Department of Physics and Astronomy
University of Heidelberg

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submitted by

Manuel Brenner

born in Bietigheim-Bissingen (Germany)

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Symmetries, GUTs and Complete Networks

This Bachelor Thesis has been carried out by Manuel Brenner at the
Institute for Theoretical Physics in Heidelberg
under the supervision of
Eran Palti

Abstract

In this thesis, the Standard Model and the Higgs Mechanism are presented and their connection to group theory and symmetries is explored. In the following chapter, the SU(5) Georgi-Glashow model is examined as a possible Grand Unified Theory (GUT) that further unifies the concepts introduced in the Standard Model. Advancing after this model, there is an ongoing search for other theories going beyond the Standard Model, possibly unifying all of physics in a "Theory of Everything". The most promising candidate for this to date is String Theory, and the rest of the thesis deals with structures called Complete Networks, which find some applications in type IIB String Theory.

In dieser Arbeit werden das Standardmodell der Teilchenphysik und der Higgsmechanismus sowie ihre Verbindungen zu Symmetrien und der damit verbundenen Gruppentheorie vorgestellt. In nachfolgenden Kapitel wird das SU(5)-Georgi-Glashow-Modell als ein mögliches Modell für eine Große Vereinheitlichte Theorie (GUT) untersucht, dessen Ziel es ist, die im Standardmodell vorgestellten Konzepte weiter zu vereinigen. Als Weiterführung dieses Modells werden weitere Theorien gesucht, deren Ziel es ist, alle physikalischen Phänomene in einer "Theorie von Allem" zu beschreiben. Der derzeit vielversprechenste Kandidat für solch eine Theorie ist durch die Stringtheorie gegeben. Die abschließenden Kapitel dieser Thesis handeln von vollständigen Netzwerken, deren Verknüpfungen zu Anwendungen in der Stringtheorie kurz vorgestellt werden.

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I. Theoretical background

"The most beautiful of all links is that which makes, of itself and of the things it connects, the greatest unity possible; and it is the proportion (summetria) which realizes it in the most beautiful way."

-Platon, *Timaeus*

I.1. The Standard Model of Particle Physics

I.1.1. History and Development

Today, there are two main branches of scientific models researching the fundamental principles of our universe and its creation, which is usually called **The Big Bang**. The first one centers around Cosmology and specializes on the origin and the evolution of the universe on a global scale and the laws that govern it. Examples for this are research on the **Cosmic Micowave Background (CMB)**, on the expansion of the universe due to dark energy, or on the form of galaxy clusters. The second one is of greater interest here, and could be summed up as Elementary Particle Physics. It studies the fundamental building blocks of matter and their interactions, and searches for the real **atoms** in the original sense meaning fundamental particles without any substructure which then could be thoroughly described by a set of quantum numbers and the way the forces corresponding to these quantum numbers interacted with each other[de Boer, 2001]. All of particle physics is built around a very sucessful model called the **Standard Model (SM)** that was brought on its way by Glashow [S. Glashow, 1961] and his efforts to unify the electromagnetic and the weak force. Further developements [see e.g.Weinberg, 1967, 21] and the incorporation of the Higgs mechanism into his model through Weinberg and Salam in 1968 were important step in the completion of the Standard Model, as it predicted massive gauge bosons and made other experimentally verifiable claims. Some details concerning this will be further discussed in this thesis, as some key concepts like the breaking of symmetries are also important for GUTs. Measurements at CERN in 1974 (the discovery of the charm quark, see [de Boer, 2001]) and over the next decades verified many of the predictions made by the Standard Model. When the Higgs boson was found in 2012 at the LHC [Aad, Abajyan,..., 2012], all doubts were cleared out that the SM is indeed a very powerful and successful theory. Despite that, there are still a couple of problems that show the need for further theories (and phenomena that cannot be explained at all, such as dark matter), which are able to incorporate the qualities of the SM yet at the same time fix its problems. The SM is therefore called an Effective Field Theory because it is effective at all energy scales that can be achieved by modern detectors, but is expected to break down (not work anymore) at higher energy scales.

I.1.2. Mathematical Foundations

There are many introductions in the structure of the Standard Model. For a more detailed analysis, [Tanovic, A., 2009], [von Steinkirch, M., 2011] and [de Boer, 2001] are recommended.

The Gauge Group

The Standard Model is formulated as a Yang-Mills-Theory, introduced in [I.2.2] , and is a non-Abelelian gauge theory built upon SU(N) and symmetry groups a $U(1)$ symmetry (see [VI.1] for a definition). Generally speaking, in such a theory, local symmetry transformations on the given system (carried out as transformations on the Lagrangian) give rise to local gauge fields. These fields can correspond, as in the case of a $U(1)$ symmetry, to the electromagnetic potential already used in the Maxwellian theory. The Standard Model is specified by the following gauge group:

$$G = SU(3) \otimes SU(2) \otimes U(1)_Y \tag{I.1}$$

where $SU(3)$ represents the strong interactions (QCD), $SU(2)$ represents the weak interactions and $U(1)_Y$ the hypercharge.
$SU(2)$ and $U(1)_Y$ are broken down spontaneously at the electroweak energy scale to $U(1)_{e.m}$ with its gauge boson being the photon. This mechanism will be looked at in more detail later on in I.2.3.

The Bosonic Content

Bosons are particles with integer spin. They are the mediators of the forces in the Standard Model. For a given symmetry group, there is a general way of finding the number of the mediators of the force, the gauge bosons, which can be found in I.2.1. QCD has 8 gluons corresponding to the 8 generators of $SU(3)$. The gluons themselves carry what is called colour charge, this being a combination of the three colours red, green, and blue (and antired, antigreen and antiblue). All three colours together add up to "white", making a particle consisting of three differently coloured quarks or two quarks with one colour and its anticolour effectively colourless.[1] Due to **confinement** taking place at low energies and densities (no single colour charge exists freely, but always pairs up to form a colourless pairing), this is observed in nature as the protons and neutrons which form, together with the electrons, the observable matter. As the generators are 3×3 matrices which do not commute, QCD is a non-abelian theory. The electroweak interaction communicates via the 3 massive bosons Z, W+ and W- and the massless A, which is the photon. They are created through the process of spontaneous symmetry breaking and are combinations of the original massless gauge fields conventionally called W1, W2, W3, and B. How this takes place will be discussed in more detail in I.2.3. The Higgs field which is necessary for the spontaneous breaking of the electroweak symmetry also has a gauge boson, the massive Higgs boson with zero spin and zero charge.

[1] This is a result of the tracelessness of the $SU(3)$ generators, which will be revisited in the context of SU(5)and the diagonal weights basis in subsection III.3.2.

The Fermionic Content

The fermionic particle content is made up of 12 particles grouped in three families. Each particle has an anti-particle. The theory of the Standard Model is a chiral theory, meaning that the fermionic field ψ can be split up in two independent components ψ^L and ψ^R via the γ^5-matrix by

$$\psi^L = \frac{1}{2}(1-\gamma_5)\psi \quad \text{and} \quad \psi^R = \frac{1}{2}(1+\gamma_5)\psi. \tag{I.2}$$

This is important to note as the different chirality components transform differently under the gauge interactions which is due to the observation that the weak force only acts on left handed particles and right-handed anti-particles (first shown in the Wu-experiment, see [Wu, Ambler, Hayward, Hoppes, & Hudson, 1957, 4]) Accounting for this, left-handed particles need to transform as doublets under a SU(2)-transformation, but right-handed-particles as singlets.
Example:

$$\text{Start with a doublet } \vec{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ where the numbers denote the content } \vec{n} = \begin{pmatrix} e \\ \mu_e \end{pmatrix}. \tag{I.3}$$

Acting on this via the SU(2)-generators, the Pauli matrices, corresponds to a SU(2)-transformation. As an example, one can rotate the doublet in the SU(2)-hyperspace by 90 degrees by acting on it with the σ_1-matrix):

$$\vec{n'} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sigma_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{I.4}$$

Interpretation:
Because the content of the doublet $\vec{n}$ changed, the electron is transformed into a neutrino, meaning that electrons and neutrinos are intrinsically connected by the symmetry of the Standard Model, in this case of the weak force. [2] As this is done via the weak interaction and only works for left handed particles, the doublet structure can not in the same way apply to right handed particles. It can also be concluded from these simple considerations that, in the boundaries of the Standard Model, there should be no right-handed-neutrino.[3]
With these informations at hand, one can construct five particles which transform in the different spaces of the gauge groups as listed in the table. There are 3 families of fermions that are formed in the same way as the first family in the table. In the second generation, up-and down-quarks are replaced by strange-and charm-quarks, the electron by the myon, the electron-neutrino by the myon-neutrino. In the third generation, we find the top-and the bottom-quark, the tauon and the tauon-neutrino. It is important to note that the particles from the higher families are far heavier and unstable, which is also not explained within the Standard Model.

[2] This result did not come out of the theory, but was rather observed, and the theory then constructed in a way that was able to describe it.

[3] If there are right-handed neutrinos, they could account for a couple of phenomena that cannot be explained in the boundaries of the Standard Model, such as neutrino osscilations, or even Dark Matter. For details, see [Drewes, 2013].

