analysis of this duality. That is, for instance, to consider how the vector bundles behave over singularities in the heterotic geometry. In addition to this, singularities in the heterotic compactification need not render the theory inconsistent, but may encode more complicated phenomena. It would therefore be of particular interest to investigate these scenarios more carefully in subsequent analyses.

polygon 3					
b_0	b_1	b_2	c_0	c_1	c_2
$-\beta+\bar{\mathcal{K}}$	$\bar{\mathcal{K}}$	$\beta+\bar{\mathcal{K}}$	$\alpha-\beta+\bar{\mathcal{K}}$	$\alpha+\bar{\mathcal{K}}$	$\alpha+\beta+\bar{\mathcal{K}}$
d_0	d_1	d_2			
$-\alpha+\bar{\mathcal{K}}$	$-\alpha+\beta+\bar{\mathcal{K}}$	$-2\alpha+\beta+\bar{\mathcal{K}}$			

polygon 6						
b_0	b_1	b_2	c_0	c_1	c_2	c_3
-2α	$\bar{\mathcal{K}}$	$2\alpha+2\bar{\mathcal{K}}$	0	$\alpha+\bar{\mathcal{K}}$	$2\alpha+2\bar{\mathcal{K}}$	$3\alpha+3\bar{\mathcal{K}}$

polygon 8					
b_0	b_1	b_2	c_0	c_1	c_2
-2α	$\bar{\mathcal{K}}$	$2\alpha+2\bar{\mathcal{K}}$	-4α	$-2\alpha+\bar{\mathcal{K}}$	$2\bar{\mathcal{K}}$

Table 15: Classes of the sections on polygon 3,6,8 (polygon 11 is given in Tate form)

Appendix A. Four dimensional duality on other polygons

One may repeat the analysis from sections 5.1 and 5.2 for the other polygons given in [1]. These are given in terms of fibrations in other ambient spaces and feature different gauge groups than those considered so far. The classes of their sections are given in tab. 15.

The constrains imposed from the effectiveness of the sections as well as turning off points of non-flat fibre, yield the results given in tab. 16. Note that one may read off from there that polygon 6 does not admit a heterotic dual in four dimensions.

While polygon 3 does not contain any extra rational sections (that is extra $U(1)$ factors), polygon 8 and 11 do. One may compute the singlet loci associated to these sections as given in tab. 17.

Next, we compute the discriminant of the heterotic elliptic fibration and investigate whether it is singular along the singlet loci or other loci. The results of which can be found in tab. 18.

Lastly, we may once again investigate different choices to turn off the singularities and compare them with the constraints derived from the effectiveness of sections as well as the absence of non-flat points. These results can be found in tab. 19.

polygon 3	top 1	top 2	top 3
non-flat point	$\{b_1 = d_0 = 0\}$ $\Rightarrow \alpha = \bar{\mathcal{K}}$	$\{b_1 = c_1 = 0\}$ $\Rightarrow \alpha = \omega - \bar{\mathcal{K}}$	$\{b_1 = c_0 = 0\}$ $\Rightarrow \beta = \alpha + \bar{\mathcal{K}}$
restriction on a_ω, b_ω	cannot be satisfied	cannot be satisfied	$a_\omega = 0$ $b_\omega = 2$
	top 4	top 7	
non-flat point	$\{b_1 = c_0 = 0\}$ $\Rightarrow \beta = \alpha + \bar{\mathcal{K}}$	$\{b_1 = b_2 = 0\}$ $\beta = \omega - \bar{\mathcal{K}}$	
restriction on a_ω, b_ω	cannot be satisfied	cannot be satisfied	

polygon 6	top 1	top 2	top 3	top 4
non-flat point			$\{b_1 = b_{0,1} = 0\}$ $\Rightarrow \alpha = -\frac{\omega}{2}$	$\{b_1 = b_2 = 0\}$ $\Rightarrow \alpha = -\bar{\mathcal{K}}$
restriction on a_ω	cannot be satisfied	cannot be satisfied	cannot be satisfied	cannot be satisfied

polygon 8	top 1
non-flat point	
restriction on a_ω	$-2 \leq a_\omega \leq -\frac{3}{4}$

Table 16: Constraints on a_ω and b_ω from the effectiveness of the $k_{i,j}$ and turning off the non-flat points for the tops on polygons 3,6 and 8. Polygon 11 has not additional degrees of freedom in form of line bundles and therefore no further constraints from the effectiveness of the sections exist.

	polygon 8	polygon 11
singlet loci	$\mathbf{1}_1 : \{b_2 = c_2 = 0\}$ $\mathbf{1}_2 : \{c_1 b_2 - b_1 c_2 = 0 = c_0 b_2^2 - b_0 b_2 c_2 + c_2^2\}$	$\{a_3 = a_4 = 0\}$

Table 17: Singlet loci for polygon 8 and 11. Polygon 3 does not contain an extra rational section and therefore no singlets. For polygon 6 there is no heterotic duality in 4D, see above.

	singular loci	Δ_{het}	f_4	g_6
polygon 3 top 3	no singlets no factor. of Δ_{het}			
polygon 8 top 1	$\mathbf{1}_1$	4	2	3
polygon 11 top 1	$f_4 = g_6 = 0$ no fibration def.			

Table 18: Singular loci in the 3 tops allowing for a heterotic dual and an $SU(5)$ singularity.

		constr. on coupling	turned off curves
polygon 3			$\bar{\mathbf{5}}_1$
top 3			
polygon 8	(✓)	$\alpha = -c_1(B_2) - 2\omega - t$	$\bar{\mathbf{5}}_1$, $\mathbf{1}_1$
top 1	(✓)	$\bar{\mathcal{K}} = \omega$	$\mathbf{1}_1$
polygon 11			
top 1			

Table 19: Different choices to turn off the singular singlets. The ✓ symbol indicates whether a particular choice agrees with the constraints derived from turning off the non-flat point and having effective sections $k_{i,j}$. The last column which curves are turned off by these restrictions.

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Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

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