Department of Physics and Astronomy
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Bachelor Thesis in Physics

Anomaly Constraints on Hypercharge Flux in F-Theory GUTs

submitted by

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Abstract

The aim of this thesis is to derive the consistent choices for hypercharge flux in the F-Theory GUT models, that were constructed in [1]. Their constraints are imposed by emerging gauge anomalies in four dimensions, so beforehand the chiral anomaly in quantum field theory as well as the the concept of flux is introduced and discussed. Initially, a short review of the representation theory of the standard model and the Georgi-Glashow model of $SU(5)$ grand unification will be given.

Das Ziel dieser Arbeit ist die Bestimmung von konsistenten Werten für den Hyperladungsfluss in F-Theorie GUT Modellen, die in [1] bestimmt wurden. Die Wahl ist durch das Aufkommen von Eichanomalien in vier Dimensionen eingeschränkt, daher wird zunächst die chirale Anomalie in der Quantenfeldtheorie sowie das konzept von Fluss eingeführt und diskutiert. Als erstes wird kurz die Struktur des Standardmodells und des Georgi-Glashow Modells für große Vereinheitlichung mittels $SU(5)$, dargestellt.

Contents

1 Introduction

...

2 Matter Representations in the Standard Model and the Georgi-Glashow Model

In this section we summarize the structure of the Standard Model of elementary Particle Physics and the Georgi-Glashow Model of $SU(5)$-Grand Unification. The fermionic matter content as well as its representations under the gauge groups are presented. Additionally we will give a brief discussion of the concept of a chiral theory and grand unification in general. Detailed investigations can be found in [2], [3] and [4].

2.1 The Standard Model

The Standard Model (SM) is a Yang-Mills theory based on the Gauge Group

$$G_{SM} = SU(3) \times SU(2)_L \times U(1)_Y, \tag{1}$$

where $SU(3)$ describes strong interactions in terms of Quantum Chromodynamics and its 8 gluons, $SU(2)_L$ corresponds to the weak Interactions via W^+, W^- and Z and $U(1)_Y$ describes Hypercharge. The fermionic matter content consists of three generations of Quarks and Leptons, all taken as left handed Weyl-Spinors in the $(\frac{1}{2}, 0)$ representation of the Lorentz Group. The latter is possible, since $i\gamma^2\psi_R^*$ transforms as a left handed field, so in this manner we can identify right handed fields by left handed ones (we will keep speaking of "right handed" fields and keep the index R, but only as an indication of the transformation behaviour with respect to the gauge group, as will be seen momentarily). The matter content is the following:
Up-type Quarks U_H^i, Downtype Quarks D_H^i, Leptons E_H^i and Neutrinos ν_H^i, where $i = 1, 2, 3$ denotes the Generation Index and the subscript H stands for the chirality of the field, in the above sense.

In order to decribe the coupling of these fields to the Gauge interactions, we have to specify their transformation behaviour under representations of the SM gauge group G_{SM}. For the $SU(2)_L$ interactions the representation is generated by the Pauli matrices τ^a and the transformation behaviour of the fields is given by

$$\begin{pmatrix} U_L^i \\ D_L^i \end{pmatrix}, U_R^i, D_R^i \tag{2}$$

for the Quark fields and

$$\begin{pmatrix} E_L^i \\ \nu_L^i \end{pmatrix}, E_R^i, \nu_R^i \tag{3}$$

for the Lepton fields. So left-handed fields transform as Douplets in the fundamental (**2**-) representation of $SU(2)_L$ whereas right-handed ones transform as singlets in the trivial representation.
In case of the strong interactions, the representation is generated by the Gell-Mann matrices λ^a and for the transformation behaviour under those we have

$$U_L^i = \begin{pmatrix} U^i_{red} \\ U^i_{green} \\ U^i_{blue} \end{pmatrix}, U_R^i = \begin{pmatrix} U^{*i}_{red} \\ U^{*i}_{green} \\ U^{*i}_{blue} \end{pmatrix}. \tag{4}$$

Similarly for the Down-type Quarks D^i_H. Left handed quarks transform as triplets in the fundamental (**3**-) representation of $SU(3)$, right handed ones in the anti-fundamental ($\overline{\mathbf{3}}-$) representation, while all leptons transform as singlets in the trivial representation of $SU(3)$.
All fermions interact with the gauge field of the Hypercharge component $U(1)_Y$ of the SM, i.e. they transform nontrivially in a representation of this gauge group. Therefore one assigns to each fermion field ψ a hypercharge Y_ψ, which happens to be the same for each of the generations $i = 1, 2, 3$. The SM is related to the Electromagnetic Gauge Group via a Spontaneous Symmetry Breaking

$$SU(2)_L \times U(1)_Y \longrightarrow U(1)_{E.M.} \tag{5}$$

By analysing the gauge interaction term of the SM Langrangian, the correspondence between hypercharge and electromagnetic charge is found to be [3]

$$Q_{E.M} = Y + T_3 \qquad T_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \tag{6}$$

where T_3 is a $SU(2)_L$ generator.
The SM is therefore a chiral theory, i.e. the left- and right handed components of the fermionic fields are in different representations of the gauge group, in this case G_{SM}. The fact, that the transformation behaviour under the gauge group depends on the chirality, has an important consequence: Consider a chiral gauge theory in the above sense of a single, massive, fermionic field with some gauge group G and Lagrangian

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i \not{D} - m)\psi \tag{7}$$

Rewritten in terms of left and right handed fields, we get for the mass-term

$$m(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L) \tag{8}$$

If the representations of ψ_L and ψ_R differ, this may not be invariant under gauge transformations, so in this case the term is forbidden if one assumes gauge invariance to hold. In consequence, we have to consider all fermions to be massless in models where this poses a problem, as long as a symmetry breaking analogous to (2) did not occur.
Lastly, we introduce the common notation in which the matter content of the SM is presented. A certain species of fermionic field is denoted as

$$(\text{representation under } SU(3),\ \text{representation under } SU(2)_L)_{\text{Hypercharge}} \tag{9}$$

Therefore, the matter content of the SM can be summarized in the following table

Field	$SU(3)$	$SU(2)_L$	$U(1)_Y$	shortcut
$\left(U^i, D^i\right)_L$	$\mathbf{3}$	$\mathbf{2}$	$\frac{1}{6}$	$(\mathbf{3},\mathbf{2})_{\frac{1}{6}}$
U_R^i	$\overline{\mathbf{3}}$	$\mathbf{1}$	$-\frac{2}{3}$	$(\overline{\mathbf{3}},\mathbf{1})_{-\frac{2}{3}}$
D_R^i	$\overline{\mathbf{3}}$	$\mathbf{1}$	$\frac{1}{3}$	$(\overline{\mathbf{3}},\mathbf{1})_{\frac{1}{3}}$
$\left(E^i, \nu^i\right)_L$	$\mathbf{1}$	$\mathbf{2}$	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{2})_{-\frac{1}{2}}$
E_R^i	$\mathbf{1}$	$\mathbf{1}$	1	$(\mathbf{1},\mathbf{1})_1$

Table 1: Matter Content of the Standard Model

2.2 The Georgi-Glashow Model

Although the low-energy behaviour of particle physics is described by the SM with very high precision, there are indications that a theory may exist, that is based on an underlying gauge group unifying all the gauge interactions present in the SM, called *Grand Unified Theory* (GUT). In general, GUTs postulate the existence of a gauge group G_{GUT} describing interactions in particle physics at high energies and is at low energies spontanously broken into the SM gauge group.
In order to be qualified, G_{GUT} has to satisfy certain conditions

- $SU(3) \times SU(2)_L \times U(1)_Y \subset G_{GUT}$ and $rank(G_{GUT}) \leq rank(SU(3) \times SU(2)_L \times U(1)_Y)$ (such that the symmetry breaking is actually possible)
- existence of complex representations of G_{GUT} (in order to allow a chiral fermion spectrum)

Additionaly, it is commonly assumed to be simple. Upon others, one of the most compelling motivations of considering such theories, is the converging behaviour of the coupling constants of the Standard Model gauge interactions for very high energies (especially in supersymmetric versions of these theories) [4]. Without going into detail about this, we will just present the model.
The simplest candidate for a theory of this type was found by Georgi and Glashow to be a $SU(5)$ GUT, $SU(5)$ being a simple, compact Lie Group satisfying all the necessary properties. In this case, the energy scale at which the grand unification is broken, would be about 10^{14} GeV.

$$SU(5) \longrightarrow SU(3) \times SU(2)_L \times U(1)_Y \tag{10}$$

The gauge bosons of the $SU(5)$ GUT are the 12 from the SM and 12 additional ones, which will be of no importance here. The fermionic matter content is the same as for the SM, but in order to understand the theory when it comes to gauge interactions, we, as before, have to describe how matter that couples to the gauge fields, is arranged in representations of $SU(5)$. As it will be seen, it is enough to do this for one generation and repeat the reasoning two more times for the other ones. Again, we will take all fermions to be left handed Weyl-fermions by identifying ψ_L' with $i\gamma^2\psi_R^*$.
One generation of SM fermions will then be composing the reducible $SU(5)$-representation $\overline{\mathbf{5}} \oplus \mathbf{10}$, build from the antifundamental- and antisymmetric representation, in the following way:

$$\overline{\mathbf{5}} = (\overline{\mathbf{3}}, \mathbf{1})_{\frac{1}{3}} \oplus (\mathbf{1}, \mathbf{2})_{-\frac{1}{2}} \longleftrightarrow \begin{pmatrix} D^*_{red} \\ D^*_{green} \\ D^*_{blue} \\ E \\ \nu \end{pmatrix} = (\psi_i)_L \tag{11}$$

$$\begin{aligned} \mathbf{10} &= (\overline{\mathbf{3}}, \mathbf{1})_{-\frac{2}{3}} \oplus (\mathbf{3}, \mathbf{2})_{\frac{1}{6}} \oplus (\mathbf{1}, \mathbf{1})_1 \\ &\longleftrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & U^*_{blue} & -U^*_{green} & U_{red} & D_{red} \\ -U^*_{blue} & 0 & U^*_{red} & U_{green} & D_{green} \\ U^*_{green} & -U^*_{red} & 0 & U_{blue} & D_{blue} \\ -U_{red} & -U_{green} & -U_{blue} & 0 & E^* \\ -D_{red} & -D_{green} & -D_{blue} & -E^* & 0 \end{pmatrix} = (\chi_{ij})_L \end{aligned} \tag{12}$$

Analogously, the right handed fields could be arranged in terms of $\mathbf{5}$ and $\overline{\mathbf{10}}$. The representation matrices are then generally 5×5 unitary matrices of the form

$$U_{ij} = \left[e^{\frac{i}{2}\alpha^a \xi^a} \right]_{ij} = \begin{pmatrix} \lambda^a & 0 \\ 0 & \tau^a \end{pmatrix} \tag{13}$$

where the ξ^a are the 24 hermitian and traceless generators, λ^a the generators of $SU(3)$ (Gell-Mann Matrices) and τ^a the generators of $SU(2)$ (Pauli Matrices). The first three indices correspond to the $SU(3)$ content, whereas the last two are identified as the $SU(2)_L$ indices. One of the generators is the (properly normalized) hypercharge operator Y. By again analysing the generators, one can find a relation bewtween Y and Q, the standard electromagnetic charge, which yields [4]

$$Q(\psi_i) = Q_i \delta_{ij} \qquad Q(\chi_{ij}) = Q(\psi_i) + Q(\psi_j) \tag{14}$$

3 The Chiral Anomaly in Quantum Field Theory

An important key in the understanding of a Quantum Field Theory (QFT) is the study of symmetries and their associated conserved currents given by Noether's Theorem

$$\partial_\mu j^\mu = 0. \tag{15}$$

However, as we will see, it may occur, that a symmetry valid in the classical field theory, is broken after the process of quantization. If this is the case, the symmetry is said to be anomalous and the current is not conserved on the operator level:

$$\partial_\mu j^\mu \neq 0. \tag{16}$$

At first glance, this should be no problem, since there are many other examples for the violation of classical laws in quantum theories. Unfortunatly, this is not true for anomalies, at least when it comes to anomalous gauge symmetries, as we will see.
In this thesis we will focus on one of the most important examples in this context, the *Chiral Anomaly*, corresponding to the global chiral symmetry in the classical theory. It is essential in the study of models that may be related to the SM at lower energies and provides a useful tool to classify these. There are many different possibilities of deriving the same result, but all of them have something in common: The ultimate origin of what we will call the Anomaly is always some subtlety involved with regularisation of a certain quantity. First, in 2.1 we briefly recover the neccessary symmetries and conservation laws in the associated classical theory. After that, in 2.2, 2.3 and 2.4 we will consider the most simple setup, Quantum Electrodynamics (QED) with only one species of fermion, which turns out to be exemplary in many aspects of the calculations and sketch the derivation of the chiral Anomaly in three different ways: A purely perturbative one by the investigation of special Feynman diagrams, another one by examining the chiral current as an operator itself and the last one by using a non-perturbative approach in the path integral formalism. Detailed investigations of all these methods can be found in [5] and [6]. In 2.5 we will generalize the result in a quite straightforward way to more general gauge theories and in 2.6 we finally consider anomalous gauge symmetries, the ones that actually cause trouble but nevertheless supply a powerful criterium for the consistency of gauge theories.

3.1 The Vector and Axial $U(1)$ Symmetries in Classical Field Theory

Before the examination of symmetries in QFT, we shortly discuss the classical symmetries and conservation laws, that will later be necessary for making the connection to anomalies. Consider an abelian gauge theory for fermionic fields given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}(\gamma^\mu\partial_\mu + ig\gamma^\mu A_\mu)\psi - m\bar{\psi}\psi \tag{17}$$

Here, as one of the most important examples in the context of anomalies, we study the global axial symmetry, so we construct the following currents:

$$\text{vector} \quad j_\mu = \bar{\psi}\gamma_\mu\psi \tag{18}$$

$$\text{axial} \quad j^5_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi. \tag{19}$$

Upon use of the equations of motion, we get the expressions

$$\partial_\mu j^\mu = 0 \tag{20}$$

$$\partial_\mu j^\mu_5 = 2imP \tag{21}$$

with $P = \bar{\psi}\gamma_5\psi$ oftentimes called Pseudoscalar-current. The vector current is actually the associated noether current to the global vector-$U(1)_V$ symmetry, obtained by taking the gauge parameter globally constant

$$\psi \longrightarrow e^{i\alpha}\psi \tag{22}$$

$$\bar{\psi} \longrightarrow \bar{\psi}e^{-i\alpha} \tag{23}$$

If only massless fields are under consideration we get

$$\partial_\mu j^\mu = \partial_\mu j^\mu_5 = 0, \tag{24}$$

So for massless fermions, the axial current is equally conserved and corresponds to the global axial-$U(1)_A$ symmetry

$$\psi \longrightarrow e^{i\alpha\gamma^5}\psi \tag{25}$$

$$\bar{\psi} \longrightarrow \bar{\psi}e^{i\alpha\gamma^5}. \tag{26}$$

Further, we construct the left- and right handed currents

$$j^L_\mu = \bar{\psi_L}\gamma_\mu\psi_L = \bar{\psi}\gamma_\mu\frac{1}{2}(1+\gamma_5)\psi = \frac{1}{2}(j_\mu + j^5_\mu) \tag{27}$$

$$j^R_\mu = \bar{\psi_R}\gamma_\mu\psi_R = \bar{\psi}\gamma_\mu\frac{1}{2}(1-\gamma_5)\psi = \frac{1}{2}(j_\mu - j^5_\mu) \tag{28}$$

for which the conservation laws (again in the massless case) are

$$\partial_\mu j^{\mu,L} = \partial_\mu j^{\mu,R} = 0 \tag{29}$$

on shell.