Table I.1.: *The fermionic content of the Standard Model*

Fermion	$SU(3)$	$SU(2)$	$U(1)$	Representation
$u^\dagger$	Triplet	Singlet	$\frac{2}{3}$	$(3,1)_{2/3}$
$d^\dagger$	Triplet	Singlet	$-\frac{1}{3}$	$(3,1)_{-1/3}$
$e^\dagger$	Singlet	Singlet	-1	$(1,1)_{-1}$
$\overline{\Psi}^\dagger$	Triplet	Doublet	$-\frac{1}{6}$	$(3,2)_{-1/6}$
$\bar{l}^\dagger$	Singlet	Doublet	$-\frac{1}{2}$	$(1,2)_{-1/2}$

Open questions

Despite its powerful predictions and great accuracy, the Standard Model still leaves some open questions to be answered by physics beyond the Standard Model. Some of these questions are:

- The SM cannot incorporate gravity as the fourth fundamental force into its model. The theory is thus expected to break down at energy scales around the Planck scale ($\sim 10^{19} GeV$) where gravity cannot be neglected anymore, but is expected to have direct consequences on interactions on the quantum scale.

- There are three independent coupling constants for the three forces described by the SM. These coupling constant are **running constants** which can be explained by the different space charge distributions of the forces leading to different radiative corrections (see [de Boer, 2001]). It remains unanswered in the SM why there should be three indepedent gauge groups and forces, and the hope is to have just one unifying coupling constants which unifies at higher energy scales, as presented in figure I.1. But as seen in the diagram, unification of the coupling constants is not achieved in the framework of the Standard Model, but requires supersymmetric extensions. More on this can be found in [Lautsch, 2014].

- Charge quantization cannot be explained. In principle, it would be possible to assign the charge generator **Y** indepedently for each representation, meaning that the quark charges and the elecron charges would not be connected by factors of $\frac{1}{3}$ or $\frac{2}{3}$, but could be connected by any arbitrary factor. In the SU(5) model, **Y** is constructed as seen in II.5, and therefore inherently connects the charges of the electron and the quarks.

- There are at least 18 free parameters in the SM (the coupling strenghts, the mixing angles and the Higgs field). This leaves much room for a underlying, unifying theory.

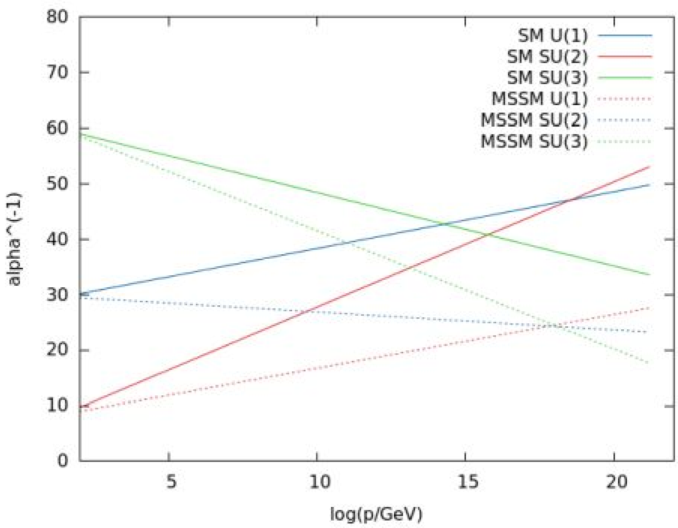

Figure I.1.: *The running of the coupling constants and their possible unification at higher energies, taken from [Lautsch, 2014]*

- As seen in the electroweak theory, a unification of forces can be achieved at higher energy scales by introducing more general symmetry groups, leading to fewer coupling constants. This is one of the main aims of GUTs, as will be discussed later.

I.2. The Higgs Mechanism

The Higgs Mechanism is responsible for the generation of the masses of the gauge bosons and the fermions in the Standard Model. It will be introduced here as the breaking of a local gauge symmetry. On first glance, this concept itself might seem contradictory, as the breaking of a gauge symmetry should not have any physical consequences, because gauge symmetries are linked to freedoms in the description of a system rather than "real" physical degrees of freedom. It is therefore always necessary to fix a gauge for a certain description of the system, which would cause the gauge symmetry to vanish and render the breaking of that symmetry meaningless. But it is possible to introduce the Higgs mechanism in a gauge invariant way, which was already done by Peter Higgs in [Higgs, 1964, 16], or can be found in more detail for example in [Dam, S., 2012].

I.2.1. Spontaneous Symmetry Breaking

Global and Local Symmetries

One can distinguish two different types of symmetries: global and local symmetries. A global symmetry is found if a system is invariant under a global transformation. This means that the element of the Lie group $U(\alpha_1, \alpha_2, ...\alpha_n, \vec{r})$ that generates the given transformation does not depend on $\vec{r}$. In general, each element of a Lie group can be written as

$$\hat{U}(\vec{a}) = e^{\alpha_k \hat{T}_k}, \tag{I.5}$$

where one can identify the generators of the Lie group $\hat{T}_k$. For making the transformation local, the dependency on $\vec{r}$ simply has to be made explicit.

Global Symmetry Breaking

To consider the breaking of a symmetry, one has to clarify what a vacuum state is. For a given Hamiltonian, the vacuum is the state of minimal energy:

$$\langle 0| H |0\rangle = minimal \tag{I.6}$$

In the case of global symmetry breaking, the Lagrangian is invariant under the symmetry transformation, but the vacuum state is not. A famous example for the occurence of symmetry breaking in nature is the ferromagnet, where all magnetizations are oriented in a random manner above the Curie temperature and therefore the whole system is invariant under rotations on a macroscopic scale. But below the Curie temperature, the energy is now minimalized if the magnetization of the entire system is above zero, so the spins align and the rotational symmetry is broken. But as the spins align in a random direction, there are infinitely many possible states of lowest energy connected by rotations. Thus all the vacuum states are degenerate and the Lagrangian is still invariant under rotations, but the vacuum state itself is not, anymore.

Goldstone Theorem

It can be proven (for a proof, look for example in [Zierenberg, 2008]) that every case of spontaneous global symmetry breaking gives rise to massless *Goldstone Bosons*. The number of arising Goldstone Bosons can be found by the following consideration: Starting from a transformation that is done by the elements of a group G, one has to search for a subgroup H in G containing all the generators in G that break the symmetry of the system whereas the vacuum is symmetric under the generators in G\H. The number of generators in G\H (the dimension of the Lie algebra of G\H) then gives the number of arising Goldstone Bosons.

Local Symmetry Breaking

By making the transformation

$$\hat{U}(\vec{a}) = e^{\alpha_k(x)\hat{T}_k} \tag{I.7}$$

locally space dependent, new physics arise from the same underlying process. In Quantum Field Theories (QFT), a covariant derivative is introduced to sustain invariance of the Lagrangian under these local transformations. (see for example [Weigand, 2013])

I.2.2. Yang-Mills-Theories

Non-Abelian gauge symmetries and their properties are studied in what is called **Yang-Mills-Theory**. They give a general way of incorporating gauge fields into Lagrange functions by the **Yang-Mills-Lagrangian**

$$\mathcal{L} = -\frac{1}{4}\sum_a F^a_{\mu\nu}F^{\mu\nu}_a + \overline{\psi}_i(i\gamma^\mu\partial_\mu - m)\psi_i - g\overline{\psi_i}\gamma^\mu A^a_\mu T^a_{ij}\psi_j \tag{I.8}$$

with g the coupling constants, A the gauge fields and T the generators of the algebra corresponding to the gauge symmetry group.
The first term is defined in a similar fashion as the familiar field strenght tensor from electromagnetism, but with an additional term taking account of the non-abelian nature of the theory:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}[A^b_\mu, A^c_\nu] \tag{I.9}$$

in which the structure constants f^{abc} occur, whose fundamental properties will be discussed in VI.1. The second and third term together form the usual Dirac Lagrangian with a covariant derivative

$$D_\mu\psi(x) = (\partial_\mu(x) + ig(\sum_a A^a_\mu(x)T^a\psi(x)). \tag{I.10}$$

From this, it becomes visible how the gauge fields arise in the theory. By that implementation, for a certain non-Abelian gauge theory, for example SU(2), one also can see that the number of gauge fields corresponds to the number of generators of the algebra of SU(2), which are three (later becoming, by additional symmetry breaking, the W^+, W^- and Z bosons). By the Goldstone theorem, they should be massless, but are in fact observed to have mass. This will be explained by the Higgs mechanism at the example of $SU(2) \otimes U(1)$ in the following section. For a more detailed introduction to Yang-Mills theories, we refer to [Weigand, 2013] or [Weinberg, 1996].