In case of a non-Abelian gauge theory, we consider Langrangians of the type

$$\mathcal{L} = -\frac{1}{4}\sum_a F^a_{\mu\nu}F^{a\mu\nu} + i\bar{\psi}(\gamma^\mu\partial_\mu + ig\gamma^\mu A^a_\mu T^a)\psi - m\bar{\psi}\psi \tag{30}$$

Analogical to the abelian case, in this setup the currents will have the form

$$\text{vector} \quad j^a_\mu = \bar{\psi}\gamma_\mu T^a\psi \tag{31}$$

$$\text{axial} \quad j^{5a}_\mu = \bar{\psi}\gamma_\mu\gamma_5 T^a\psi. \tag{32}$$

Upon use of the Dirac equations, one gets the the conservation laws

$$\mathcal{D}_\mu j^{\mu a} = 0 \tag{33}$$

$$\mathcal{D}_\mu j^{5\mu a} = 2imP^a. \tag{34}$$

in terms of the covariant derivative $\mathcal{D} = \partial_\mu - igA^a_\mu T^a$ and the non-abelian pseudoscalar current $P^a = \bar{\psi}\gamma_5 T^a\psi$. Again as in the abelian case, for massless fermions these will become the conservation laws

$$\mathcal{D}_\mu j^{\mu a} = \mathcal{D}_\mu j^{5\mu a} = 0 \tag{35}$$

associated to the global vector $SU(N)_V$ symmetry and the global axial $SU(N)_A$ symmetry.

The left- and right handed currents are defined similarly as

$$j^{\mu La} = \bar{\psi}_L\gamma_\mu T^a\psi_L = \frac{1}{2}(j^{\mu a} + j^{\mu 5a}) \tag{36}$$

$$j^{\mu Ra} = \bar{\psi}_R\gamma_\mu T^a\psi_R = \frac{1}{2}(j^{\mu a} - j^{\mu 5a}) \tag{37}$$

For these, there is likewise a conservation law, stating

$$\mathcal{D}_\mu j^{\mu La} = \mathcal{D}_\mu j^{\mu Ra} = 0. \tag{38}$$

3.2 Triangle Diagrams

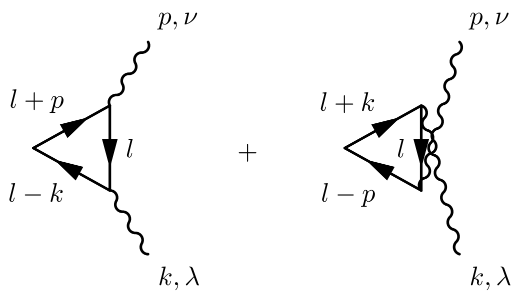
In QFT, however, the validity of conservation laws is described in terms of the Ward-Identity

$$\partial_\mu\langle\Omega|Tj^\mu\prod_{i=1}^n\phi(x_i)|\Omega\rangle = \underbrace{\langle\Omega|T\partial_\mu j^\mu\prod_{i=1}^n\phi(x_i)|\Omega\rangle}_{=0} \\ - i\sum_{i=1}^n\langle\Omega|T\phi(x_1)...\phi(x_{i-1})\delta\phi(x_i)\delta^{(4)}(x-x_i)\phi(x_{i+1})...\phi(x_n)|\Omega\rangle \tag{39}$$

which is the basic equation when it comes to the discussion of symmetries in quantized theories. Now we study, how an anomaly emerges in a purely perturbative approach, by the investigation of the QED matrix element of j_5^μ between the vacuum and a two photon state at one-loop level. So consider the correlation function in momentum space

$$\int d^4 x e^{-iqx} \langle p,k|j^{\mu 5}(x)|\Omega\rangle = (2\pi)^4 \delta^{(4)}(p+k-q)\epsilon_\nu^*(p)\epsilon_\lambda^*(k)\mathcal{M}^{\mu\nu\lambda}(p,k), \quad (40)$$

which can be visualized in terms of Feynman Diagrams as



(41)

to first order. This is the case, why this anomaly is often called *Triangle Anomaly*. Upon use of the Feynman rules we get for the amplitude the value

$$\mathcal{M}^{\mu\nu\lambda} = (-1)(-ie)^2 \int \frac{d^4 l}{(2\pi)^4} \left\{ tr\left[\gamma^\mu\gamma^5 \frac{i(\not{l}-\not{k})}{(l-k)^2}\gamma^\lambda \frac{i\not{l}}{l^2}\gamma^\nu \frac{i(\not{l}+\not{p})}{(l+p)^2}\right] + tr\left[\gamma^\mu\gamma^5 \frac{i(\not{l}-\not{p})}{(l-p)^2}\gamma^\nu \frac{i\not{l}}{l^2}\gamma^\lambda \frac{i(\not{l}+\not{k})}{(l+k)^2}\right]\right\}. \quad (42)$$

To check, if the Ward identity is satisfied, we analyse the equation

$$\int d^4 x e^{-iqx} \langle p,k|\partial_\mu j^{\mu 5}(x)|\Omega\rangle = \int d^4 x e^{-iqx} \partial_\mu \langle p,k|j^{\mu 5}(x)|\Omega\rangle = (2\pi)^4 \delta^{(4)}(p+k-q)\epsilon_\nu^*(p)\epsilon_\lambda^*(k) i q_\mu \mathcal{M}^{\mu\nu\lambda}(p,k) \quad (43)$$

which is true due to partial integration and the non-occurance of contact terms. Therefore, the Ward identity will be satisfied, if

$$i q_\mu \mathcal{M}^{\mu\nu\lambda} = 0. \quad (44)$$

Using the identity

$$q_\mu \gamma^\mu \gamma^5 = (\not{l}+\not{p}-\not{l}+\not{k})\gamma^5 = (\not{l}+\not{p})\gamma^5 + \gamma^5(\not{l}-\not{k}), \quad (45)$$

transforming the integration variable via $l \longrightarrow (l+k)$ or $l \longrightarrow (l+p)$ and after rearrangement of some gamma matrices, both terms in (42) would cancel. However, a more careful analysis reveals that there is a problem with this reasoning: Since the integrals are linearly divergent, the shift is actually not allowed, so we have to take a regularisation procedure into account. There are many possibilities to do this, but here only dimensional regularisation will be presented (for other methods we refer to [6]). So we first evaluate the integral in N dimensions and all the divergences occur as poles in dimension $n = 4$. These may cancel in the end by taking the limit $n \longrightarrow 4$. In the following, we use the conventions:

$$\{\gamma_5, \gamma_\mu\} = 0 \qquad ; \mu = 0, 1, 2, 3 \tag{46}$$

$$[\gamma_5, \gamma_\mu] = 0 \qquad ; \mu = 4, 5, ..., n-1. \tag{47}$$

with the definition $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. All other properties of the γ matrices are inherited from the four dimensional ones. In this setup, the external momenta p and k as well as the indices ν and λ are four dimensional, whereas the internal loop-momentum l takes values in all dimensions $i = 0, ..., n$. Next, we have to modify the integral in a suitable way, so we split the internal momentum into components l for $n = 0, 1, 2, 3$ and L for $i = 4, ..., n$, such that the measure in (42) becomes

$$\int \frac{d^4 l}{(2\pi)^4} \int \frac{d^{n-4} L}{(2\pi)^{n-4}}. \tag{48}$$

To repeat our previous aproach, we have to modify the identity (45) to

$$q_\mu \gamma^\mu \gamma^5 = (\not{l} + \not{L} + \not{k})\gamma^5 + \gamma^5(\not{l} + \not{L} - \not{p}) - 2\gamma^5 \not{L} \tag{49}$$

where we used the conventions (46) and (47). By applying the same logic as before, the shift of the integral variables is now actually allowed due to the regularisation, but the additional term in (49) yields a further constribution, which can be explicitly evaluated and found to be non-vanishing, the origin of the anomaly. We end up with

$$iq_\mu \mathcal{M}^{\mu\nu\lambda} = \mathcal{A}^{\nu\lambda} \tag{50}$$

with the *Anomaly*

$$\mathcal{A}^{\nu\lambda} = -\frac{e^2}{2\pi^2}\epsilon^{\alpha\nu\beta\lambda} p_\alpha k_\beta, \tag{51}$$

where the ϵ tensor stems from the trace over the involved γ matrices.
This contribution to the integral spoils the Ward Identity of the classical global axial symmetry in the quantized theory and (50) is called the *anomalous axial Ward identity.* To get this into a form, which we can compare to our results in the following chapters, consider

$$\langle p,k|\partial_\mu j^{5\mu}|\Omega\rangle = -\frac{e^2}{2\pi^2}\epsilon^{\alpha\nu\beta\lambda}(-ip_\alpha)(-ik_\beta)\epsilon^*_\nu(p)\epsilon^*_\lambda(k) = \\ -\frac{e^2}{16\pi^2}\langle p,k|\epsilon^{\alpha\nu\beta\lambda}F_{\alpha\nu}F_{\beta\lambda}|\Omega\rangle, \quad (52)$$

or formulated as an operator equation

$$\boxed{\partial_\mu j^{5\mu} = -\frac{e^2}{16\pi^2}\epsilon^{\alpha\nu\beta\lambda}F_{\alpha\nu}F_{\beta\lambda}.} \quad (53)$$

This final result is called the *Adler-Bell-Jackiw (ABJ-)Anomaly* and it is actually independent of the chosen regularisation method.
The next question is of course, whether there are further radiative corrections in higher perturbation orders. The answer can be stated as follows:

In QED, the ABJ-Anomaly is one-loop exact, i.e. it recieves no contributions in higher order perturbation theory.

This was proven by Adler and Bardeen and is also true for other theories, e.g. QCD. In more general ones, this is not necessarily the case.

3.3 The Point Splitting Method

Another method of deriving the ABJ-Anomaly is to investigate the current and its conservation on an operator level. Consider massless QED, such that $U(1)_A$ is a symmetry and therefore

$$\partial_\mu j^{5\mu} = 0. \quad (54)$$

As in the perturbative approach, there is a subtle problem with the regularisation involved: Taking the dependence on spacetime coordinates and the dirac spinor inidices into account, we get

$$j^{5\mu}(x) = \bar\psi(x)\gamma^\mu\gamma^5\psi(x) = \psi^\dagger_A(x)(\gamma^0)^A_B(\gamma^\mu)^B_C(\gamma^5)^C_D\psi^D(x) \\ = \psi^\dagger_A(x)\psi^D(x)(\gamma^0)^A_B(\gamma^\mu)^B_C(\gamma^5)^C_D = \delta^D_A\delta^{(3)}(x-x)(\gamma^0)^A_B(\gamma^\mu)^B_C(\gamma^5)^C_D \\ -\psi^D(x)\psi^\dagger_A(x)(\gamma^0)^A_B(\gamma^\mu)^B_C(\gamma^5)^C_D \quad (55)$$

Because of the delta-function the chiral current operator is actually a singular object and is therefore reliant on regularisation. This can be done by using the so called *Point-Splitting Method*: we define a regularized version of the current operator by splitting the spacetime dependence of the fermion field apart by some small distance ϵ^μ, using a Wilson Line. The latter is needed in order to retain gauge invariance. So we define

$$j^{5\mu}(x,\epsilon) = \bar{\psi}(x+\frac{\epsilon}{2})\gamma^\mu\gamma^5\psi(x-\frac{\epsilon}{2})e^{ie\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} dy^\nu A_\nu(y)} \tag{56}$$

and the regularized chiral current

$$j^{5\mu}_{reg}(x) = \lim_{\epsilon\to 0} j^{5\mu}(x,\epsilon). \tag{57}$$

To test the conservedness of the regularised current, we will first perform the calculation with non-vanishing ϵ and in the end take the limit $\lim_{\epsilon\to 0}\partial_\mu j^{5\mu}(x,\epsilon)$. We get

$$\begin{aligned}\partial_\mu j^{5\mu}(x,\epsilon) = \bar{\psi}(x+\frac{\epsilon}{2})\overleftarrow{\not\partial}\gamma^5\psi(x-\frac{\epsilon}{2})e^{ie\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} dy^\nu A_\nu(y)}\\ - \bar{\psi}(x+\frac{\epsilon}{2})\gamma^5\not\partial\psi(x-\frac{\epsilon}{2})e^{ie\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} dy^\nu A_\nu(y)}\\ + \bar{\psi}(x+\frac{\epsilon}{2})\gamma^\mu\gamma^5\psi(x-\frac{\epsilon}{2})\partial_\mu e^{ie\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} dy^\nu A_\nu(y)}\end{aligned} \tag{58}$$

and by using the Dirac equations for massless fields

$$(i\not\partial + e\not A)\psi = 0 \tag{59}$$

$$\bar{\psi}(i\overleftarrow{\not\partial} - e\not A) = 0 \tag{60}$$

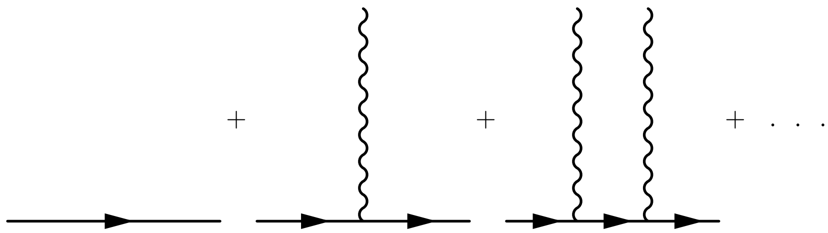
we obtain

$$\begin{aligned}\partial_\mu j^{5\mu}(x,\epsilon) &= -iej^{5\mu}(x,\epsilon)\left[-\partial_\mu\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} dy^\nu A_\nu(y) + A_\mu(x+\frac{\epsilon}{2}) - A_\mu(x-\frac{\epsilon}{2})\right]\\ &= -iej^{5\mu}(x,\epsilon)\epsilon^\nu\left[\partial_\nu A_\mu(x) - \partial_\mu A_\nu(x)\right]\\ &= -iej^{5\mu}(x,\epsilon)\epsilon^\nu F_{\mu\nu}\end{aligned} \tag{61}$$

where we expanded the fields A_μ to first order. It is not yet possible to take the limit $\lim_{\epsilon\to 0}\partial_\mu j^{5\mu}(x,\epsilon)$ since the expression is still singular for $\epsilon \longrightarrow 0$. For the further investigation, we look at the expectation value (the quantity we actually care about) of (61) and treat A_μ not as a gauge field, but rather as an external field (because we are not interested in the dynamics of A_μ itself), we find

$$\langle\Omega|\partial_\mu j^{5\mu}(x,\epsilon)|\Omega\rangle = -ie\langle\Omega|j^{5\mu}(x,\epsilon)|\Omega\rangle\epsilon^\nu F_{\mu\nu}. \tag{62}$$

The expectation value on the right hand side is basically the fermion propagator between $x-\frac{\epsilon}{2}$ and $x+\frac{\epsilon}{2}$ in presence of the background field A_μ. It will turn out to be very useful, to expand this quantity in terms of A_μ, which can be diagramatically depicted as



(63)

The leading contribution of this series will be the second term, for it being still divergent and all other either do not supply the right amount of γ's or are regular.