I.2.3. A practical example of symmetry breaking: The GWS model

This is the mechanism that is responsible for the way the electroweak force and its gauge bosons appear on low energy scales: all observations lead to the conclusion that there are 3 massive gauge bosons for the weak force and one massless boson for the electromagnetic force, the proton. The model is known as the **Glashow-Weinberg-Salam model** of the electroweak theory.
We start by considering a ϕ^4-theory with the Lagrangian

$$\mathcal{L} = \partial^\mu\phi\partial^\mu\phi^* + m^2\phi^*\phi - \lambda^2(\phi^*\phi)^2 = \partial^\mu\phi\partial^\mu\phi^* - \boldsymbol{V} \tag{I.11}$$

and implement the SU(2) gauge theory and the $U(1)$ theory according to the general way found for Yang-Mills theories in I.2.2. Both additions to the usual derivative are simply added up

$$D_\mu = \partial_\mu + igW_\mu + \frac{i}{2}g'A_\mu, \tag{I.12}$$

where $W_\mu = T^a W^a_\mu$. The gauge fields of the SU(2) symmetry are now called W, and the gauge field of the $U(1)$ symmetry is called A [4]. Furthermore, contractions of the gauge

[4] Note that the factor of $\frac{1}{2}$ will lead to the fact that the Higgs fields have hypercharge $\frac{1}{2}$.

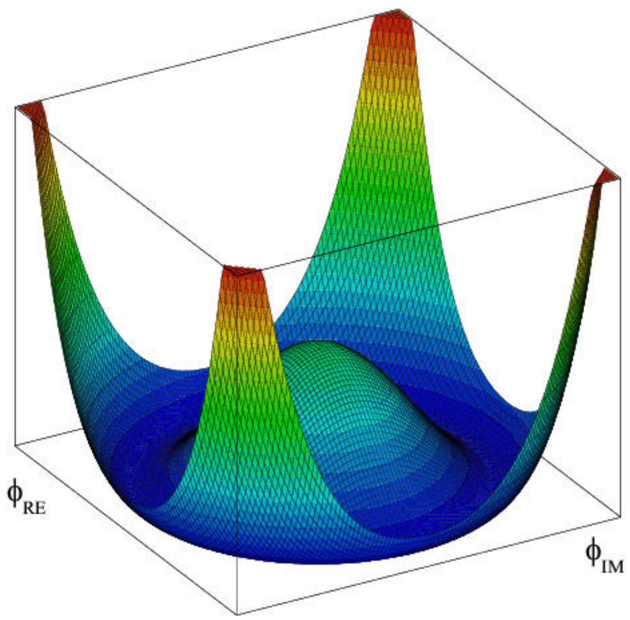

Figure I.2.: *The shape of the Higgs potential reminiscent of a Mexican hat, taken from [Wikipedia, 2011].*

field tensors $G^{\mu\nu}$ (corresponding to SU(2)) and $F^{\mu\nu}$ (corresponding to $U(1)$) which were set up according to equation I.9, are added, leading to a Lagrangian of the form:

$$\mathcal{L} = D^{\mu}\phi D^{\mu}\phi^{*} - \boldsymbol{V} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}G^{a\mu\nu}G^{a}_{\mu\nu} \tag{I.13}$$

The form of the potential

$$\boldsymbol{V} = -m^{2}\phi^{*}\phi + \lambda^{2}(\phi^{*}\phi)^{2} \tag{I.14}$$

determines the theory described by this Lagrangian. Differentiation of the potential leads to the condition that for $m^2 > 0$, the potential will have a minimum at 0, and leads to the usual Yang-Mills-theory with massless gauge bosons.

But if $m^2 < 0$, the minimum is reached for $\phi^2 = -\frac{m^2}{2\lambda^2} = a^2$. In this case, there are infinitely many degenerate vacuum states connected by $SU(2)$ transformations (in figure I.2, they correspond to rotations in the x-y-plane). The potential is a Mexican Hat shaped potential, shown in figure I.2 A choice can be made for a particular vacuum state, for example

$$\phi_0 = \begin{pmatrix} 0 \\ a \end{pmatrix}. \tag{I.15}$$

This vacuum expectation value breaks three of the four symmetries present, the only remaining symmetry being a linear combination between the $U(1)_Y$ of hypercharge and a $U(1)$ subgroup of SU(2) (conventionally the one generated by σ_3), which will later turn out to be $U(1)_{em.}$, generated by the linear combination $Q = Y + T_3$ (Y is the generator of the $U(1)$ symmetry called the **Hypercharge generator**, and T_3 is the third generator of the $SU(2)$ symmetry). This is due to the fact that a combined symmetry transformation

$$\phi(x) \to exp(\frac{i}{2}\theta(x))exp(\frac{i}{2}\theta(x)\sigma_3) = \begin{pmatrix} exp(i\theta(x) & 0 \\ 0 & 1 \end{pmatrix}\phi(x), \tag{I.16}$$

with $\theta(x)$ an arbitrary function, leaves the vacuum state ϕ_0 invariant.
The next step is to expand the field around this vacuum state by introducing the Higgs fields as

$$\phi = \begin{pmatrix} \chi_1 + i\chi_2 \\ (a+\chi_3) + i\chi_4 \end{pmatrix} \tag{I.17}$$

which are doublets under $SU(2)$ and have hypercharge $\frac{1}{2}$ under $U(1)$ (For more details, see [Kaplunovsky, V., 2012]). It is now possible to apply local $SU(2)$ transformations to this fields to eliminate 3 of the 4 Higgs fields

$$\phi = \begin{pmatrix} 0 \\ (a+\chi_3) \end{pmatrix}. \tag{I.18}$$

This fixes a gauge, and leaves one physical scalar field χ_3, which will later occur in the Lagrangian with a mass term, and can be interpreted as the **Higgs boson** that was detected at the LHC in 2012 almost 50 years after its first prediction. This scalar field can be plugged into the Lagrangian I.13, and written out. This won't be done in detail here, but can be found in many papers and textbooks, see for example in [Zierenberg, 2008]. Central is the generation of masses for the bosons that were massless before symmetry breaking. From the kinetic term

$$D_\mu\phi D^\mu\phi^* = \frac{1}{2}(\partial_\mu\chi_3)^2 + \frac{(a+\chi_3)^2}{8}(g'B_\mu - gW_\mu^3)^2 + \frac{g^2(a+\chi_3)^2}{8}((W_\mu^1)^2 + (W_\mu^2)^2)) \tag{I.19}$$

the masses of the W_μ^1 and W_μ^2 vector fields are read off as being $M_W = \frac{ga}{2}$, while the masses of the W_μ^3 and B_μ fields are still coupled. This again motivates what was already mentioned earlier: decoupling both fields through

$$\begin{aligned} Z_\mu(x) &= cos(\theta)W_\mu^3(x) - sin(\theta)B_\mu \\ A_\mu(x) &= sin(\theta)W_\mu^3(x) + cos(\theta)B_\mu \end{aligned} \tag{I.20}$$

leads to a massless vector field that corresponds to the electromagnetic field A_μ with the massless photon and to a massive vector field Z_μ with the gauge boson Z. The angle

$$\theta = arctan(\frac{g'}{g}) \tag{I.21}$$

is called the **weak mixing angle** or **Weinberg angle**. It can be measured experimentally by determination of the masses of the W-and Z-bosons, and is found to be $sin^2(\theta) \approx 0,23$. In a similar fashion, linear combinations can be formed from the W_μ^1 and W_μ^2 fields getting to get the charged weak fields W_μ^+ and W_μ^- through

$$\begin{aligned} W_\mu^+(x) &= \frac{W_\mu^1(x) + iW_\mu^2(x)}{\sqrt{2}} \\ W_\mu^-(x) &= \frac{W_\mu^1(x) - iW_\mu^2(x)}{\sqrt{2}} \end{aligned} \tag{I.22}$$

which are electrically charged as +1 or -1 respectively.

I.2.4. On the philosophy and aesthetics of symmetries

Symmetries play a very important role in physics, which becomes apparent in several places in this thesis.
For a long time in the history of human thought, symmetries have been connected with beauty. Going back to Ancient Greek, Plato states in the Timaeus: *"The most beautiful of all links is that which makes, of itself and of the things it connects, the greatest unity possible; and it is the proportion (summetria) which realizes it in the most beautiful way".*
Symmetries connect elements that appear in a different way by allowing for the interchange of these elements in certain ways. By that, symmetries connect some, and do not connect other elements of the physical world.
For every symmetry that is being perceived, a higher symmetry has to be broken, because (first formulated that way in the studies of Pierre Curie in 1894), a symmetry can only appear in an environement that has at least the symmetry of the phenomenon. This can be tracked "up the chain" towards an absolute symmetry behind the physical world. But in a world with this symmetry, phenomena could not occur, because everything would be interchangeable, everything would "look" the same. In other words, the way some symmetries are broken and some remain determines the way in which the physical world appears, something can only distinguish itself through the interplay of the lack and the presence of its symmetries.
This can be an idea of great intellectual appeal. Because phenomena exist through the breaking of underlying symmetries, there is an order behind the world that is unapparent while one explores the vastness of possible physical phenomena, but might be found by looking at phenomena, and trying to deduce, by abstract considerations, in which way they might be connected.
In mathematics, symmetries are closely linked to group theory, because symmetries are defined as invariances under certain symmetry transformation generated by group elements of a symmetry group. The search for a GUT group that is exemplified at the example of the $SU(5)$ group in the following chapter is therefore not only the search for a better physical theory, but can at the same time be understood as an attempt to find more general, or even the most general symmetry behind the world. To clarify what this could mean, in the example of the $SU(5)$ GUT, electrons and quarks would no longer be entirely different particles, but would be connected on a more fundamental level through $SU(5)$ symmetry transformations, and could in fact be interchanged, manifesting itself in proton decay.
For detailed reviews on this, and more intuitive and philosophical insights into symmetries and the breaking of symmetries, we refer to [Brading & Castellani, 2003 and Katzir, 2004].