The evaluation of this expression via the Feynman rules yields

$$\langle\Omega|\partial_\mu j^{5\mu}(x,\epsilon)|\Omega\rangle = -\frac{e^2}{4\pi^2}\epsilon^{\alpha\beta\mu\gamma}F_{\alpha\beta}F_{\mu\nu}\frac{\epsilon_\gamma\epsilon^\nu}{\epsilon^2} \tag{64}$$

The final step of the regularisation procedure is now to take the limit $\epsilon \longrightarrow 0$. However, this has to be done with caution: Because $j^{5\mu}(x,\epsilon)$ has to exhibit the same transformation behaviour under Lorentz transformations as $j^{5\mu}(x)$ for all ϵ, we have to take the limit in a symmetric way and therefore require

$$\lim_{\epsilon\to 0}\left\{\frac{\epsilon_\gamma\epsilon^\nu}{\epsilon^2}\right\} = \frac{g_\gamma^\nu}{n}, \tag{65}$$

where g_γ^ν is the metric tensor and n the spacetime dimension, in our case $n = 4$. Performing the limit in this manner and thereby finishing the analysis, one ends up with the final result as an operator equation

$$\boxed{\partial_\mu j^{5\mu}(x,\epsilon) = -\frac{e^2}{16\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}.} \tag{66}$$

This non-vanishing, finite expression is exactly the same anomalous non-conservation of the axial current that we already encountered in the study of triangle diagrams, the Adler-Bell-Jackiw Anomaly. It was again a proof by Adler and Bardeen, which ensures that equally in this method there are no further contributions by higher perturbation orders in (63) at least, as before, in QED. In this case the ABJ-Anomaly is a correct expression up to all orders in perturbation theory.

3.4 The Fujikawa Method

In the path integral formalism of quantum field theory, the fundamental quantity is the generating functional for the Green function, a path integral containing the classical action. But since the action is invariant under symmetry transformations, it must be the measure, which destroys the classical symmetry and originates the anomaly. The derivation of the chiral Anomaly in this context was introduced by Fujikawa [7] and is called the *Fujikawa Method*.

The Ward-Identity only holds for symmetries, that leave the measure invariant, in this way, as we will see, the non-invariance of the measure under chiral transformations will cause the non conservedness of the chiral current. As always, the transformed measure requires a regularization process in order to be welldefined, which will be the cause of the problem. In this way of dealing with the anomaly, we will not have to rely on perturbation theory, therefore this method is also called the non-perturbative approach. Consider massless QED

$$\mathcal{L} = i\bar{\psi}(\gamma^\mu \partial_\mu - ig\gamma^\mu A_\mu)\psi = i\bar{\psi}\not{\mathcal{D}}\psi, \tag{67}$$

where the dynamics of the gauge field A_μ is omitted, since it is irrelevant for the following.
In order to guarantee the path integrals convergence, we first perform a Wick Rotation to euclidian space

$$ix^0 = x^4, \tag{68}$$

define the metric to be

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -\delta^{\mu\nu} \tag{69}$$

and all the γ-matrices anti-hermitian (which is always possible), such that the Dirac operator is hermitian

$$\not{\mathcal{D}}^\dagger = (\gamma^\mu \mathcal{D}_\mu)^\dagger = (g^{\mu\nu}\gamma_\nu \mathcal{D}_\mu)^\dagger = -\mathcal{D}_\mu \gamma_\mu^\dagger = \mathcal{D}_\mu \gamma_\mu = \not{\mathcal{D}} \tag{70}$$

As oultined in the beginning, we study the generating functional (disregarding for now $\mathscr{D}A$ and the source terms)

$$Z = \int \mathscr{D}\psi \mathscr{D}\bar{\psi} e^{-\int d^4x \; i\bar{\psi}\not{\mathcal{D}}\psi}. \tag{71}$$

By applying the infinitesimal version of the chiral transformation with the gauge parameter taken dependent on spacetime coordinates for the moment $\beta \longrightarrow \beta(x)$

$$\psi \longrightarrow (1 + i\beta(x)\gamma^5)\psi \tag{72}$$

$$\bar{\psi} \longrightarrow \bar{\psi}(1 + i\beta(x)\gamma^5) \tag{73}$$

we get the implication

$$\delta S = 0 \Rightarrow \partial_\mu j^{5\mu} = 0, \tag{74}$$

by varying with respect to $\beta(x)$.
The corresponding Ward identity can be derived by varying with respect to the sources in the generating functional. For the latter, however, we further

have to check, if the path integral measure $\mathscr{D}\psi\mathscr{D}\bar{\psi}$ stays invariant under the transformations (72) and (73).
In order to do this, we will bring into into a more accessible form by decomposing ψ and $\bar{\psi}$ into eigenfunctions of $\not{D}$

$$\not{D}\varphi_n(x) = \lambda_n \varphi_n(x). \tag{75}$$

Because $\not{D}$ is hermitian, the λ_n are real and the $\{\varphi_n\}$ form a orthonormal and complete eigenbasis

$$\int d^4x \varphi_i^\dagger(x)\varphi_j(x) = \delta_{ij} \tag{76}$$

$$\sum_i \varphi_i(x)\varphi_i(y)^\dagger = \delta(x-y) \tag{77}$$

such that the spinors can bewritten in terms of $\{\varphi_n\}$ as

$$\psi(x) = \sum_i a_i \varphi_i(x) \tag{78}$$

$$\bar{\psi}(x) = \sum_j \varphi_j^\dagger(x)\bar{b}_j \tag{79}$$

where a and $\bar{b}$ are independent, Grassmann valued coefficients and we leave the spinor indices of $\varphi_i^A(x)$ and $\varphi_{iA}^\dagger$ implicit. The sum runs over the indices of the infinite Hilbertspace basis. The path integral measure can then be defined as

$$\mathscr{D}\psi\mathscr{D}\bar{\psi} = \prod_i da_i d\bar{b}_i \tag{80}$$

by choosing the arbitrary normalisation factor to be unity. We find, that under the chiral transformations (72) and (73) the measure transforms as

$$\begin{aligned}\mathscr{D}\psi'\mathscr{D}\bar{\psi}' = \prod_i da'_i d\bar{b}'_i = \frac{1}{det(C)^2}\prod_i da_i d\bar{b}_i = \frac{1}{det(C)^2}\mathscr{D}\psi\mathscr{D}\bar{\psi} \\ = J[\beta]\mathscr{D}\psi\mathscr{D}\bar{\psi}\end{aligned} \tag{81}$$

with the Jacobian $J[\beta]$ and keeping in mind that Grassmann measures transforming with the inverse determinant of the transformation matrix. The Jacobian is given by

$$J[\beta] = e^{-2i\int d^4x\ \beta(x)\sum_i \varphi_i^\dagger(x)\gamma^5\varphi_i(x)} \tag{82}$$

which immediately causes trouble, since upon use of the completeness relation of the eigenbasis (77) the sum becomes

$$\sum_i \varphi_i^\dagger(x)\gamma^5\varphi_i(x) = tr(\gamma^5)\delta^{(4)}(0), \tag{83}$$

where the trace comes from the implicit spinor indices. Despite the naive guess to assume $tr(\gamma^5) = 0$, therefore the Jacobian to be trivial, the measure to transform identically and finally the Ward Identity to hold and not to be anomalous, the Delta function in (83) is responsible for the integral in (82) not to be well-defined, so it actually needs to be regularised properly in a gauge invariant way, which ultimately causes the anomaly in this method.

One possibility to do so, is to introduce the Gaussian cutoff $e^{-\frac{\lambda_i^2}{\Lambda^2}}$, which, for finite Λ regulates the contributions of modes with large eigenvalues. So we rewrite the sum as

$$\sum_i \varphi_i^\dagger(x)\gamma^5\varphi_i(x) = \lim_{\Lambda\to\infty}\sum_i \varphi_i^\dagger(x)\gamma^5 e^{-\frac{\lambda_i^2}{\Lambda^2}}\varphi_i(x), \tag{84}$$

or equivalently

$$\sum_i \varphi_i^\dagger(x)\gamma^5\varphi_i(x) = \lim_{\Lambda\to\infty}\sum_i \varphi_i^\dagger(x)\gamma^5 e^{-\frac{\not{D}^2}{\Lambda^2}}\varphi_i(x), \tag{85}$$

which exactly gives (84) by acting with $\not{D}^2$ on the right and left respectively, so this method is gauge invariant, as required. In Fourier space, the calculation leads to the Jacobian

$$J[\beta] = e^{-\int d^4x\ \beta(x)\mathcal{A}(x)} \tag{86}$$

with

$$\mathcal{A}(x) = \frac{-ie^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \tag{87}$$

or after performing another Wick Rotation to Minkowski Space

$$\mathcal{A}(x) = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}. \tag{88}$$

So the generating functional after the full tranformation of the action *and* the measure will be of the form

$$Z = \int \mathscr{D}\psi\mathscr{D}\bar{\psi}e^{-\int d^4x\left[\ i\bar{\psi}\not{D}\psi+\beta(x)\left(\partial_\mu j^{5\mu}+\mathcal{A}(x)\right)\right]}, \tag{89}$$

by varying, we get the ABJ-Anomaly as a final result, like in the perturbative approaches before, as an exact expression:

$$\boxed{\partial_\mu j^{5\mu} = -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}.} \tag{90}$$

The standard way of calculating the Green function out of the generating functional leads to the corresponding, non-fulfilled Ward Identity.

3.5 Generalisation

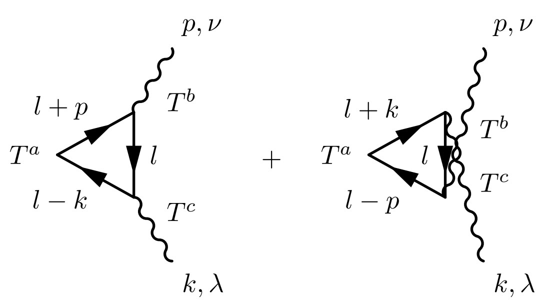
Until now, we only considered QED, so the next logical step is to expand these concepts to more general theories, i.e. theories with some arbitrary gauge group G. It may be, as in QED, Abelian, but with more than one generator, or non-Abelian with more than one generator and non-trivial commutation relations $\left[T^a, T^b\right] = f^{abc}T^c$. As before, the three different approaches presented before are applicable and will yield the same result. The most convenient and straightforward way to arrive at the general result for the anomaly is to use the perturbative method in terms of triangle diagrams, so consider the gauge theory of massless fermions

$$\mathcal{L} = -\frac{1}{4}\sum_a F^a_{\mu\nu}F^{a\mu\nu} + i\bar{\psi}\not{\mathcal{D}}\psi \tag{91}$$

with $\mathcal{D}_\mu = \partial_\mu - ieA^a_\mu T^a$ and the generators T^a_R (in a certain representation of the Lie Algebra) of the non-Abelian gauge group G, satisfying the standard commutation relations. The associated currents to the global vector and axial symmetry are given by (31) and (32) and in the classical massless theory both are covariantly conserved (35). In order to compute the Anomaly we look at the expectation value of the non Abelian curren $j^{5\mu a}$ between the vacuum and a state of two gauge bosons

$$\langle p, k|j^{\mu 5a}(x)|\Omega\rangle \tag{92}$$

and depict it in terms of Feynman Diagrams to first order in perturbation theory as



(93)

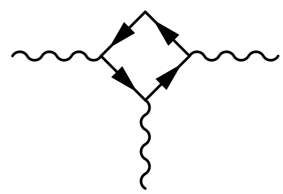
where now each vertex is equipped with an additional generator of the Lie-Algebra $Lie(G)$ of G in some representation R. Upon use of the Feynman rules we get for the Amplitude

$$\begin{aligned}\mathcal{M}^{\mu\nu\lambda} = (-1)(-ie)^2\int\frac{d^4l}{(2\pi)^4}\Bigg\{&tr[T^a_RT^b_RT^c_R]tr\left[\gamma^\mu\gamma^5\frac{i(\not{l}-\not{k})}{(l-k)^2}\gamma^\lambda\frac{i\not{l}}{l^2}\gamma^\nu\frac{i(\not{l}+\not{p})}{(l+p)^2}\right]\\ &+tr[T^a_RT^c_RT^b_R]tr\left[\gamma^\mu\gamma^5\frac{i(\not{l}-\not{p})}{(l-p)^2}\gamma^\nu\frac{i\not{l}}{l^2}\gamma^\lambda\frac{i(\not{l}+\not{k})}{(l+k)^2}\right]\Bigg\},\end{aligned} \tag{94}$$

so after a similar evaluation as in the QED case, it only changes up to the symmetric factor $tr\left[T_R^a\left\{T_R^b, T_R^c\right\}\right]$. The further calculation also proceeds as in QED and by taking the definition of the non-Abelian Field Strength $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - ef^{abc}\left[A^b_\mu, A^c_\nu\right]$ into account, the *general ABJ Anomaly* takes the form

$$\boxed{\mathcal{D}_\mu j^{5\mu a} = -\frac{e^2}{16\pi^2} tr\left[T_R^a\left\{T_R^b, T_R^c\right\}\right]\epsilon^{\mu\nu\alpha\beta}F^b_{\mu\nu}F^c_{\alpha\beta}.} \tag{95}$$