II. Grand Unified Theories

II.1. Introduction

II.1.1. A heuristic introduction to GUTs

Grand Unified Theories, or GUTs, are built in the hope to solve the problems of the SM and other parts of physics as mentioned above: some main points are to get rid of the ″messy″ structure of the gauge groups of the SM, to explain the hypercharges and other patterns as the symmetry between quarks and leptons, and the chiral nature of the theory. For these purposes, concepts from group theory seem extraordinarily well suited, as will become apparent in the following chapter.
The next important step after unifying the concepts of the Standard Model is to explain phenomena that are not a part of the SM, such as, most prominently, gravity in connection with Quantum Theories, which is the aim of a **Theory of Everything (TOE)**. [1]

II.1.2. GUTs from a formal standpoint

We will introduce GUTs now from a formal mathematical standpoint. The central consideration is the following: If a space V is a representation space of a group G and the gauge group of the Standard Model is a subgroup of this group $G_{SM} \subset G$, then V is also a representation space of G_{SM}. The aim of a GUT is then to find a homomorphism from G_{SM} to G and an isomorphism from V_{SM} to V so that the diagram shown in figure II.1 commutes:

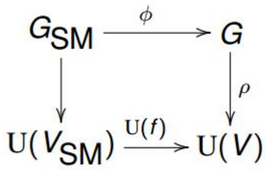

Figure II.1.: *If mappings exist so that this diagram commutes, the group G is a candidate for a GUT (taken from [Baez & Huerta, 2009])*

The hope is now that V breaks down in more irreducible representations of G_{SM} than G, simplifying the theory (in the SU(5) theory, the number of irreducible representations becomes two as opposed to the five from the Standard Model, listed in table [I.1]. For a detailed analysis, we refer to [Baez & Huerta, 2009].

[1]It has to be noted here that, despite the common and widespread conception that General Relativity and Quantum Mechanics are incompatible or even contradictory, a successful description of GR with an effective field theory works very well at all acessible experimental energy scales, but is expected to break down at higher energy scales. (Look for example in [Donoghue, 1995]

II.2. The Georgi-Glashow Minimal-SU(5)

II.2.1. Developement of a SU(5) GUT

The idea of a GUT based on the SU(5)-group was first introduced in the 1974 paper "Unity of all Elementary-Particle Forces" by H. Georgi and S. L. Glashow, see [Georgi & Glashow, 1974, 8]
In the SM, the weak and the electromagnetic force are not really united because they have two indepedent coupling constants. A true unification would require a gauge group of $V = SU(3) \otimes W$, with W being a group containing $SU(2) \otimes U(1)$ with a single unique coupling constant. But considering a model of the form $V = SU(3) \otimes W$ leads to ambiguities: the only possibilities for W would be $SU(3)$, $SU(3) \otimes SU(3)$ or $SU(6)$, but this does not work because the tracelessness of the generator responsible for the electrical charge would require the sum of the quarks to have zero charge. This leads to the conclusion that a unfication of both the electroweak and the strong interaction at the same time is necessary. [2]

Reasons for SU(5) as a unifying gauge group

The unifying theory has to be at least of rank four because it has to contain at least the four commuting generators, which are conventionally called T_3, T_8 (the elements of the Cartan subalgebra of $SU(3)$),R_3 (the choice for $SU(2)$, which has rank one, and S (the $U(1)$ generator). This leaves 9 possible Lie groups. Two of them do not contain $SU(3)$ and can be ruled out, $SU(3)^2$ was already ruled out earlier. One now has to consider the the Standard Model representation of the right-handed creation operators as lined out earlier, which is the representation for $SU(3) \times SU(2) \times U(1)$:

$$(3,1)_{2/3} \oplus (3,1)_{-1/3} \oplus (1,1)_{-1} \oplus (\overline{3},2)_{-1/6} \oplus (1,2)_{1/2} \tag{II.1}$$

In contrast to this, one finds the representation of the left-handed creation operators as:

$$(\overline{3},1)_{-2/3} \oplus (\overline{3},1)_{1/3} \oplus (1,1)_{1} \oplus (3,2)_{1/6} \oplus (1,2)_{-1/2} \tag{II.2}$$

Here it was used that the $SU(2)$ representation is real. One can see that these two representations are not the same and the representation is complex. This can be related to the fact that the electroweak interaction violates parity.
This has an important consequence for the possible Lie groups, because it leaves just one rank 4 group, the one which has a complex representation. This is the SU(5).

II.2.2. Embedding of the Particles in SU(5) representations

The 5

SU(5) has a five-dimensional representation, the fundamental representation, denoted by a 5, and its complex conjugate $\overline{5}$. The aim is now to find a $SU(2) \times U(1)$-subgroup

[2] A direct result of this is that the leptons and the quarks would lie in the same irreducible representation of the group, and, in the next step, that mixing between them is possible. This gives a first intuitive grasp on why the protons lifetime becomes limited, because in the SM theory, a proton decay is impossible, but becomes possible through breaking of the baryon number symmetry, which is not a true symmetry anymore in the SU(5) model. This can be and was used as a tool for verification or falsification of the theory.

so that the 5 transforms as a subgroup of the creation operators (expressed in a more intuitive way, this simply means that one fits some of the particle content of the Standard Model in this representation). The only possibility for this is the

$$(3,1)_{-1/3} \oplus (1,2)_{1/2} \tag{II.3}$$

This gives 5 particles, the first are the three down-quarks ($SU(3)$-triplet) with charge -1/3, the second one the left-handed leptons. The version with up-quarks instead of down-quarks

$$(3,1)_{2/3} \oplus (1,2)_{1/2} \tag{II.4}$$

does not work because S would not be traceless in it (the sum of the charges is not zero). The way one writes the particles into the vectors depends on the choice for the generators of the $SU(3) \times SU(2) \times U(1)$ subgroups in $SU(5)$. An easy way of doing it is writing the $SU(3)$ generators as $\begin{pmatrix} T_a & 0 \\ 0 & 0 \end{pmatrix}$ where the T_a are the $SU(3)$ generators (3×3 matrices), the $SU(2)$ generators (the R_a are 2×2 matrices) are $\begin{pmatrix} 0 & 0 \\ 0 & R_a \end{pmatrix}$ and, for the $U(1)$, the matrix:

$$Y = \begin{pmatrix} -I/3 & 0 \\ 0 & I/2 \end{pmatrix} \tag{II.5}$$

All of the three matrices obviously commute with each other, which is required by them forming subgroups.

By this choice, the 5 representation is the following: $\begin{pmatrix} d_1^\dagger \\ d_2^\dagger \\ d_3^\dagger \\ \overline{e}^\dagger \\ \overline{\nu}^\dagger \end{pmatrix}$

The 10

3 of the Standard Model creation operators remain: the 3 up-quark states, the right-handed-electron and the quark-$SU(2)$-doublet-states. This leaves ten creation operators, and suggests the use of a 10-dimensional representation, which is an antisymmetric tensor product of two 5s. This can be written as a 5x5 antisymmetric matrix (with $\frac{5^2-5}{2} = 10$ independent components). The antisymmetric tensor product is defined as a linear map $A : V \otimes V \rightarrow V \otimes V$ with

$$A(v_1 \otimes v_2) := v_1 \otimes v_2 - v_2 \otimes v_1 \tag{II.6}$$

For $A(5 \otimes 5)$, one finds, after some calculations which can be found in [Tanovic, A., 2009, that this transforms as follows:

$$(3,1)_{2/3} \oplus (1,1)_{-1} \oplus (3,2)_{-1/6} \tag{II.7}$$

But taking the antisymmetric tensor product of the 5 from above gives:

$$(\overline{3},1)_{-2/3} \oplus (1,1)_{1} \oplus (\overline{3},2)_{1/6} \tag{II.8}$$

which is the complex conjugate, the $\overline{10}$.
The leptons can be placed in the 10 in the following way (note that this in fact antisymmetric):

- $\chi^{ab\dagger} = \epsilon^{abc} u_c^\dagger$ for a,b,c from 1 to 3
- $\chi^{a4\dagger} = \overline{\Psi}_2^{a\dagger} = \overline{u}^{a\dagger}$ for a,b,c from 1 to 3
- $\chi^{a5\dagger} = \overline{\Psi}_1^{a\dagger} = \overline{d}^{a\dagger}$ for a,b,c from 1 to 3
- $\chi^{45\dagger} = e^\dagger$

As a conclusion, this means that the fermionic content of the $SU(5)$ GUT is contained in the **5** and the $\overline{\mathbf{10}}$, which are the two irreducible representations.

II.2.3. The generators of SU(5)

There is a general way of constructing the generators of $SU(5)$ taken from [Georgi, 1999]: As noted above, the matrices have to be hermitian and traceless. The aim is thus to find 24 linearily independent 5×5 matrices that fulfill this requirement. Consider the matrices E_{mn} with a 1 at position P:

- All matrices $E_{mn} + E_{nm}$ $(1 \leq m < n \leq N)$
- All matrices $iE_{mn} - iE_{nm}$ $(1 \leq m < n \leq N)$
- All matrices $E_{nn} - I/N$ $(1 \leq n < N)$

This gives $\frac{N(N-1)}{2} + \frac{N(N-1)}{2} + N - 1 = N^2 - 1$ linearly independent matrices that are traceless and hermitian, as can be directly seen in the way they are constructed. This is obviously not the only way to construct the generators. In fact, there are infinitely many possibilities connected by the Cartan subalgebra, introduced in the following section.