This is actually no exact result, contrary to the results in QED. There are contributions from higher order loops like for example



(96)

However, since we are only interested in gauge anomalies, that we demand to vanish completely anyway, this poses no further complication: If the triangle anomaly vanishes, so do all higher order loop anomalies, therefore we do not have to care about them [6].
If the gauge group is of the form

$$G = G_1 \times ... \times G_k, \tag{97}$$

for some $k \in \mathbb{N}$, like it is the case in the SM and the theories we consider later, the situation is like this:
If all the groups in (97) are connected (what is true for all groups we will discuss), then $Lie(G) = Lie(G_1) \times ... \times Lie(G_k)$, so for a representation ρ of $Lie(G)$ we have

$$\rho(Lie(G)) = \rho(Lie(G_1)\times ... \times Lie(G_k)) = \rho_1(Lie(G_1))\otimes ... \otimes \rho_k(Lie(G_k)), \tag{98}$$

where $\rho_1 ... \rho_k$ are independent representations of the different components. A gauge theory based on such a gauge group may also produce triangle anomalies of the type (93), but now the Lie-Algebra generators could possibly stem from different groups in the product (98). If this is the case and e.g. $T^a \in G_1$, $T^b \in G_2$ and $T^c \in G_3$, the triangle anomaly is said to be a $G_1 - G_2 - G_3$-Anomaly and is denoted as $\mathscr{A}_{G_1-G_2-G_3}$.
In more realistic models, where more than one species of fermionic fields are present, i.e. the fermionic part of the Lagrangian is of the form

$$\mathcal{L}_\psi = i \sum_i \bar{\psi}_i \not{D} \psi_i, \tag{99}$$

the axial current is given by

$$j_5^{\mu a} = \sum_i j_{i5}^{\mu a} = \sum_i \bar{\psi}_i \gamma^\mu \gamma_5 T_R^a \psi_i. \tag{100}$$

Likewise, its conservation is anomalous. Except for the factor $tr\left[T_R^a\left\{T_R^b, T_R^c\right\}\right]$, the single results coincide, so we end up with

$$\mathcal{D}_\mu j_5^{\mu a} = -\sum_R tr\left[T_R^a\left\{T_R^b, T_R^c\right\}\right] \frac{e^2}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}^b F_{\alpha\beta}^c. \tag{101}$$

Here, the sum over R is to be understood as a formal some over all fermions ψ_i, in some representation of the gauge group R, that contribute to the Anomaly. The factor $tr\left[T_R^a\left\{T_R^b, T_R^c\right\}\right]$ is of great usefullness and denoted as $\mathcal{A}(R)$. It deserves further treatment, which will become helpful later and will be done in the next chapter.

3.6 Gauge Anomalies

All until now, although we saw that the conservation laws of these currents were spoiled, they gave not rise to conceptual problems at all. But finally we are capable of building the bridge to gauge anomalies and explaining the role of the chiral anomaly in this context, using these general expressions. To do this, we have to look at chiral theories, so consider a gauge theory of Dirac fermions and the fermionic part of the Lagrangian

$$\mathcal{L}_\psi = i\bar{\psi}\not{D}\psi = i\bar{\psi}\not{D}\frac{1}{2}(1-\gamma^5)\psi + i\bar{\psi}\not{D}\frac{1}{2}(1+\gamma^5)\psi = i\bar{\psi}_L\not{D}\psi_L + i\bar{\psi}_R\not{D}\psi_R. \tag{102}$$

In case of a chiral theory, the ψ_L transform in a representation R_L of the gauge group and the ψ_R in a different representation R_R. As in the SM and the $SU(5)$-GUT, we will take all fermions as left handed Weyl-fermions by identifying ψ_L' with $-i\gamma^2\psi_R^*$. In this manner we can rewrite (102) as

$$\mathcal{L}_\psi = i\bar{\psi}_L\not{D}\psi_L, \tag{103}$$

where in this setup, ψ_L is in a reducible representation $R = R_L \oplus \bar{R}_R \neq \bar{R}_L \oplus R_R = \bar{R}$. So the final Lagrangian will be

$$\mathcal{L} = -\frac{1}{4}\sum_a F_{\mu\nu}^a F^{a\mu\nu} + i\bar{\psi}_L\not{D}\psi_L, \tag{104}$$

which enjoys a local vector gauge symmetry $U(1)_V$ with associated current

$$j_L^{\mu a} = \bar{\psi}_L \gamma^\mu T_R^a \psi_L, \tag{105}$$

the left handed vector current (37). As in (37), this can be written in terms of the vector-current and axial-current as

$$j_L^{\mu a} = j^{\mu a} + j_5^{\mu a} = \frac{1}{2}\bar{\psi}\gamma^\mu T_R^a \psi - \frac{1}{2}\bar{\psi}\gamma^\mu \gamma_5 T_R^a \psi. \tag{106}$$

This is the point, where the axial current and its anomaly becomes a problem:
By investigating the conservation of the involved $j^{\mu a}$ and $j_5^{\mu a}$, we get, as before

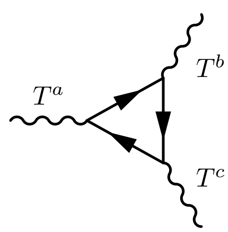
$$\mathcal{D}_\mu j^{\mu a} = 0 \tag{107}$$

$$\mathcal{D}_\mu j_5^{\mu a} = -\frac{e^2}{16\pi^2} tr\left[T_R^a \left\{T_R^b, T_R^c\right\}\right] \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}^b F_{\alpha\beta}^c, \tag{108}$$

and consequently, a non-conservation of the left handed current (105)

$$\begin{aligned}\mathcal{D}_\mu j_L^{\mu a} &= -\frac{1}{2}\frac{e^2}{16\pi^2} tr\left[T_R^a \left\{T_R^b, T_R^c\right\}\right] \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}^b F_{\alpha\beta}^c \\ &= -\frac{e^2}{32\pi^2} tr\left[T_R^a \left\{T_R^b, T_R^c\right\}\right] \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}^b F_{\alpha\beta}^c, \end{aligned}\tag{109}$$

corresponding to the triangle anomaly of left handed fields



(110)

Note, that the generators T^a, T^b and T^c could possibly belong to different groups in a product like (97). The triangle anomaly would then couple to the respective gauge bosons of these groups. Since this left handed current is the associated Noether current to the global version of the gauge symmetry, its anomalous non-conservation corresponds to an anomaly of the latter, a *Gauge Anomaly*. This is not tolarable, because the gauge invariance is necessary for the ghost states, that arrise in the quantisation of the theory, to decouple from all physical processes and therefore for the consistency of the theory. Consequently, we have to claim, that all gauge Anomalies of this type have to vanish in the following sense, such that the gauge symmetry is maintained:

A chiral gauge theory is consistent only if $\mathcal{A}(R)$ vanishes

$$\boxed{\mathcal{A}(R) = tr\left[T_R^a \left\{T_R^b, T_R^c\right\}\right] = 0.} \tag{111}$$

This is called the *Anomaly Consistency Condition.*

If more than one fermionic field are charged under the corresponding symmetries and contribute to the anomaly, we may use the expression (101) and the Anomaly consistency condition gets modified to

$$\boxed{\sum_R \mathcal{A}(R) = \sum_R tr\left[T_R^a\left\{T_R^b, T_R^c\right\}\right] = 0.} \tag{112}$$

Some comments are in order:

- If $T_R^a \in Lie(G)$ is a generator in a representation R, then it is in the conjugate representation $\overline{R}$

$$T_{\overline{R}}^a = -(T_R^a)^* = -(T_R^a)^T, \tag{113}$$

  so the $\mathcal{A}(\overline{R})$ can be written as

$$\begin{aligned}\mathcal{A}(\overline{R}) = tr\left[T_{\overline{R}}^a\left\{T_{\overline{R}}^b, T_{\overline{R}}^c\right\}\right] &= tr\left[-(T_R^a)^T\left\{-(T_R^a)^T, -(T_R^a)^T\right\}\right] \\ &= -tr\left[\left\{T_R^c, T_R^b\right\}T_R^a\right] = -\mathcal{A}(R),\end{aligned} \tag{114}$$

  by using the cyclicity of the trace. So if the representation is real or pseudoreal $R = \overline{R}$, the corresponding Anomaly will vanish.

- Generally, if the gauge group is $SU(n)$ for $n \geq 3$, $\mathcal{A}(R)$ is proportional to a totally symmetric invariant of the group which is independent of the representation,

$$\mathcal{A}(R) = \frac{1}{2}A(R)d^{abc}. \tag{115}$$

  $A(R)$ is called the *Anomaly Coefficient* and groups which do not provide such a nonzero invariant are “safe” when it comes to anomalies to begin with. If this is not the case, the anomaly coefficient can be expressed as the cubic Casimir operator of this representation and if one of the generators in $\mathcal{A}(R)$ stems from another group, it is proportional to the quadratic Casimir operator $C(R)$. For the Lie groups, that commonly occur and for later purposes, we give these invariants in the following table:

Representation R	$dim(R)$	$C(R)$	$A(R)$
F	N	$\frac{1}{2}$	1
$\overline{F}$	N	$\frac{1}{2}$	-1
A	$\frac{N(N-1)}{2}$	$\frac{N+2}{2}$	$N-4$
$\overline{A}$	$\frac{N(N-1)}{2}$	$\frac{N+2}{2}$	$-N+4$

Table 2: Quadratic and cubic Casimirs for $SU(N)$, $N \geq 3$, for the fundamental, antifundamental, antisymmetric and the conjugate antisymmetric representations

For all gauge groups, for which complex representations exist and $d^{abc} \neq 0$ there is a possibility for gauge anomalies to exist, therefore the anomaly consistency condition poses a constraint on the actual matter content, that is allowed in chiral gauge theories. It is therefore the main result in this section and will be used later in order to pose constraints on the models from F-Theory.

Examples

- The Standard Model:
  Since in the SM gauge group the conditions for the existence of gauge anomalies are satisfied, one has to check explicitly if the contributions to the chiral anomaly indeed cancel each other. Happily, this is the case: For a detailed calculation see [8] or [9].

- The Georgi-Glashow Model:
  Also here, gauge anomalies do occur and have to cancel. Since d^{abc} is an invariant we can show that this model is anomaly free in the following way: Take as $T^a = T^b = T^c = Q$ the charge operator. Then

$$\frac{\mathcal{A}(\overline{\mathbf{5}})}{\mathcal{A}(\mathbf{10})} = \frac{tr\ Q^3(\psi_i)}{tr\ Q^3(\chi_{ij})} = -1, \tag{116}$$

  by inserting the corresponding charges. And therefore

$$\mathcal{A}(\overline{\mathbf{5}}) + \mathcal{A}(\mathbf{10}) = 0, \tag{117}$$

  which is independent of the choose of generator or representation, so the gauge anomalies cancel for each family respectively.

4 Chirality and Flux

Before discussing the models from F-Theory, it will be neccessary to explain some concepts, that will be very important in this context.
As we saw in the SM and in the $SU(5)$ GUT, one of the most distinctive properties of realistic models of nature is, that they provide a chiral spectrum of fermionic matter. However, in string theory this statement translates into having a chiral spectrum of fermionic matter *in four dimensions*, since in order for these theories to be consistent one has to introduce a certain amount of additional dimensions, in our cases six. What can then be observed directly by a four dimensional physicist are only those phenomena taking place in four dimensions. Spacetime is then described as a ten dimensional manifold $\mathcal{M}$, which in the simplest case can be thought of as a product

$$\mathcal{M} = M^{3,1} \times X^6, \tag{118}$$

where $M^{3,1}$ is usually Minkowskian spacetime and X^6 is a Calabi-Yau manifold, i.e. a compact Kähler manifold satisfying some further properties (6 is the real dimension).
As it turned out, the presence of chirality in $M^{3,1}$ is strongly related to the topology of X^6, which essentially originates from the statement of the Atiyah-Singer Index Theorem. Although all of the following could be equally done for six dimensions, for our purpose it is sufficient to restrict on a two dimensional subspace of X^6 which will in general have the form of a compact Riemann surface. The reason for this will be given in the next section.
After discussing what it means to be a fermion in six dimension in 3.1 as well as outlining the general idea, we will have to switch to a more suitable formulation in terms of differential geometry in 3.2. As it will be seen, the central object of study is the Dirac operator in curved spacetime, its connection to topology is given in 3.3. Lastly, the correspondence with the mathematical formulation of magnetic monopoles will be shown in 3.4.