II.2.4. The SU(5) Cartan subalgebra

The **Cartan subalgebra** of an algebra is a subset of as many commuting hermitian matrices as possible. The generators are called **Cartan generators** with the following properties:

- $H_i = H_i^\dagger$
- $[H_i, H_j] = 0$

Their number is independent of the chosen basis, and is called the **rank** of the algebra. A useful property of the generators is that, because they commute, they all can be diagonalized at the same time. This leads to the fact that every state corresponding to a representation D can be classified by the eigenvalues of the Cartan subalgebra.

$$H_i \left|\mu, x, D\right\rangle = \mu_i \left|\mu, x, D\right\rangle \tag{II.9}$$

The components of the vector μ_i are the eigenvalues of the states under application of the generator H_i. The vector formed by the components is called **weight vector**. This will now be demonstrated at the example of the SU(5) Cartan subalgebra.

The generators

There is a general way of constructing the Cartan generators for SU(N) which makes sure that they are normalized by

$$\mathrm{Tr}(T_a T_b) = \frac{1}{2}\delta_{ab} \tag{II.10}$$

The N-1 Cartan generators (because the rank of the Cartan subalgebra is always N-1) are

$$[H_m]_{ij} = \frac{1}{\sqrt{2m(m+1)}} \sum_{k=1}^{m} \delta_{jk}\delta_{ik} - m\delta_{i,m+1}\delta_{j,m+1} \tag{II.11}$$

In the basis where all the elements of the Cartan subalgebra are already diagonalized, one finds the 4 commuting matrices for the rank 4 Cartan subalgebra of SU(5):

1. $$\begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

2. $$\begin{pmatrix} 1/\sqrt{12} & 0 & 0 & 0 & 0 \\ 0 & 1/\sqrt{48} & 0 & 0 & 0 \\ 0 & 0 & -1/\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3. $$\begin{pmatrix} 1/\sqrt{24} & 0 & 0 & 0 & 0 \\ 0 & 1/\sqrt{24} & 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{24} & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{\frac{3}{2}}/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

4. $$\begin{pmatrix} 1/\sqrt{40} & 0 & 0 & 0 & 0 \\ 0 & 1/\sqrt{40} & 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{40} & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{40} & 0 \\ 0 & 0 & 0 & 0 & -2/\sqrt{5} \end{pmatrix}$$

The weights

One can find the weights by multiplying the ordinary basis vectors with all the Cartan generators.

- $w_1 = (1/2, 1/\sqrt{12}, 1/\sqrt{24}, 1/\sqrt{40})$
- $w_2 = (-1/2, 1/\sqrt{12}, 1/\sqrt{24}, 1/\sqrt{40})$
- $w_3 = (0, -2/\sqrt{12}, 1/\sqrt{24}, 1/\sqrt{40})$
- $w_4 = (0, 0, -3/\sqrt{24}, 1/\sqrt{40})$
- $w_5 = (0, 0, 0, -2/\sqrt{5})$

This gives 5 weights for the 5 linearilly independent basis vectors.

The roots

The roots are the weights of the adjoint representation. As introduced in section VI.1, in the adjoint representation, the linear operators in the form of the representation matrices are found with the commutators of the algebra (and therefore, the structure constants). This way, a vector space is formed with the generators of the algebra as basis vectors. The roots are now the eigenvalues of the basis vectors (which correspond to the generators).

The simple roots are the roots that can not be written as linear combinations of other roots. They are also basis dependent. In a basis where all Cartan generators are diagonal, the simple roots are the vectors that transport from one weight to the next one, so are four simple roots corresponding to the four Cartan generators for a rank four subalgebra. In this example, they can be gained by computing $w_i - w_{i+1}$:

- $r_1 = (1, 0, 0, 0)$
- $r_2 = (-1/2, 3/\sqrt{12}, 0, 0)$
- $r_3 = (0, -2/\sqrt{12}, 4/\sqrt{24}, 0)$
- $r_4 = (0, 0, -3/\sqrt{24}, 5/\sqrt{40})$

Consequently, all other roots can be written as linear combinations of simple roots. We will look at the roots in the weight basis in the following steps.
Remark:
The concept of roots is of great importance in the classification of algebras. They allow for a geometric approach to the structure of semi-simple algebras, which is due to relation that puts strong constraints on the angles between the roots (see for example [Georgi, 1999] and [Cahn, 2014] for a detailed introduction). This in turn makes it possible to find a visual representation of the structure of the algebra, the so called **Dynik Diagrams**.

II.2.5. Predictions of the SU(5) GUT

Some central points about the SU(5) GUT were already mentioned:

- It explains charge quantization.
- It unifies.
- It predicts proton decay.

Another point made by one of its inventors, Georgi Glashow, is caught in the following quote: *"The most interesting thing about it is the way everything fits."* Although aesthetic principles cannot be the sole guideline in the search for theories, it is nevertheless undoubtable that the SU(5) structure fits very well to the structure of the Standard Model, making its treatment interesting.
Despite that, $SU(5)$ has already been ruled out as a GUT for some time now, because the lifetime of the proton could experimentally be tested to be far above the maximum the theory predicted (see in [(Georgi, 1999)]. This motivated further expansions of the theory (for example, supersymmetric extensions) and the search for other GUTs and TOEs.

III. Advancing after SU(5)

III.1. Motivation

The SU(5) GUT was only one of many possible GUTs that were tested in the course of the twentieth century. Parallel to that, a new, even more fundamental theory called **String Theory** was discovered, which manages to unify gravity and the other three forces, making it a so called **Theory of Everything**. The fundamental assumption of String Theory is that the fundamental building stones of the physical world are no longer pointlike particles, but one-dimensional objects called **Strings**.
String theory is currently the most promising applicant for a Theory of Everything, making predictions ranging from particle physics to general relativity, but has not yet been experimentally verified or falsified. This led and still leads to much controversy among physicists as to whether working on a theory that may not ever be tested experimentally (as the size of the strings is extremely small) is not against the principles of science. Setting this aside, the advances in String Theory have created many connections between mathematics and physics, and even led to fundamentally new discoveries in mathematics (for an introduction, see e.g. in [Weigand, 2011a]).
There is a branch of research in String Theory focusing on trying to recreate the physics of the world currently observable out of String Theory. This is generally called **String Phenomenlogy**. An introduction on this is given for example in [Louis, J., 2015].
The structure called complete networks that will be discussed in chapter IV plays a role in **Type IIB** String Theory. This connection will be sketched in section IV.1.1.

III.2. Weights and representations

III.2.1. The weights of different representations

Weights and charges

The concept of weights is of physical significance because the weights are the eigenvalues correponding to the Cartan generators. In a physical sense, these eigenvalues can be seen as charges. An example for this from Quantum Field Theory (see [Weigand, 2013]) is the conserved Noether charge associated with global $U(1)$ transformations. Applying this to creation operators in the **Fock space** (corresponding to physical particles) gives eigenvalues ± 1, which are the electric charges (under $U(1)$). This can be generalized for bigger symmetry groups, like $U(1)^4$, which has four entries in the charge vector.

Diagonal weights basis

Another way of writing the generators of the Cartan subalgebra is by embedding them in $U(1)^4 \to S[U(1)]^5$. One can then use the five matrices

- $t^1 = diag(1,0,0,0,0)$
- $t^2 = diag(0,1,0,0,0)$
- ...

as a basis and form the Cartan generators with the traceless constraint

$$\sum_{i=1}^{5} a_i t^i \;\;\forall\;\; \sum_i a_i = 0. \tag{III.1}$$

A matrix fulfilling these two constraints is the most general form of a $U(1)$ symmetry of the system and will be looked at in further detail later. Notice that these generators are generally not normalized, but normalization has to be carried out additionally.
The weights associated with this are simply the row vectors

- $t_1(1,0,0,0,0)$
- $t_2(0,1,0,0,0)$
- ...

This can be called a **diagonal weights basis**, because the basis is chosen so that the weights are as simple as possible.
The t's contract by

$$t^i t_j = \delta^i_j \tag{III.2}$$

In this new basis, the weights for the fundamental, the antisymmetric and the adjoint representation can be constructed. Notice that they are similar to the weights defined in II.2.4, where fulfillment of the traceless condition $\sum_i w_i = 0$ is easily checked.

Fundamental representation

For the fundamental representation, the weights are just the vectors t_1 to t_5 as written above.

Antisymmetric representation

In general, a tensor product of representations has weights which are just the sum of the weights of the representations (from Shapiro, J., 2015). The antisymmetric representation was introduced earlier in the context of finding the necessary representations for the lepton content of the Standard Model. Thus, as the antisymmetric is the tensor product of two fives, one simply adds up two weights. Ten different weights are found for the ten different components. This is done by the prescription

$$t_i + t_j \text{ with } i<j \tag{III.3}$$

which gives ten weight vectors, all linear combinations of the weight basis vectors.