4.1 Fermions in Six Dimensions

In the sequel, we will consider six dimensional spacetime with a topology

$$\mathcal{M} = M^{3,1} \times \Sigma_g, \tag{119}$$

where Σ_g is a compact Riemann surface of genus g and $M^{3,1}$ is Minkowskian spacetime. The labeling of the coordinates is given by $M, N = 0, .., 5$, $\mu, \nu = 0, ...3$ and $i, j = 4, 5$.
The Tangent space of $T_p\mathcal{M} = T_pM^{3,1} \times T_p\Sigma_g$ on each point $p \in \mathcal{M}$ is a vector space with orthogonal transformations given by the groups

- $SO(3, 1)$ for $T_pM^{3,1}$
- $SO(2)$ for $T_p\Sigma_g$

- $SO(5,1)$ for $T_p\mathcal{M}$

so in order to specify the matter content of the theory when it comes to the spin of the fields in $\mathcal{M}$, we have to choose a representation ρ of $SO(5,1)$. The main interest lies on the spinor representation, so we introduce the gamma matrices in six spacetime dimensions Γ_M and demand them to satifsy the anti-commutation relations

$$\left\{\Gamma^M, \Gamma^N\right\} = 2g^{MN}. \tag{120}$$

With this choice, the spinor representation of $SO(5,1)$ will be analogously to the four dimensional case given by

$$S_{MN} = \frac{i}{4}\left[\Gamma_M, \Gamma_N\right], \tag{121}$$

where the Dirac spinors are elements of the corresponding representation space of dimension $2^3 = 8$.
The representation of the Clifford Algebra $Cliff(5,1)$ itself, can be chosen in such a way, that for the components in Minkowskian spacetime Γ^μ the representation matrices $S^{\mu\nu}$ coincide with the usual ones in four dimensions. Therefore, fields transforming in the spinor representation of $SO(5,1)$ will also transform in the spinor representation of $SO(3,1)$. By an analogous argument, these fields will also transform in the spinor representation of $SO(2)$.
Now, what we are really interested in, are the irreducible representations of $Spin(5,1)$ in order to specify the matter content. We therefore have to take a brief discussion of chirality into account.
The chirality operators are defined as

$$\begin{aligned} \Gamma^{(6)} &= \Gamma_1 ... \Gamma_6 && \text{six dimensional chirality operator} \\ \Gamma^{(4)} &= i\Gamma_1 ... \Gamma_4 && \text{four dimensional chirality operator} \\ \Gamma^{(K)} &= -i\Gamma_5\Gamma_6 && \text{"internal" chirality operator.} \end{aligned} \tag{122}$$

Their eigenvalues determine the chiralities of the six, four and two dimensional subspaces respectively. The conventional factors of $\pm i$ has been chosen in such a way, that

$$\left(\Gamma^{(6)}\right)^2 = \left(\Gamma^{(4)}\right)^2 = \left(\Gamma^{(K)}\right)^2 = 1 \tag{123}$$

$$\Gamma^{(6)} = \Gamma^{(4)}\Gamma^{(K)}. \tag{124}$$

Because of $\left(\Gamma^{(6)}\right)^2 = 1$, the eigenvalues of $\Gamma^{(6)}$ must be ± 1. So by specifiying the chirality of a given fermion as $+1$ in six dimensions, we have

$$\Gamma^{(4)} = \Gamma^{(K)}, \tag{125}$$

so the four dimensional and internal chiralities have to coincide [10]. As always, the positive and negative chirality subspaces of the Dirac spinor representation will be irreducible representations of $Spin(5,1)$.

It turns out, that there is a remarkable correspondence between mass of fermions in four dimensions and the dynamics of fermions on the Riemann surfaces, that can be seen by analysing the Dirac equations in the respective components of $\mathcal{M}$. In the following, $\mathcal{D}_M$ is the Dirac operator in curved spacetime, as it is defined later. A Dirac fermion Ψ_6 in six dimensions obeys the Dirac equation (possible mass terms in six dimensions are ignored)

$$i\sum_{M=1}^{6}\Gamma^M\mathcal{D}_M\Psi_6 = i\not{\mathcal{D}}_6\Psi_6 = 0, \tag{126}$$

which can be written in four and six dimensional components as

$$i(\not{\mathcal{D}}_4 + \not{\mathcal{D}}_K)\Psi_6 = i(\sum_{\mu=1}^{4}\Gamma^\mu\mathcal{D}_\mu + \sum_{k=5}^{6}\Gamma^k\mathcal{D}_k)\Psi_6 = 0. \tag{127}$$

To solve this equation, we introduce the operators

$$\tilde{\not{\mathcal{D}}}_4 = \Gamma^{(4)}\mathcal{D}_4 \qquad \tilde{\not{\mathcal{D}}}_K = \Gamma^{(4)}\mathcal{D}_K \tag{128}$$

which commute and can therefore be diagonalized simultanously. They are equivalent to the operators before, by a redefinition of the Γ matrices. One can rewrite (127) as

$$i(\tilde{\not{\mathcal{D}}}_4 + \tilde{\not{\mathcal{D}}}_K)\Psi_6 = 0 \tag{129}$$

and its solutions are given by

$$\Psi_6(x^\mu, y^k) = \sum_i \psi_i(x^\mu)\phi_i(y^k), \tag{130}$$

which satisfy the equations

$$i\tilde{\not{\mathcal{D}}}_K\phi_i(y^k) = m_i\phi_i(y^k) \tag{131}$$

$$(i\tilde{\not{\mathcal{D}}}_4 + m_i)\psi_i(x^\mu) = 0. \tag{132}$$

So by comparing this to the standard Dirac equation

$$(i\not{\mathcal{D}}_4 + m)\psi_4 = 0, \tag{133}$$

it is natural to interpret $\not{\mathcal{D}}_K$ as an operator whose eigenvalues are the masses of fermionic fields, measured in four dimensions.
Although we have not introduced gauge fields yet, it is assumed, that the physics in four dimensions is governed by the gauge group $SU(3) \times SU(2)_L \times U(1)_Y$ or $SU(5)$, dependent on the energy scale, but anyhow, as we mentioned in the discussion of the SM, until the corresponding gauge symmetries are not broken, we have to assume that the fermions in these theories are massless, in order for gauge invariance to hold. By this reasoning, what we are actually interested in, are the solutions of these equations, that correspond to massless fermions, i.e.

$$i\not{D}_K \phi_i^0(y^k) = 0 \tag{134}$$

such that the Dirac equation for massless fermions in four dimensions

$$i\not{D}_4 \psi_i(x^\mu) = 0 \tag{135}$$

is satisfied. Our ultimate goal is to explain how chirality for these massless fermions arises, for which we have to investigate the projection onto negative and positive chirality eigenspaces

$$i\not{D}_K \phi_{i\pm}^0 = \frac{1}{2} i\not{D}_K (1 \pm \Gamma^{(K)}) \phi_i^0 = 0. \tag{136}$$

The integer quantity

$$\#\{\phi_{i+}\} - \#\{\phi_{i-}\} = n_+ - n_- := ind(i\not{D}_K), \tag{137}$$

where $\#\{\phi_{i\pm}\}$ denotes the cardinality of the bases of eigenspaces of positive and negative chirality solutions, is called the *Index of the Dirac Operator.* In general the index is a characteristic invariant of a differential operator, that is strongly connected to the topology of the space over which it is defined. So in order to further discuss this dependence on the topology of Σ_g, we have to reformulate all this concepts in the more suitable terms of differential geometry.

4.2 The Fiber Bundle Viewpoint

It turned out, that the geometric properties of fiberbundles mirror interesting physical phenomena especially for gauge theories. The occurence of chirality will reveal itself to be one of those. In this setup, the union of all tangent spaces defines the tangent bundle

$$\bigsqcup_{p \in \Sigma_g} T_p\Sigma_g = T\Sigma_g \tag{138}$$

which is together with the canonical projection $T\Sigma_g \xrightarrow{\pi} \Sigma_g$ a vector bundle with fiber metric g given by the metric tensor on $T_p\Sigma_g$. Since Riemann surfaces are orientable, the structure group of this bundle is given by $SO(2)$, rather than $O(2)$. Each bundle of this type has an associated frame bundle $L\Sigma_g$ in which the fibers at $p \in \Sigma_g$ are the sets of all vector space bases of $T_p\Sigma_g$. This is found to be a principal bundle over Σ_g.
The transition functions of the bundle $t_{ij} : U_i \cap U_j \to SO(2)$ satisfy the consistency conditions

$$t_{ij}t_{jk}t_{ki} = \mathbb{I} \quad \text{and} \quad t_{ii} = \mathbb{I}. \tag{139}$$

The manifold Σ_g is said to admit a *spin structure* if for $\tilde{t}_{ij} \in Spin(2)$ the consistency conditions are equally satisfied and additionally the diagramm

$$\begin{array}{ccc} & & Spin(2) \\ & \overset{\tilde{t}_{ij}}{\nearrow} & \downarrow \varphi \\ U_i \cap U_j & \xrightarrow{t_{ij}} & SO(2) \end{array} \tag{140}$$

commutes, i.e. $\varphi \circ \tilde{t}_{ij} = t_{ij}$, where $\varphi : Spin(2) \to SO(2)$ is the double covering map.
Such a lift $\tilde{t}_{ij}$ may in general not exists, but for Riemann surfaces, it does (there is a criterium for this using the Stiefel-Whitney characteristic classes [11]). The set of the $\tilde{t}_{ij}$ define a principal $Spin(2)$ bundle over Σ_g, called the *Spin Bundle* $S(\Sigma_g)$. A dirac spinor is an element of a section of this bundle $\psi \in \Delta(\Sigma_g) = \Gamma(\Sigma_g, S(\Sigma_g))$. These can again be seperated according to their eigenvalues

$$\Gamma^{(K)}\psi^{\pm} = \pm\psi^{\pm} \qquad \psi^{\pm} \in \Delta^{\pm}(\Sigma_g), \tag{141}$$

so we have a splitting into eigenspaces

$$\Delta(\Sigma_g) = \Delta^{+}(\Sigma_g) \oplus \Delta^{-}(\Sigma_g) \tag{142}$$

by the standard projection operators.

Including the gauge symmetries can also be done in this context quite similar. Omitting for now the spin structure on Σ_g, we consider a gauge theory with some gauge group G on Σ_g. In the geometrical language, we have a principal fiber bundle $P \xrightarrow{\pi} \Sigma_g$, where the structure group as well as the fiber at each point $p \in \Sigma_g$ is given by the gauge group G. The further properties of P are as always determined by the transition functions $t_{ij} : U_i \cap U_j \to G$.
In order to specify the representation of the matter content, we have to consider the associated vector bundle. Let V be a vector bundle, for which the group G has a representation ρ on the fibers F of V. The associated vector bundle is then given by

$$E(\Sigma_g) = P \times_G V = P \times V/\sim, \tag{143}$$

where $(pg, f) \sim (p, \rho(g)f)$, for $p \in P$, $g \in G$ and $f \in E$. the projection is defined by $\pi_E(p, v) = \pi(p)$ and the transition functions are given by $\rho(t_{ij})$.

Both the spin structure and the gauge theory structure can be incorporated simultanously by taking at each point $p \in \Sigma_g$ the tensor product of the fibers of $S(\Sigma_g)$ and $E(\Sigma_g)$ which yields the product bundle $F(\Sigma_g, S(\Sigma_g) \otimes E(\Sigma_g))$. The *Dirac operator in curved spacetime* is defined to be

$$\mathcal{D}_k\psi = [\partial_k + \Omega_k + \mathcal{A}_k]\,\psi, \tag{144}$$

by gauging with respect to local Lorentz transformations as well as local gauge transformations. Ω_k is the connection 1-form of the spin bundle, called *spin connection*, and $\mathcal{A}_k$ the connection 1-form of the bundle describing the gauge theory. In local coordinates, this can be written as [11]

$$\mathcal{D}_k \psi = \left[\partial_k + \frac{1}{2} i \Gamma_k^{\alpha\beta} S_{\alpha\beta} + A_k \right] \psi, \tag{145}$$

where $\Gamma_k^{\alpha\beta}$ are the Christoffel symbols of the Levi-Civita connection on Σ_g, $S_{\alpha\beta}$ is the spinor representation of the Lorentz group and A_k is the Yang-Mills gauge field.

4.3 Index Theory of the Dirac operator

In general, the Dirac operator is a map

$$\not{\mathcal{D}}_k : \Gamma(\Sigma_g, S(\Sigma_g) \otimes E(\Sigma_g)) \to \Gamma(\Sigma_g, S(\Sigma_g) \otimes E(\Sigma_g)), \tag{146}$$

and since all the vector bundles are equipped with fiber metrics, the adjoint Dirac operator $\not{\mathcal{D}}_k^\dagger$ exists. The analytical index can now be defined as

$$ind(\not{\mathcal{D}}_k) = dim(ker(\not{\mathcal{D}}_k)) - dim(ker(\not{\mathcal{D}}_k^\dagger)), \tag{147}$$

which is a finite integer quantity, at least for compact spaces like Σ_g (because then $\not{\mathcal{D}}_k$ is a Fredholm operator). However, as in the Fujikawa method for deriving the chiral Anomaly, we can choose the metric and the gamma matrices in such a way, that the Dirac operator is hermitian $\not{\mathcal{D}}_k = \not{\mathcal{D}}_k^\dagger$, which immediately implies, that the index vanishes $ind(\not{\mathcal{D}}_k) = 0$. So studying the index of the Dirac operator itself, will not give any further insight and since the aim is to explain the origin of chirality anyway, it is natural to introduce the following operators:

$$\not{\mathcal{D}}_k^+ := \not{\mathcal{D}}_k \mathcal{P}_+ \qquad \not{\mathcal{D}}_k^- := \not{\mathcal{D}}_k \mathcal{P}_- \tag{148}$$

Where the $\mathcal{P}_\pm$ denote the standard projection operators. These are called *Weyl Operators* and their adjoints are given by

$$(\not{\mathcal{D}}_k^+)^\dagger = (\not{\mathcal{D}}_k \mathcal{P}_+)^\dagger = (\mathcal{P}_+)^\dagger (\not{\mathcal{D}}_k)^\dagger = \mathcal{P}_+ \not{\mathcal{D}}_k = \not{\mathcal{D}}_k \mathcal{P}_- = \not{\mathcal{D}}_k^- \tag{149}$$

and equally $(\not{\mathcal{D}}_k^-)^\dagger = \not{\mathcal{D}}_k^+$.
The Weyl operators are mappings

$$\not{\mathcal{D}}_k^+ : \Delta^+(\Sigma_g) \otimes E(\Sigma_g) \to \Delta^-(\Sigma_g) \otimes E(\Sigma_g) \tag{150}$$

$$\not{\mathcal{D}}_k^- : \Delta^-(\Sigma_g) \otimes E(\Sigma_g) \to \Delta^+(\Sigma_g) \otimes E(\Sigma_g), \tag{151}$$

as can be seen by application on an eigenfunction and using the anticommutativity of $\Gamma^{(K)}$ and $\not{\mathcal{D}}_k^\pm$. One ends up with a two component complex

$$\Delta^+(\Sigma_g) \otimes E(\Sigma_g) \underset{\not{\mathcal{D}}_k^-}{\overset{\not{\mathcal{D}}_k^+}{\rightleftarrows}} \Delta^-(\Sigma_g) \otimes E(\Sigma_g) \tag{152}$$

called *twisted spin complex*. The index of the Weyl operator yields

$$ind(\not{D}_k^+) = dim(ker(\not{D}_k^+)) - dim(ker((\not{D}_k^+)^\dagger))$$
$$= dim(ker(\not{D}_k^+)) - dim(ker(\not{D}_k^-)) = n_+ - n_-, \quad (153)$$

which is exactly the difference between positive and negative chirality eigenstates, as before.
The index as it is defined in this context turned out to be a topological invariant of the fiber bundle on which the Dirac operator (or generally, every Fredholm operator) is defined. In fact, it can be completely written in terms of certain characteristic classes of the bundle, which is the remarkable statement of the *Atiyah-Singer Index Theorem* (ASIT). For an arbitrary Fredholm operator, it is much more general than we actually need, so we will only give the result for our setup [11]:

Theorem 1 (Atiyah-Singer Index Theorem)**.** *For a twisted spin complex over a compact manifold M, the index of $\not{D}_k^+$ is given by*

$$ind(\not{D}_k^+) = \int_M \hat{A}(TM)ch(E(M))\big|_{vol}, \quad (154)$$

where $\hat{A}(TM)$ is the $\hat{A}$-roof genus of the tangent bundle and $ch(E(M))$ the total Chern character of the vector bundle $E(M)$.

The total Chern character can be written in terms of the curvature 2-form $\mathcal{F}$ as

$$ch(E(M)) = tr\left(exp\left[\frac{i}{2\pi}\mathcal{F}\right]\right). \quad (155)$$

The $\hat{A}$-roof genus is a topological invariant of the tangent bundle, that in the case of compact Riemann surfaces will always have the value $\hat{A}(TM) = 1$. The reason is, that it can be written as a polynomial in the Pontrjagin characteristic classes in a form like

$$\hat{A}(TM) = 1 - \frac{1}{24}p_1 + \frac{1}{5760}(7p_1^2 - 4p_2) + ... \quad (156)$$

However, the Pontrjagin classes are elements of the cohomology $p_i \in H^{4i}(M;\mathbb{R})$ and since $dim(\Sigma_g) = 2$ for all g, the cohomology groups vanish in all degrees bigger than 2. Therefore, the final result supplied by the ASIT is

$$\boxed{ind(\not{D}_k^+) = \int_{\Sigma_g} ch(E(M))\big|_{vol} = \int_{\Sigma_g} ch_1(E(M)) = \frac{i}{2\pi}\int_{\Sigma_g} tr\left(\mathcal{F}\right),} \quad (157)$$

with $ch_1(E(M))$ beeing the first term in the Taylor series of the exponential function called the first Chern character. It is the only term in (155) we have to consider, since in the integration only forms of the same degree as the dimension of M are picked up. It is worth mentioning, that in the case where no gauge interactions are present at all, the contribution of the total Chern character drops out and there are no forms of suitable degree, so the index will vanish in this case.

It is useful to further study the representation of the gauge group G. If the fermions are in some representation R of G, the index for this specified representation is denoted as $ind_R(\not{D}_k^+)$. The index in the representations R and $\overline{R}$ are related by

$$ind_R(\not{D}_k^+) = -ind_{\overline{R}}(\not{D}_k^+), \tag{158}$$

which can be seen, by noting that the curvature 2-form is Lie-Algebra valued, i.e. $\mathcal{F} \in \Lambda^2(E(\Sigma_g)) \otimes Lie(G)$, so it may be written as $\mathcal{F} = \sum_a T_R^a \mathcal{F}^a$ in case of the representation R. However, the generators of the Lie-Algebra are conventionally chosen in such a way, that $T_R^a = -(T_{\overline{R}}^a)^T$. Since the transposition has no impact on the trace, the expression (157) will change by a sign. So for a real or pseudoreal representation of the fermions the index will always vanish, because $R = \overline{R}$. Additionally, if the group G is semisimple, the Lie-Algebra is a sub algebra of the special linear Lie-Algebra $\mathfrak{sl}_n(\mathbb{R})$. In this case, the generators $T_R^a \in \mathfrak{sl}_n(\mathbb{R})$ are traceless, so the trace in (157) vanishes, as does the index. Consequently, the only interesting results are given by non-semisimple Lie groups, in our case by $U(1)$.
Until now, we only found a relation between the topology of the additional dimensions and the difference between positive and negative chirality eigenstates, however by being more explicit with the representations, a similar correspondence can be found for the chiral asymmetry in four dimensions. Let G be the gauge group of the theory in six dimensions and A_k be the gauge fields corresponding to a subgroup $J \subset G$. If these fields are assumed to have vacuum expectation values $\langle A \rangle \neq 0$, these will only be nonzero in the compactified dimensions. This is true, because there exists a Lorentz transformation $\langle A \rangle \xrightarrow{\Lambda} \langle -A \rangle$ and by Lorentz invariance $\langle A \rangle = \langle -A \rangle = 0$. According to [10] such fields appear as Higgs-like bosons in four dimensions and break the gauge group G to the subgroup $H \subset G$ that commutes with J. H is the gauge group governing the four dimensional gauge interactions. Fermions in six dimensions are required to be in the adjoint representation of G, because the gauge bosons are and supersymmetry is assumed to hold. The adjoint representation has a decomposition

$$A \cong \oplus_i L_i \otimes Q_i \tag{159}$$

with representations L_i of H and Q_i of J. It can then be argued, that a fermion transforming in a representation L_i of H, transforms in a representation Q_i of J and their chiralities coincide because of (125). So with these arguments follows, that massless, four dimensional fermions in the L_i representation of H arise as zero-modes of the Dirac operator on Σ_g in the representation Q_i of J.

If one defines the *chiral asymmetry* of some fermionic state in a representation L_i as

$$\chi\left[\psi(L_i)\right] = \#\left\{\psi(L_i)\right\} - \#\left\{\psi(\overline{L_i})\right\}, \tag{160}$$

the final result is given by

$$\boxed{\chi\left[\psi(L_i)\right] = ind_{Q_i} \not{D}_K.} \tag{161}$$

So for a given representation of the fermion fields, the chiral asymmetry can be exclusively expressed in terms of topological invariants of some bundle over a compact space, in this case a compact Riemann surface. By specifying the actual type of this bundle, one has now the possibility to influence the chiral fermion spectrum of a given theory when building models in string theory. However, as we saw before, the resulting chiral spectrum might give rise to gauge anomalies that we demand to vanish in one way or the other. We will use this final result later to classify which choices are therefore allowed in this sense for given models.

4.4 Connection to Magnetic Monopoles

This setup, of a $U(1)$-bundle over some Riemann surface Σ_g is actually a generalisation of the geometric description of a magnetic monopole in the following sense. Under the assumption of the existence of magnetic monopoles, we get a modification of the maxwell equations

$$\vec{\nabla}\vec{B} = \rho_M. \tag{162}$$

For a magnetic charge density of $\rho_M = 4\pi m \delta^{(3)}(\vec{x})$, the magnetic field will be of the form

$$\vec{B} = \frac{m}{r^2}\hat{e}_r. \tag{163}$$

If we now seek a vector potential $\vec{A}$ on $\mathbb{R}^3 - \{0\}$, such that $\vec{\nabla} \times \vec{A} = \vec{B}$, a problem arises: After some analysis, it turns out, that it is impossible, to construct a vector potential that is singularity free everywhere in $\mathbb{R}^3 - \{0\}$. On the other hand, however, the magnetic field strength $\vec{B}$ is defined globally on $\mathbb{R}^3 - \{0\}$, so it is natural to assume, that there might be some mathematical description in which these non-physical singular expressions will not be involved. The solution to this problem is, to define potentials $A_\pm$, that are defined on certain overlapping subspaces of $\mathbb{R}^3 - \{0\}$, but *not* globally. Again, to further analyse this problem, we will switch our viewpoint to differential geometry.
We ultimately want to use our previous result for the index (157), given by the integral over the first Chern character. Since the first Chern character is an element of the de-Rham cohomology $ch_1(E) \in H^*_{dR}(\mathbb{R} - \{0\}; \mathbb{R})$, which is a homotopy invariant, we have the freedom to simplify our problem by using that three dimensional Euclidian space, with the origin removed, is homotopic to the two sphere,

$$\mathbb{R}^3 - \{0\} \simeq S^2, \tag{164}$$

so, at least up to homotopy, we can describe this gauge theory by a principal $U(1)$ bundle over S^2, $P(S^2, U(1))$. Note, that S^2 is just the compact Riemann surface of genus 0, Σ_0. S^2 can be covered by two charts, H_N and H_S being the northern and the southern hemispheres (with a little overlap over the equator). Since these are homotopy equivalent to a point $H_{NS} \simeq pt.$ they provide local subspaces for which the bundle is trivial.The orientations are chosen as shown in the picture.

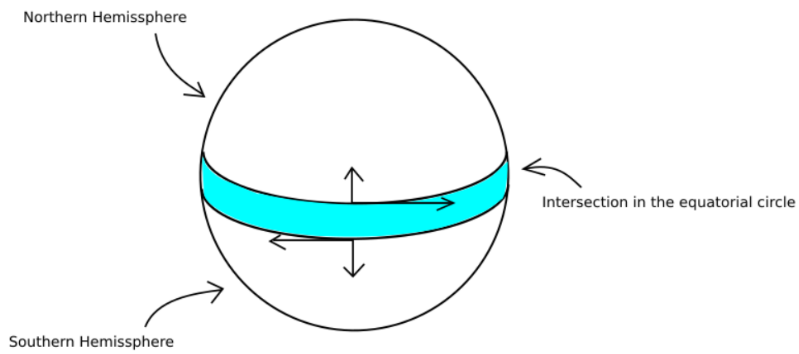


Figure 1: Covering of the Sphere S^2

Therefore we have local sections

$$s_{N/S} : H_{N/S} \to P \tag{165}$$

and if ω is an Ehresmann connection on P, we get the local potential 1-forms

$$\mathcal{A}_{N/S} = s^*_{N/S}\omega. \tag{166}$$

At this point one could give explicit expressions for these potentials, as it is done in [12]. However, for our purpose this will not be necessary. Since $U(1)$ is abelian, the curvature 2-form is in terms of the connection 1-form just

$$\mathcal{F} = d\mathcal{A}_{N/S} \tag{167}$$

So $\mathcal{F}$ is closed, but *not* exact, because the $\mathcal{A}_{N/S}$ are not defined globally.
The intersection of the northern and southern hemispheres is given by the equatorial cirlce $H_N \cap H_S \simeq S^1$, parametrized by the angle φ. So the transition functions $t_{N/S}(\varphi)$ are given by

$$t_{N/S} : H_N \cap H_S \simeq S^1 \longrightarrow U(1) \cong S^1. \tag{168}$$

They are of the form $t_{N/S} = e^{in\varphi}$ with integer n, such that $t_{N/S}(2\pi) = id_{S^1}$ and the fibers are glued together in the right way. The gauge transformation (compatibility condition) of the $\mathcal{A}_{N/S}$ are

$$\mathcal{A}_N = t_{N/S}^{-1}\mathcal{A}_S t_{N/S} + t_{N/S}^{-1} d\ t_{N/S} = \mathcal{A}_S + in\ d\varphi, \tag{169}$$

by using the Abelianity of $U(1)$ and applying the differential d.
The mappings defined by the transition functions can be, again up to homotopy, classified by an integer quantity, the *winding number*, according to $\pi_1(U(1)) = \pi_1(S^1) = \mathbb{Z}$, since these mappigs exactly coincide with the fundamental group. This can also be seen by explicitly evaluating the Chern character:

$$\begin{aligned} -\, ind(\not{D}_k^+) &= -\frac{i}{2\pi}\int_{S^2}\mathcal{F} = -\frac{i}{2\pi}\left[\int_{H_N} d\mathcal{A}_N + \int_{H_S} d\mathcal{A}_S\right] \\ = -\frac{i}{2\pi}\left[\int_{\partial H_N}\mathcal{A}_N + \int_{\partial H_S}\mathcal{A}_S\right] &= -\frac{i}{2\pi}\int_{S^1}(\mathcal{A}_N - \mathcal{A}_S) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\ \ n = n, \end{aligned} \tag{170}$$

by using Stokes' theorem and the orientation defined above. It is worth mentioning, that the classifying integer n does not depend on the potenials at all. It is only determined by the gauge transformation in between, i.e. the transition functions, that govern the actual topological properties of the bundle. In the trivial case, where $n = 0$, the transition functions are equal to the identity and therefore the trivial sections of the bundle coincide, which means that the bundle is globally trivial, in our case $P = S^2 \times S^1$. The integral over the Chern character yields

$$\frac{i}{2\pi}\int_{S^2}\mathcal{F} = \frac{i}{2\pi}\int_{S^2} d\mathcal{A} = \frac{i}{2\pi}\int_{\partial S^2=\emptyset}\mathcal{A} = 0, \tag{171}$$

because now $\mathcal{A}$ is globally defined and $\mathcal{F}$ exact. Replacing $\mathcal{F}$ with $\vec{B}$ and $\mathcal{A}$ with the vector potential $\vec{A}$, this means that the magnetic flux through S^2 vanishes. If $n \neq 0$, the transition functions are not trivial, the bundle is "twisted", i.e. its topology is nontrivial and the integral over the Chern character gives a result analogous to (170). Translating into to magnetic monopole context, this means that there is a nonzero magnetic flux through S^2 characteristic for the topology of the bundle. That is the reason, why the phenomenon of obtaining chirality in four dimensions through a nontrivial $U(1)$ fiberbundle over some compact surface is referred to as "Flux".
This procedure could be done for Riemann surfaces of higher genus quite similar. In case of the torus, $g = 1$, one can think of a quotient space or lattice $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. By covering $\mathbb{R}^2$ in a suitible way by open sets $\{U_i\}$, such that after taking the quotient, these are homotopically trivial $U_i \simeq pt.$, one can analyse

the transition functions in a similar way and obtains an analogous result for the flux. For higher genus, $g \geq 2$, there is a slight complication: A Riemann surface of this genus can be viewed as a $4g$-gon, with edges identified in the right way, but in this case it is not generally possible to build a non-overlapping lattice as for the torus. Therefore, in this cases we have to take the quotient in the hyperbolic plane, i.e. $\Sigma_g = \mathbb{H}/\mathbb{Z}^{2g}$. However, the further treatment is essentially the same.

5 Constraints on F-Theory GUTs

Finally, we are able to discuss the models of the $SU(5)$ Georgi Glashow GUTs in the F-Theory context, that were derived in [1] and calculate the restrictions by gauge anomalies in four dimensions on the involved flux.
To do that, we first give an informal describtion of how these models are actually constructed in 4.1. In 4.2 the contraints originating from the requirement of gauge anomaly cancellation in four dimensions are derived and the results for the contraints on the hypercharge flux are presented in the appendix.

5.1 String Theory Description of $SU(5)$ GUTs

A gauge theory is described in this context in the following way: The open strings are propagating through ten-dimensional spacetime $\mathcal{M} = M^{3,1} \times X^6$ according to certain equations of motion. These require, that the endpoints satisfy Dirichlet boundary conditions, which restrict the movement of the endpoints to hyperplanes of spatial dimension p which are subspaces of $\mathcal{M}$. In our case the dimension will be $p = 7$ and these geometrical objects are called $D7$ branes. The strings are specified by their endpoint's position as well as by their orientation if they are stretched between different branes.