Adjoint representation

The weights of the adjoint representation as defined in (VI.1) are just the roots. One expects, from the 24 generators of the adjoint representation, to get four uncharged states with zero weight for the Cartan subalgebra and 20 charged states because the roots are just differents of weights, and for every root α, $-\alpha$ is also a root. With this at hand, one can easily find the non-zero-weights by the prescriptions

$$\begin{aligned} t_i - t_j \text{ with } i<j \\ \text{and } t_i - t_j \text{ with } j<i. \end{aligned} \tag{III.4}$$

By adding the respective This gives 20 different weights and four trivial charges $q = 0$ in the case of $i = j$, meaning that they are uncharged under the Cartan subalgebra. For this, consider:

$$H_i \left| H_j \right\rangle = \langle [H_i, H_j] \rangle = 0 \tag{III.5}$$

So as all Cartan generators commute, they have zero weights (zero roots). It has to be noted again that the weights are basis dependent and have five components instead of four, which is due to the fact that the additional traceless constraint on the Cartan generators has not been carried out.

Adding additional singlets

Besides the singlets of the form $t_i - t_j$, there is a second form of singlets that can be introduced. This is helpful because singlets can secure gauge invariant couplings between pairings of $\mathbf{5}$s and $\mathbf{10}$s through $\mathbf{5}\,\bar{\mathbf{5}}\,\mathbf{1}$ and $\mathbf{10}\,\overline{\mathbf{10}}\,\mathbf{1}$ couplings. They are introduced as

$$\boldsymbol{X} = perm(1, 1, -1, -1, 0), \tag{III.6}$$

so $\boldsymbol{X}_{ijkl} = t_i + t_j - t_k - t_l$ is one possible singlet. Summing up all possibilities leads to 15 additional singlets.

III.3. A case of E_8 breaking

E_8 plays an important role in the mathematics of String Theory, and has a variety of applications. The term "E_8" can actually mean a number of things, depending on the mathematical context in which it is examined and applied. Here, it suffices to say that E_8 is the name for some closely related Lie groups and Lie algebras of dimension 248 which are the largest possible exceptional Lie groups. Some notes on this will be made in the following chapters.

III.3.1. An E_8 decomposition

E_8 contains a $SU(5) \times SU(5)$ subgroup that can be labeled as $SU(5)_{GUT} \times SU(5)_{perp}$. One can specify the breaking pattern for the adjoint representation of E_8 in the following way (taken from [Maharana & Palti, 2012]):

$$\mathbf{248} \;\rightarrow\; (\mathbf{24}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{24}) \oplus (\mathbf{10}, \mathbf{5}) \oplus (\bar{\mathbf{5}}, \mathbf{10}) \oplus (\overline{\mathbf{10}}, \bar{\mathbf{5}}) \oplus (\mathbf{5}, \overline{\mathbf{10}}) \tag{III.7}$$

Where the first number in each bracket stands for $SU(5)_{GUT}$ and the second one for $SU(5)_{perp}$. This means for example that the GUT fundamental representations are in the 10-representation of the perpendicular SU(5). The reason why this is interesting lies in the various possible breaking patterns of SU(5) into factors of $U(1)$. In our case, we consider the breaking pattern

$$E_8 \;\rightarrow\; SU(5) \times S[U(1)^5] \tag{III.8}$$

which is the purely Cartan way of breaking the $SU(5)$. This is exactly the situation that was considered when writing the weights in the diagonal weights basis with the additional traceless constraint. What this means is that one can for example consider a **10**-multiplet from the $SU(5)$ GUT group. These are in the fundamental representation of the perpendicular $SU(5)$. One then knows from the considerations above how the fundamentals are charged under the 4 $U(1)'s$ or the $S[U(1)^5]$ respectively, and also knows how the other representations are charged.

- Q(5)=t_i
- Q(10)=$t_i + t_j$ with $i<j$
- Q(1)=$t_i - t_j$ with $i<j$ and their negatives.

III.3.2. Yukawa Couplings and Gauge invariance

Yukawa couplings are, in the reference frame of QFT, cubic couplings between a scalar field and two fermionic fields.
In this thesis, we will consider the couplings of any of the three possible combinations of representations of the $SU(5)_{GUT}$. As we know how they are charged under the $SU(5)_{perp}$, it is possible to discuss gauge invariant couplings between them (couplings where the charges add up to zero). The possible couplings are:

- 1. **5 10 10** (up type coupling)
- 2. $\overline{\mathbf{5}}$ $\overline{\mathbf{5}}$ **10** (down type coupling)
- 3. **1 5** $\overline{\mathbf{5}}$
- 4. **1 10** $\overline{\mathbf{10}}$
- 5.**1 1 1**

The **5** is charged under $\overline{\mathbf{10}}$, the **10** under the **5** and the **1** under the adjoint. With this at hand, one can construct cubic couplings and find the gauge invariance conditions for every case:

- 1. $-t_i - t_j + t_k + t_l \stackrel{!}{=} 0$
- 2. $t_i + t_j + t_k + t_l + t_m \stackrel{!}{=} 0$
- 3. $t_i - t_j - t_k - t_l + t_m + t_n \stackrel{!}{=} 0$

- 4. $t_i - t_j + t_l - t_m \stackrel{!}{=} 0$
- 5. $t_i - t_j + t_k - t_l + t_m - t_n \stackrel{!}{=} 0$

For couplings 1,3,4 and 5, the invariance condition is easily found as there is an equal number of parings t_i and $-t_j$, so the term is zero if pairings can be found for all charges. For coupling number 2, this is not so obvious. But remembering the tracelessless condition on the weight basis $\sum_i t_i = 0$, this is indeed fulfilled if $i \neq j \neq k \neq l \neq m$. Looking back at $SU(3)$, this is the same structure that makes three quarks of different coulour colourless when brought together.

III.4. Higgsing down from $SU(5) \times U(1)^4$ using GUT singlets

III.4.1. "Higgsing" in this context

"Higgsing" as the expression of the application of the Higgs mechanism I.2 was already introduced in I.2.1 as the spontaneous breaking of certain symmetries. It is from a different perspective that this is inspected here. As an example, we will discuss the Higgsing down from a $SU(5) \times U(1)^4$ to a theory with just one $U(1)$ using GUT singlets by two different chains (a different order in which the singlets are applied).
After the Higgs mechanism is applied to a system, the remaining symmetries are the ones under which the Higgs is uncharged. This can be made clear by looking at an example from the Standard Model Lagrangian. The Higgs field couples to the gauge field terms in the following way:

$$|(\delta_\mu + iqA_\mu)\Phi|^2 \tag{III.9}$$

So to leave the coupling to the gauge fields in the Lagrangian invariant, the terms in which the Higgs field occur have to vanish, so

$$q^2\Phi^2 A_\mu A^\mu = 0 \text{ and } \delta_\mu q A_\mu \Phi = 0 \tag{III.10}$$

As q appears in both terms, the contribution of the Higgs field vanishes for charge zero. In our case, the Higgs can be two different types of singlets formed by

$$\mathbf{1} : t_i - t_j \quad \textbf{and} \quad \boldsymbol{X} : t_i + t_j - t_l - t_k, \tag{III.11}$$

where our applications will focus on the first type of singlets, the **1**s. A most general $U(1)$ symmetry was found to be generated by a Cartan generator fulfilling

$$\sum_{i=1}^{5} a_i t^i \ \ \forall \ \ \sum_i a_i = 0 \tag{III.12}$$

Higgsing a system with a certain set of $U(1)$ symmetries, the remaining symmetries are in the same way as above the symmetries under which the Higgs is uncharged. As an example, when the Higgs is a singlet of the form $< \mathbf{1} >= t_i - t_j$, for a symmetry to remain unbroken, the condition reads