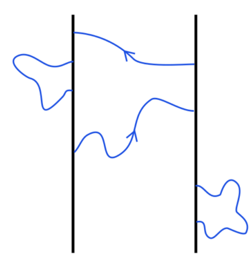


Figure 2: Open Strings and Branes

Those strings, whose endpoints are moving along the same $D7$ brane correspond to the massless $U(1)$ gauge fields in the adjoint representation of $U(1)$, whereas those with endpoints on different $D7$ branes correspond to massive fields in the bifundamental representation of $U(1)$. Whether the string has mass or not, essentially depends on the "tension" between the endpoints. This description can be generalised by considering a stack of N different $D7$ branes. In this case, all strings with endpoints on one of these branes will be massless, since they have "no tension" and after taking the orientation of these strings into account one ends up with N^2 different types of massless strings, matching the dimension of the adjoint representation of $U(N)$. Strings connecting two such stacks of branes, are massive fields in the bifundamental $(\mathbf{N},\overline{\mathbf{M}})$ representation of $U(N)$ and $U(M)$.

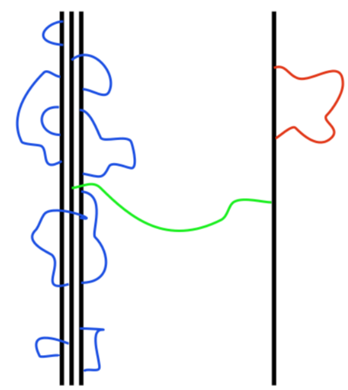


Figure 3: Stacks of Branes

The models that are studied in [1] describe gauge theories with gauge group $SU(5) \times \prod_i U(1)^i$ and an initially massless, fermionic matter content as described in the first section. To obtain such models in this language consider a stack of five $D7$ branes and for the sake of simplicity only one additional $D7$ brane, that both cover whole of $M^{3,1}$ and two different subspaces of the Calabi-Yau manifold X^6. These subspaces are assumed to be 4-cycles, in particular divisors (subvarieties of complex codimension 1) in order to maintain supersymmetry, according to [13]. Both of these 4-cycles intersect in a 2-cycle or complex curve, that in this case has the form of a compact Riemann surface Σ_g. This intersection may be schematically illustrated as follows:

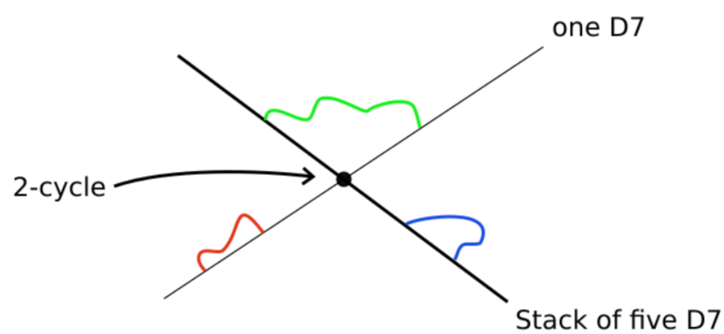


Figure 4: Intersection in a Riemann Surface

The open strings streching between the stack of branes and the single brane describe massive fields in the bifundamental representation (**5**,Q), i.e. they transform in the fundamental representation of $U(5)$ and are charged under the additional $U(1)$. However, those whose endpoints stay on the intersection surface again have "no tension" and are therefore massless. So finally, these fields describe the massless spectrum which is governed by the gauge group $U(5) \times U(1)$. After identifying $U(5) = SU(5) \times U(1)$ and forgetting about this new $U(1)$, since it plays no role in the sequel, this describes the desired gauge theory given by $SU(5) \times \prod_i U(1)^i$ by generalising to more than one additional $U(1)$. Until now, this only includes the matter which is in the **5** of $SU(5)$, but in order to have a realistic model, one also has to explain how the antisymmetric representation **10** arises in this formulation, because it also contains a subset of the SM matter. To do this one has to include certain orientifolds in this picture and describe the connection between those and the divisors wrapped by the $D7$ branes. These intersections will also be complex curves, but in general not the same ones as for the $\overline{\mathbf{5}}$. They are called $\overline{\mathbf{5}}$-matter curves and **10**-matter curves respectively. A detailed investigation can be found in [13].
This is actually the reason, why we could restrict ourselves to the simpler case of $6 = 4+2$ dimensions in the discussion around flux. Since the matter content, which we are interested in, is localised on a subspace given by Σ_g, this restriction provides a toy model for these theories in which many problems can be more easily handled with.

5.2 Anomaly Constraints

A theory that is constructed in such a way is apriori non chiral [13]. The four dimensional spectrum, can be influenced by introducing suitable fluxes, which determine the specific chirality. But consequently, they may lead to the emergence of possible gauge anomalies which are required to cancel in some way, in order for the theory to be consistent. As in the standard Georgi-Glashow model for the $SU(5)$ GUT, these models are related to the SM by a symmetry breaking

$$SU(5) \times \prod_i U(1)^i \longrightarrow \left[\underbrace{SU(3) \times SU(2)_L \times U(1)_Y}_{SM}\right] \times \prod_i U(1)^i. \tag{172}$$

So in the geometrical language, one obtains a vectorbundle associated to the principal $SU(3) \times SU(2)_L \times U(1)_Y \times \prod_i U(1)^i$ fiberbundle over $\mathcal{M}$, i.e. a product bundle of the respective associated vectorbundles. By virtue of the arguments before, we ignore all contributions from the non-Abelian groups, since they all are simple and the flux will vanish. So it is sufficient to look at the vectorbundle $\mathcal{L}_Y \bigotimes_i F_{i,5}$ for the **5**-matter curves and $\mathcal{L}_Y \bigotimes_i F_{i,10}$ for the **10**-matter curves. Here $\mathcal{L}_Y$ denotes the bundle for the hypercharge $U(1)_Y$, the $F_{i,10}$ and $F_{i,5}$ denote the bundles for the additional $U(1)$s along both matter curves. "Turning flux on" along these curves, i.e. making a choice how the bundle looks topologically will yield an integer number A, which is now a sum of the integrals over the curvature 2-forms. As we saw before, it equals the chirality of a certain species of matter on those curves. So we get for the **5** curves after the GUT breaking the flux

$$\begin{aligned} \chi[(\mathbf{3},\mathbf{1})^{Q_5^j}_{-\frac{1}{3}}] &= -\frac{1}{3}N + \sum_j Q_5^j M_5^j = A \\ \chi[(\mathbf{1},\mathbf{2})^{Q_5^j}_{\frac{1}{2}}] &= \frac{1}{2}N + \sum_j Q_5^j M_5^j = A' \end{aligned} \tag{173}$$

where the Q_5^js are the charges under the additional $U(1)$s, N is the flux associated to the hypercharge, called *hypercharge flux*, M_5^j is the flux associated to the j'th additional $U(1)$ for the **5** and we now take the coupling constant, i.e. the charge into account.
However, there is a problem involved: If the hypercharge flux is chosen wrong, it may induce chirality of the additional gauge bosons, that are present after breaking the $SU(5)$ GUT symmetry which should not be the case. We will not go into detail about this and just state, that this can be solved by choosing the hypercharge flux in such a way, that it would equally be well defined as flux for the bundle $\mathcal{L}^{\frac{5}{6}}Y$ (without additional $U(1)$s) where the representation of $U(1)_Y$ is fixed as $\pm\frac{5}{6}$, which is the charge of these bosons [14]. To achieve this in the above expressions we have to demand

$$N = \frac{5}{6}N'. \tag{174}$$

But under this assumption, we cannot be sure anymore, whether the above fluxes A and A' are still integers at all. As we saw, they must be, so we have to adjust the choice of the fluxes along the additional $U(1)$s in such a way, that they are. A little algebra reveals, that the combination

$$\begin{aligned} \chi[(\mathbf{3},\mathbf{1})^{Q_5^j}_{-\frac{1}{3}}] &= A \\ \chi[(\mathbf{1},\mathbf{2})^{Q_5^j}_{\frac{1}{2}}] &= A + N_5 \end{aligned} \tag{175}$$

will do it. The same logic applies to the **10** curves and the possible fluxes are

$$\begin{aligned} \chi[(\mathbf{3},\mathbf{2})_{\frac{1}{6}}] &= B \\ \chi[(\overline{\mathbf{3}},\mathbf{1})_{-\frac{2}{3}}] &= B - N_{10} \\ \chi[(\mathbf{1},\mathbf{1})_{1}] &= B + N_{10}, \end{aligned} \tag{176}$$

where B is the chirality of $(\mathbf{3},\mathbf{2})_{\frac{1}{6}}$ given by the flux of the bundle $\mathcal{L}_Y \bigotimes_i F_{i,10}$. The notation N_5 and N_{10} is defined as the restriction to the respective curves $\mathcal{C}_{5,10}$

$$N_{5/10} = \int ch_1(\mathcal{L}_Y^{\frac{5}{6}})\big|_{\mathcal{C}_{5,10}}. \tag{177}$$

There are several constraints on these fluxes and their charges that where derived in a geometrical way originally [15], but as it was pointed out in [16], they are also strongly related to gauge anomaly cancellation in four dimensions. The triangle anomalies that may be present could contain SM gauge groups and a combination of the additional $U(1)$s. However, following the arguments in [16][17][15], one finds that most of them will not pose any constraints on the spectrum.

First of all, it is important that all anomalies that are present after turning on flux and the GUT symmetry breaking, must be proportional to the corresponding ones before. It turnes out, that the only interesting ones, that impose the constraints on the spectrum are

$$\mathscr{A}_{SU(3)^2-U(1)}, \mathscr{A}_{SU(2)^2-U(1)}, \mathscr{A}_{U(1)_Y^2-U(1)} \tag{178}$$

$$\mathscr{A}_{SU(3)^3}, \mathscr{A}_{SU(2)^3} \tag{179}$$

$$\mathscr{A}_{U(1)_Y-U(1)^A-U(1)^B} \tag{180}$$

which satisfy

$$\mathscr{A}_{SU(3)^2-U(1)} \propto \mathscr{A}_{SU(2)^2-U(1)} \propto \mathscr{A}_{U(1)_Y^2-U(1)} \propto \mathscr{A}_{SU(5)^2-U(1)} \tag{181}$$

$$\mathscr{A}_{SU(3)^3} \propto \mathscr{A}_{SU(2)^3} \propto \mathscr{A}_{SU(5)^3} = 0 \tag{182}$$

$$\mathscr{A}_{U(1)_Y-U(1)^A-U(1)^B} \propto \mathscr{A}_{SU(5)-U(1)^A-U(1)^B} = 0. \tag{183}$$

In (178), there is no requirement of cancellation in the standard way, since this taken care of by a string theory process, called the *Green-Schwarz anomaly cancellation mechanism* (GSM), which results in the vanishing of these anomalies and provide the gauge bosons of the $U(1)$s with a so called *Stückelberg mass*, what is no problem in this case. At the GUT level, the anomalies (179) must vanish automatically, because the GUT is anomaly free. However, after the breaking and turning on flux, this may not be true and the GSM will help us neither, because it does not effect the non-Abelian groups. So we have to demand these anomalies to cancel in the standard way to ensure the proportionality (182). For (180), it is a similar situation, since the GSM would give rise to a Stückelberg mass of the $U(1)_Y$ gauge boson, the photon, which we should avoid. Consequently, because of (183) and the vanishing trace of any generator of $SU(5)$, we have to demand the cancellation of this anomaly, also in the standard way.

Additionaly, there is a simplification when it comes to the fluxes in (175) and (176):

The fluxes given by the integers A and B do not contribute to these anomalies at all. Turning on one of them will give a full representation of $SU(5)$ and by using these proportionalities they will not have any effect and we have the freedom to set them to zero. Therefore, we are only interested in the spectrum, that is induced by the hypercharge flux defined in (177), which is given by

$$\begin{aligned} \mathcal{C}_5^j &: \chi\left[(\mathbf{1},\mathbf{2})_{+\frac{1}{2}}^{Q_5^j}\right] = N_5^j \\ \mathcal{C}_{10}^i &: \chi\left[(\overline{\mathbf{3}},\mathbf{1})_{-\frac{2}{3}}^{Q_{10}^i}\right] = -N_{10}^i \\ \mathcal{C}_{10}^i &: \chi\left[(\mathbf{1},\mathbf{1})_{+1}^{Q_{10}^i}\right] = N_{10}^i, \end{aligned} \tag{184}$$

for a given number of **5** and **10** matter curves indexed by i and j.
From the proportionalities (180) one can derive the requirement

$$\sum_{\mathcal{C}_{10}^i} Q_{10}^i N_{10}^i + \sum_{\mathcal{C}_5^j} Q_5^j N_5^j = 0, \tag{185}$$

which is a constraint on the hypercharge flux, that also occures in the geometrical description. This is equally true for

$$\sum_{\mathcal{C}_{10}^i} N_{10}^i = \sum_{\mathcal{C}_5^j} N_5^j = 0, \tag{186}$$

which origins in the anomaly cancellation approach by requiring the cancellation of the SM anomalies (179). The last constraint, which is for now only found by considering gauge anomaly cancellation is obtained by simply applying the anomaly consistency condition for the anomaly (179) in case of this spectrum:

$$3\sum_{\mathcal{C}_{10}^i}(Q_{10}^i)^A(Q_{10}^i)^B N_{10}^i + \sum_{\mathcal{C}_5^j}(Q_5^j)^A(Q_5^j)^B N_5^j = 0. \tag{187}$$

So all in all, since the models in [1] involve one, two or three, additional $U(1)$s one has respectively four, six or nine constraints on the introduced hypercharge flux supplied by the necessity of anomaly cancellation. These are given by the system of equations in N_{10}^i and N_5^j

$$\begin{cases} 3\sum_{\mathcal{C}_{10}^i}(Q_{10}^i)^A(Q_{10}^i)^B N_{10}^i + \sum_{\mathcal{C}_5^j}(Q_5^j)^A(Q_5^j)^B N_5^j = 0 \\ \sum_{\mathcal{C}_{10}^i} Q_{10}^i N_{10}^i + \sum_{\mathcal{C}_5^j} Q_5^j N_5^j = 0 \\ \sum_{\mathcal{C}_{10}^i} N_{10}^i = 0 \\ \sum_{\mathcal{C}_5^j} N_5^j = 0. \end{cases} \tag{188}$$

The matter content, together with the charges of the $\overline{\mathbf{5}}$ and $\mathbf{10}$ representations in these models is taken from [1]. Plugging these data in the system (188) gives the results in the appendix for the possible choices of hypercharge flux and the requirement of consistency when it comes to anomalies.