$$Q : (a_i - a_j) = 0 \tag{III.13}$$

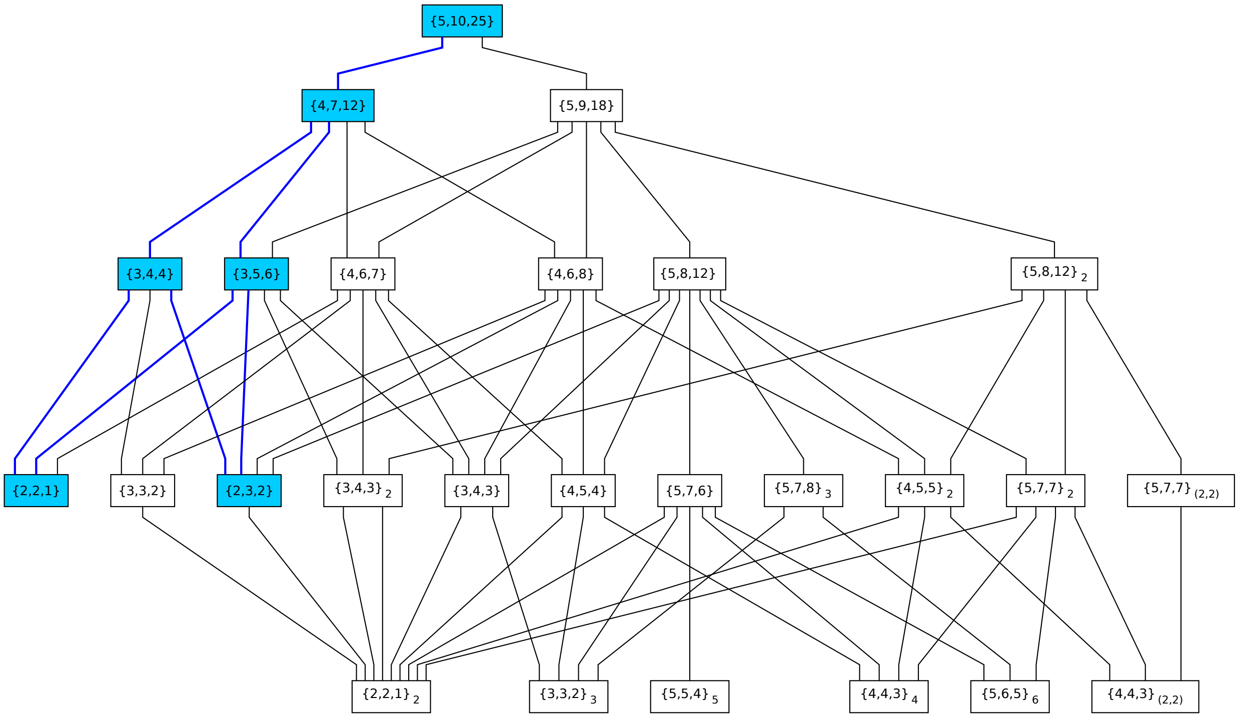

Figure III.1.: *The tree diagram for all possible Higgsing paths, taken from [F. Baume, Palti, & Schwieger, 2015]*

So the remaining $U(1)$s after Higgsing satisfy a new traceless condition. As an example, with the Higgs being the singlet $< 1 >= t_1 - t_2$, the traceless condition becomes

$$2a_1 + a_3 + a_4 + a_5 = 0 \tag{III.14}$$

III.4.2. The spectrum of different Higgsing chains

Starting from a full set of states [5,10,25], where 5 denotes the number of differently charged **10**s, the 10 of differently charged **5**s and the 25 of differently charged singlets (10 **1**s and 15 $\boldsymbol{X}s$, one can apply different singlets (it is also possible to use the additional singlets introduced in III.2.1, but this will not be done here) and Higgs the system in different ways.

A thorough analysis of this leads to the tree structure depicted in figure III.1. Note that the choices of basis for the $U(1)$s is arbitrary, so all $U(1)$s satisfying the traceless conditions and being independent from each other are possible $U(1)$s. In the same way, the manner in which the **10**s and the **5**s are named is also dependent on the individual choice.

Two examples will be considered here: the $\langle \mathbf{1}_{12}\mathbf{1}_{13}\mathbf{1}_{45} \rangle$ and the $\langle \mathbf{1}_{12}\mathbf{1}_{13}\mathbf{1}_{14} \rangle$ Higgsing chains, where the $\mathbf{1}'s$ denote the singlets used. These two chains lead to two different sets of states, a [2,2,1] and a [2,3,2], which both can form a structure called "complete network", which will be further explored in the rest of this thesis.

Table III.1.: *The spectrum after Higgsing with a singlet of the form* $\langle \mathbf{1}_{12} \rangle$

$\mathbf{10}$	$\bar{\mathbf{5}}$	$\pm\mathbf{1}$
t_3	$t_3 + t_4$	$\pm t_3 - t_4$
t_4	$t_3 + t_5$	$\pm t_3 - t_5$
t_5	$t_4 + t_5$	$\pm t_4 - t_5$
$-\frac{1}{2}(t_3 + t_4 + t_5)$	$-(t_3 + t_4 + t_5)$	$\pm - \frac{1}{2}(t_3 + t_4 + 3t_5)$
	$\frac{1}{2}(t_3 - t_4 - t_5)$	$\pm - \frac{1}{2}(t_3 + 3t_4 + t_5)$
	$\frac{1}{2}(-t_3 + t_4 - t_5)$	$\pm - \frac{1}{2}(3t_3 + t_4 + t_5)$
	$\pm\frac{1}{2}(-t_3 - t_4 + t_5)$	

Table III.2.: *The spectrum after Higgsing with singlets* $\langle \mathbf{1}_{12} \rangle$ *and* $\langle \mathbf{1}_{13} \rangle$

$\mathbf{10}$	$\bar{\mathbf{5}}$	$\pm\mathbf{1}$
t_3	$2t_3$	$\pm(t_3 - t_5)$
t_5	$-2t_3 - t_5$	$\pm(3t_3 + 2t_5)$
$-3t_3 - t_5$	$t_3 + t_5$	$\pm(4t_3 + t_5)$
	$-3t_3$	

First level

The first choice of a singlet is arbitrary because all of the possibilities are connected by permutations of the basis vectors. So without loss of generality, one can set $\langle \mathbf{1}_{12} \rangle$ as the first singlet. This leaves the set of states depicted in table III.1.

For the $\mathbf{10}$s, this is seen as follows: starting from the 5 independent $\mathbf{10}$s t_i, after the first Higgsing ($t_1 = t_2$), there remain the unaffected states t_3,t_4 and t_5, and the fourth is restricted by the condition $t_1 = t_2 = -\frac{1}{2}(t_3 + t_4 + t_5)$. For the $\mathbf{5}$s, the principle followed is the same. Starting off with the three combinations only depending on t_3, t_4 and t_5, the following ones are in the same way combinations where $t_1 = t_2 = \frac{1}{2}(t_3 + t_4 + t_5)$ was used, for example $t_1 + t_2 = -(t_3 + t_4 + t_5)$. Note that, in 3 cases, this leads to an arbitrariness in the states, because, $t_1 + t_i = t_2 + t_i$ for i=3,4,5. After following the same procedure for the singlets, we end up with a [4,7,6] set of states. Note here that two singlets which only differ by a minus sign are not counted seperately in this convention.

Second level

There are two ways to continue now that lead to a different set of states. The first is to choose a singlet that has either a 1 or a 2 in it, or to use a different singlet, like $\langle \mathbf{1}_{34} \rangle$ or $\langle \mathbf{1}_{45} \rangle$. For the purpose of this thesis, we continue with the singlet $\langle \mathbf{1}_{13} \rangle$. On the third level, two different choices of singlet will be examined. The set of states after the second Higgsing is depicted in table III.2. The procedure that was followed here is similar to the one used for the first level.

Table III.3.: *The spectrum after Higgsing with singlets* $\langle \mathbf{1}_{12} \rangle$, $\langle \mathbf{1}_{13} \rangle$ *and* $\langle \mathbf{1}_{14} \rangle$

$\mathbf{10}$	$\bar{\mathbf{5}}$	$\pm\mathbf{1}$
t_5	$-\frac{1}{2}t_5$	$\pm\frac{5}{4}t_5$
$-\frac{1}{4}t_5$	$-\frac{3}{4}t_5$	

Table III.4.: *The spectrum after Higgsing with singlets* $\langle \mathbf{1}_{12} \rangle$, $\langle \mathbf{1}_{13} \rangle$ *and* $\langle \mathbf{1}_{45} \rangle$

$\mathbf{10}$	$\bar{\mathbf{5}}$	$\pm\mathbf{1}$
t_5	$2t_5$	$\pm\frac{5}{3}t_5$
$-\frac{2}{3}t_5$	$\frac{1}{3}t_5$	
	$-\frac{4}{3}t_5$	

Third level

Here we will look at two different examples, the first one being a Higgsing by the singlet $\langle \mathbf{1}_{14} \rangle$ with the spectrum depicted in III.3, and the second one by Higgsing with the singlet $\langle \mathbf{1}_{45} \rangle$, for which the spectrum can be found in table III.4. The resulting spectrums for the Higgsing chains are a [2,2,1] network and a [2,3,1] spectrum. We are now interested at how they are charged under the remaining $U(1)$ symmetry in order to be able to look at (gauge invariant) Yukawa couplings.
It is straightforward to read the charges off. For simplicity, we define the $U(1)$ coupling in a way so that all charges are integers, so

$$U(1) = 4t_5 \tag{III.15}$$

As the tracelessness constraint was already incorporated in the sets of state, it does not have to be included in the $U(1)$ symmetry. Applying the states to this symmetry gives the following charges for the [2,2,1]:

$\mathbf{10}$	$\bar{\mathbf{5}}$	$\pm\mathbf{1}$
4	-2	± 5
-1	3	

and for the [2,3,1]:

$\mathbf{10}$	$\bar{\mathbf{5}}$	$\pm\mathbf{1}$
3	6	± 5
-2	1	
	-4	

The way these charges are distributed form what was already referred to as a complete network.

The additional singlets

The additional singlets $\boldsymbol{X}$ were left out of the spectrums, but their calculation can be done in a similar manner as for the other charges. The most vital conclusions for this thesis are that the [2,3,1] network becomes a [2,3,2] network, with the additional singlet carrying charge 10, and that the $[5, 10, 25]$ four dimensional network formed with these additional singlets is complete (see [Baume, n.d.] for a thorough analysis).

IV. Complete Networks

IV.1. Introduction

IV.1.1. Connections to String theory

The details of the connections between complete networks and String Theory exceed the scope of this thesis, but a short overview will be given in the following section.