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A Consistent Choices for the Hypercharge Flux

A.1 One $U(1)$ Models

$\{2,2,1\}$		
Curve	Charge	Flux
$\mathcal{C}_{10}^1$	-4	0
$\mathcal{C}_{10}^2$	1	0
$\mathcal{C}_{5}^1$	3	0
$\mathcal{C}_{5}^2$	-2	0

$\{2,3,2\}$		
Curve	Charge	Flux
$\mathcal{C}_{10}^1$	-3	N_1
$\mathcal{C}_{10}^2$	2	$-N_1$
$\mathcal{C}_{5}^1$	6	0
$\mathcal{C}_{5}^2$	1	N_1
$\mathcal{C}_{5}^3$	-4	$-N_1$

$\{3,3,2\}$		
Curve	Charge	Flux
$\mathcal{C}_{10}^1$	-1	N_1
$\mathcal{C}_{10}^2$	0	N_2
$\mathcal{C}_{10}^3$	1	$-N_1-N_2$
$\mathcal{C}_{5}^1$	1	N_1+2N_2
$\mathcal{C}_{5}^2$	0	$-3N_2$
$\mathcal{C}_{5}^3$	-1	$-N_1+N_2$

$\{3,4,3\}$		
Curve	Charge	Flux
$\mathcal{C}_{10}^1$	-4	N_1
$\mathcal{C}_{10}^2$	1	N_2
$\mathcal{C}_{10}^3$	6	$-N_1-N_2$
$\mathcal{C}_{5}^1$	8	N_3
$\mathcal{C}_{5}^2$	3	$3N_1+3N_2-3N_3$
$\mathcal{C}_{5}^3$	-2	$-4N_1-5N_2+3N_3$
$\mathcal{C}_{5}^4$	-7	$N_1+2N_2-N_3$

$\{4,5,4\}$		
Curve	Charge	Flux
$\mathcal{C}_{10}^1$	-8	N_1
$\mathcal{C}_{10}^2$	-3	N_2
$\mathcal{C}_{10}^3$	2	N_3
$\mathcal{C}_{10}^4$	7	$-N_1 - N_2 - N_3$
$\mathcal{C}_5^1$	11	N_4
$\mathcal{C}_5^2$	6	N_5
$\mathcal{C}_5^3$	1	$3N_1 + 5N_2 + 4N_3 - 6N_4 - 3N_5$
$\mathcal{C}_5^4$	4	$-3N_1 - 8N_2 - 7N_3 + 8N_4 + 3N_5$
$\mathcal{C}_5^5$	9	$3N_2 + 3N_3 - 3N_4 - N_5$

$\{5,7,6\}$		
Curve	Charge	Flux
$\mathcal{C}_{10}^1$	-2	N_1
$\mathcal{C}_{10}^2$	-1	N_2
$\mathcal{C}_{10}^3$	0	N_3
$\mathcal{C}_{10}^4$	1	N_4
$\mathcal{C}_{10}^5$	2	$-N_1 - N_2 - N_3 - N_4$
$\mathcal{C}_5^1$	3	N_5
$\mathcal{C}_5^2$	2	N_6
$\mathcal{C}_5^3$	1	N_7
$\mathcal{C}_5^4$	0	N_8
$\mathcal{C}_5^5$	-1	$10N_1 + 12N_2 + 11N_3 + 7N_4 - 15N_5 - 10N_6 - 6N_7 - 3N_8$
$\mathcal{C}_5^6$	-2	$-16N_1 - 21N_2 - 20N_3 - 13N_4 + 24N_5 + 15N_6 + 8N_7 + 3N_8$
$\mathcal{C}_5^7$	-3	$6N_1 + 9N_2 + 9N_3 + 6N_4 - 10N_5 - 6N_6 - 3N_7 - N_8$

$\{3,4,3\}_2$		
Curve	Charge	Flux
$\mathcal{C}_{10}^1$	-4	0
$\mathcal{C}_{10}^2$	1	N_1
$\mathcal{C}_{10}^3$	1	$-N_1$
$\mathcal{C}_5^1$	3	N_2
$\mathcal{C}_5^2$	3	$-N_2$
$\mathcal{C}_5^3$	-2	N_3
$\mathcal{C}_5^4$	-2	$-N_3$

$\{4,5,5\}_2$		
Curve	Charge	Flux
$\mathcal{C}_{10}^1$	-3	N_1
$\mathcal{C}_{10}^2$	-3	N_2
$\mathcal{C}_{10}^3$	2	N_3
$\mathcal{C}_{10}^4$	2	$-N_1 - N_2 - N_3$
$\mathcal{C}_5^1$	6	0
$\mathcal{C}_5^2$	1	N_4
$\mathcal{C}_5^3$	1	$N_1 + N_2 - N_4$
$\mathcal{C}_5^4$	-4	N_5
$\mathcal{C}_5^5$	-4	$-N_1 - N_2 - N_5$

$\{5,7,7\}_2$		
Curve	Charge	Flux
$\mathcal{C}_{10}^1$	-4	N_1
$\mathcal{C}_{10}^2$	-4	N_2
$\mathcal{C}_{10}^3$	1	N_3
$\mathcal{C}_{10}^4$	1	N_4
$\mathcal{C}_{10}^5$	6	$-N_1 - N_2 - N_3 - N_4$
$\mathcal{C}_5^1$	8	N_5
$\mathcal{C}_5^2$	3	N_6
$\mathcal{C}_5^3$	3	$3N_1 + 3N_2 + 3N_3 + 3N_4 - 3N_5 - N_6$
$\mathcal{C}_5^4$	-2	N_7
$\mathcal{C}_5^5$	-2	$-4N_1 - 4N_2 - 5N_3 - 5N_4 + 3N_5 - N_7$
$\mathcal{C}_5^6$	-7	N_8
$\mathcal{C}_5^7$	-7	$N_1 + N_2 + 2N_3 + 2N_4 - N_5 - N_8$

$\{5,7,8\}_3$		
Curve	Charge	Flux
$\mathcal{C}_{10}^1$	-3	N_1
$\mathcal{C}_{10}^2$	-3	N_2
$\mathcal{C}_{10}^3$	2	N_3
$\mathcal{C}_{10}^4$	2	N_4
$\mathcal{C}_{10}^5$	2	$-N_1 - N_2 - N_3 - N_4$
$\mathcal{C}_5^1$	6	0
$\mathcal{C}_5^2$	1	N_5
$\mathcal{C}_5^3$	1	N_6
$\mathcal{C}_5^4$	1	$N_1 + N_2 - N_5 - N_6$
$\mathcal{C}_5^5$	-4	N_7
$\mathcal{C}_5^6$	-4	N_8
$\mathcal{C}_5^7$	-4	$-N_1 - N_2 - N_7 - N_8$

$\{5,7,7\}_{2,2}$		
Curve	Charge	Flux
$\mathcal{C}_{10}^1$	-4	0
$\mathcal{C}_{10}^2$	1	N_1
$\mathcal{C}_{10}^3$	1	N_2
$\mathcal{C}_{10}^4$	1	N_3
$\mathcal{C}_{10}^5$	1	$-N_1 - N_2 - N_3$
$\mathcal{C}_5^1$	3	N_4
$\mathcal{C}_5^2$	3	N_5
$\mathcal{C}_5^3$	3	N_6
$\mathcal{C}_5^4$	3	$-N_4 - N_5 - N_6$
$\mathcal{C}_5^5$	-2	N_7
$\mathcal{C}_5^6$	-2	N_8
$\mathcal{C}_5^7$	-2	$-N_7 - N_8$

A.2 Two $U(1)$s Models

$\{3,4,4\}$		
Curve	Charge	Flux
$\mathcal{C}_{10}^1$	(-3,-1)	0
$\mathcal{C}_{10}^2$	(1,0)	0
$\mathcal{C}_{10}^3$	(0,1)	0
$\mathcal{C}_5^1$	(3,0)	0
$\mathcal{C}_5^2$	(2,1)	0
$\mathcal{C}_5^3$	(-1,-1)	0
$\mathcal{C}_5^4$	(-2,0)	0

$\{3,5,6\}$		
Curve	Charge	Flux
$\mathcal{C}_{10}^1$	(-2,-2)	0
$\mathcal{C}_{10}^2$	(1,0)	N_1
$\mathcal{C}_{10}^3$	(0,1)	$-N_1$
$\mathcal{C}_5^1$	(2,1)	$-N_1$
$\mathcal{C}_5^2$	(1,2)	N_1
$\mathcal{C}_5^3$	(-1,-1)	0
$\mathcal{C}_5^4$	(-2,0)	0
$\mathcal{C}_5^5$	(0,-2)	0

$\{4,6,7\}$		
Curve	Charge	Flux
$\mathcal{C}_{10}^1$	(-1,2)	N_1
$\mathcal{C}_{10}^2$	(0,-4)	0
$\mathcal{C}_{10}^3$	(1,0)	N_2
$\mathcal{C}_{10}^4$	(0,1)	$-N_1 - N_2$
$\mathcal{C}_5^1$	(1,2)	N_3
$\mathcal{C}_5^2$	(1,-3)	$-N_1 - 2N_2 - N_3$
$\mathcal{C}_5^3$	(0,3)	$N_1 - N_2 - 2N - 3$
$\mathcal{C}_5^4$	(0,-2)	$2N_1 + 4N_2 + 2N_3$
$\mathcal{C}_5^5$	(-1,4)	$-N_1 + N_2 + N_3$
$\mathcal{C}_5^6$	(-1,-1)	$-N_1 - 2N_2 - N_3$

$\{4,6,8\}$		
Curve	Charge	Flux
$\mathcal{C}_{10}^1$	(-2,-2)	N_1
$\mathcal{C}_{10}^2$	(0,1)	N_2
$\mathcal{C}_{10}^3$	(1,0)	N_3
$\mathcal{C}_{10}^4$	(3,3)	$-N_1 - N_2 - N_3$
$\mathcal{C}_5^1$	(4,4)	$N_1 + 2N_2 + 2N_3$
$\mathcal{C}_5^2$	(2,1)	$-N_2 - 2N_3$
$\mathcal{C}_5^3$	(1,2)	$-2N_2 - N_3$
$\mathcal{C}_5^4$	(-1,-1)	$-N_1 + N_2 + N_3$
$\mathcal{C}_5^5$	(-3,-4)	0
$\mathcal{C}_5^6$	(-4,-3)	0

$\{5,8,12\}$		
Curve	Charge	Flux
$\mathcal{C}_{10}^1$	(-4,6)	N_1
$\mathcal{C}_{10}^2$	(-1,1)	N_2
$\mathcal{C}_{10}^3$	(0,1)	N_3
$\mathcal{C}_{10}^4$	(2,-4)	N_4
$\mathcal{C}_{10}^5$	(3,-4)	$-N_1 - N_2 - N_3 - N_4$
$\mathcal{C}_5^1$	(5,-7)	N_5
$\mathcal{C}_5^2$	(4,-7)	N_6
$\mathcal{C}_5^3$	(2,-2)	$N_1 + 2N_2 + 2N_3 + 3N_4 - 2N_5$
$\mathcal{C}_5^4$	(1,-2)	$-3N_4 - N_5 - 3N_6$
$\mathcal{C}_5^5$	(-1,3)	$-N_2 - 2N_3 - 2N_4 + N_5$
$\mathcal{C}_5^6$	(-2,3)	$-2N_2 - N_3 + 2N_4 + 2N_5 + 3N_6$
$\mathcal{C}_5^7$	(-3,3)	0
$\mathcal{C}_5^8$	(-5,8)	$-N_1 + N_2 + N_3 - N_5 - N_6$

$\{5,8,12\}_2$		
Curve	Charge	Flux
$\mathcal{C}_{10}^1$	(-2,-2)	0
$\mathcal{C}_{10}^2$	(1,0)	N_1
$\mathcal{C}_{10}^3$	(0,1)	N_2
$\mathcal{C}_{10}^4$	(1,0)	N_3
$\mathcal{C}_{10}^5$	(0,1)	$-N_1-N_2-N_3$
$\mathcal{C}_5^1$	(2,1)	N_4
$\mathcal{C}_5^2$	(1,2)	N_5
$\mathcal{C}_5^3$	(-1,-1)	N_6
$\mathcal{C}_5^4$	(2,1)	$-N_1-N_3-N_4$
$\mathcal{C}_5^5$	(1,2)	$N_1+N_3-N_5$
$\mathcal{C}_5^6$	(-1,-1)	$-N_6$
$\mathcal{C}_5^7$	(-2,0)	0
$\mathcal{C}_5^8$	(0,-2)	0

A.3 Three $U(1)$s Models

$\{4,7,12\}$		
Curve	Charge	Flux
$\mathcal{C}_{10}^1$	(-2,-1,-1)	0
$\mathcal{C}_{10}^2$	(1,0,0)	0
$\mathcal{C}_{10}^3$	(0,1,0)	0
$\mathcal{C}_{10}^4$	(0,0,1)	0
$\mathcal{C}_5^1$	(2,1,0)	0
$\mathcal{C}_5^2$	(2,0,1)	0
$\mathcal{C}_5^3$	(1,1,1)	0
$\mathcal{C}_5^4$	(-1,-1,0)	0
$\mathcal{C}_5^5$	(-1,0,-1)	0
$\mathcal{C}_5^6$	(0,-1,-1)	0
$\mathcal{C}_5^7$	(-2,0,0)	0

$\{5, 9, 18\}$		
Curve	Charge	Flux
$\mathcal{C}_{10}^1$	(-2,-2,0)	0
$\mathcal{C}_{10}^2$	(1,0,0)	N_1
$\mathcal{C}_{10}^3$	(0,1,0)	N_2
$\mathcal{C}_{10}^4$	(0,0,1)	N_3
$\mathcal{C}_{10}^5$	(1,1,-1)	$-N_1 - N_2 - N_3$
$\mathcal{C}_5^1$	(2,2,-1)	$2N_1 + 2N_2 + N_3$
$\mathcal{C}_5^2$	(2,1,0)	$-2N_1 - N_2$
$\mathcal{C}_5^3$	(1,2,0)	$-N_1 - 2N_2$
$\mathcal{C}_5^4$	(-1,-1,0)	0
$\mathcal{C}_5^5$	(-1,0,-1)	0
$\mathcal{C}_5^6$	(0,-1,-1)	0
$\mathcal{C}_5^7$	(1,1,1)	$N_1 + N_2 - N_3$
$\mathcal{C}_5^8$	(-1,-2,1)	0
$\mathcal{C}_5^9$	(-2,-1,1)	0