Type IIB String Theory

As was mentioned earlier, the fundamental objects of String theory are the one-dimensional **Strings**. The vibrational patterns of these strings are responsible for the physical properties with which they are measured in experiments (there is, for example, a certain vibrational pattern of a closed string corresponding to the graviton).
Type IIB String theory is one of the 5 string theories that were unified in the second superstring revolution to form **M-theory** (look for example in [Elfman, 2011b] for more details).

D-branes

In the framework of type IIB String Theory, other fundamental objects called **D-branes** are introduced as objects on which open strings can end. They are multi-dimensional surfaces located in spacetime. Only branes with odd spatial dimensions and one time dimension (D1, D3, D5, D7 and D9) are present in the theory (see for example in [Elfman, 2011a] for more insights).
Of special interest are D7-branes because they allow for the formation of gauge theories based on the U(N) groups (the groups of all unitary matrices which do not need to have determinant one as opposed to elements of SU(N)). This can be visualized by imagining branes and the way the strings can form connections between them. For $N = 1$ and $N = 2$, this is depicted in figure IV.1. The number of possible string connections corresponds to the number of arising states, and it can easily be checked that this is always the number of generators of the U(N) algebra N^2.

Intersecting D-branes

Of further interest is the study of intersections of D-branes, because this gives additional possibilities for states to arise at the junction of the branes. Roughly speaking, a D-brane intersecting five D-branes stacked above each other leads to a $U(5) \times U(1) \cong SU(5) \times U(1) \times U(1)$ theory, which gives an idea as to how the theory of intersecting D-branes becomes related to the other topics studied in this thesis.
The subject of intersecting D-branes is a very rich one. Much more profound insights can be found for example in [Buznego, 2003].

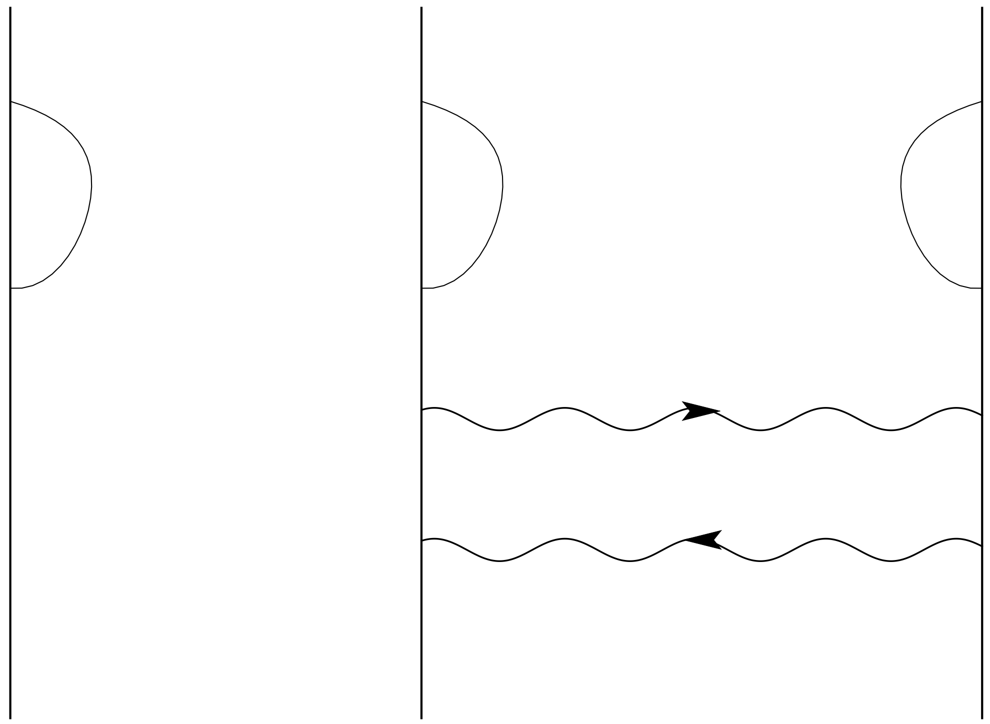

Figure IV.1.: *The connections of strings with branes for N=1 and N=2*

Matter curves and complete networks

We consider a six dimensional complex **Calabi-Yau-Manifold** (for an introduction on these manifolds, see for example [Dennis, 2008]). The Calabi-Yau-Manifold can be specified by three complex parameters giving six real dimensions.
On the manifold, we specify 4-cycles, which are four-dimensional products of four linearilly independent cycles (as the lower dimensional analogon, a circle is a 1-cycle and a donut is a 2-cycle) by an arbitrary constraint, e.g. $z_i = 0$ without loss of generality. Two 4-cycles then generically[1] intersect in a 2-cycle, described on the manifold by $z_i = z_j = 0$. In our framework, these 2-cycles are called **matter curves** and associated with $SU(5)$ representations (see III.3.1). Because of that, their intersections are connected to complete networks. Two matter curves generically intersect in a point given by $z_1 = z_2 = z_3 = 0$ without loss of generality. In these points, the matter curves can couple to each other by Yukawa couplings. For a theory defined that way to be gauge invariant, all pairings of matter curves have to be able to form gauge invariant couplings, requiring that there always exists a third matter curve that makes the coupling gauge invariant. This leads to the structure examined in this chapter.

[1]There still can be a class of exceptions. Consider for example two lines in a plane which intersect if they are not parallel. But as them being parallel is an additional constraint giving a hypersurface in probability space, this means that the probability for the lines not to intersect is given by zero as the surface does not add to the spatial integral. Therefore, the term ”generically” means that for an arbitrary choice, all exceptions have zero probability.

IV.1.2. Definition

Recalling the possible Yukawa couplings for the three types of representations:

- 1. **5 10 10** (up type coupling)
- 2. $\overline{\mathbf{5}}\ \overline{\mathbf{5}}$ **10** (down type coupling)
- 3. $\mathbf{1}\ \mathbf{5}\ \overline{\mathbf{5}}$
- 4. $\mathbf{1}\ \mathbf{10}\ \overline{\mathbf{10}}$
- 5. **1 1 1**

one can define a complete network in the following way:

- *A coupling of a set of representations charged under a number of $U(1)$s is called gauge invariant if all the charges add up to zero:*

$$\vec{q_1} + \vec{q_2} + ... + \vec{q_n} = 0 \tag{IV.1}$$

- *A **complete network** is a structure in which, for every possible pairing of two (**5** and/or **10**) representations, there is a third one which makes the coupling gauge invariant.* [2]

Example:
As the simplest example, one assumes that there is only one $U(1)$ charge present. The representations are characterised by this charge. Take now a set of states [1,1,0] where the charge of the **10** is 1 and the charge of the $\overline{\mathbf{5}}$ is 2. The possible pairings between the two are the up type and the down type coupling. As

$$\mathbf{10} + \mathbf{10} + \mathbf{5} = 1 + 1 - 2 = 0 \tag{IV.2}$$

the network is indeed complete. Advancing from this simplest example, the questions whether the complete networks could be classified and whether the number of possible states could be extended towards infinity was faced.

IV.1.3. The simplest [2,2,1] network

We started to think about the simplest "complete network" structure with all 3 charges present.
Trying to construct this in a strictly general way leads to a long list of conditions that need to be satisfied for the network to be complete. It is therefore impractical to start from a merely formal standpoint, and more useful to look at the charges of the [2,2,1] network we found above by Higgsing down to a single $U(1)$ (considerations on why this network is complete can be found in V.2. This leads to the following

[2]This is not required for the singlets. Rather, one can simply leave out singlets that are not necessary for making any of the parings gauge invariant.

Proposition:

The [2,2,1] network formed by the charges $[-1,4\,;\,-2,3\,;\,5]$ is a **complete network**.

Proof:

- Starting off with the 1 and pairing it with the -4, we see that, by a type 4 coupling, gauge invariance can be achieved for the 1 being a $\overline{\mathbf{10}}$, the -4 a **10** and the 5 a singlet, with the charges adding up by $-1-4+5=0$. This also takes care of the pairing of 1 with 5.

- The second pairing is the 1 with the 2. Using the second $\overline{\mathbf{5}}$, by a type 2 coupling, this gives 1+2-3=0, and also takes care of the next pairing 1 with -3.

- In the same way, all other pairings can be checked and found to have a third charge that cancels them to zero, making the structure a complete network.

IV.1.4. The [2,3,2] network

The same examination can be carried out for the $[2,3,2]$ network that was found above for the $\langle \mathbf{1}_{12}\mathbf{1}_{13}\mathbf{1}_{45}\rangle$ Higgsing chain. It is straightforward to check that the network formed by the charges $[3,-2\,;\,6,1,-4\,;\,5,10]$ is indeed also a complete network.

IV.2. Extending the networks to infinity

An important claim that can be made is the following: It is possible to extend both complete networks to infinity by adding states in differences of 5.

IV.2.1. Extending the [2,2,1] network

First step

Starting off with the charge distribution $[-1,4\,;\,-2,3\,;\,5]$, it is possible to add a **10** with charge -6, a $\overline{\mathbf{5}}$ with charge -7, and a **1** with charge 10, leading to a $[3,3,2]$ network with charges $[-1,4,-6\,;\,-2,3,-7\,;\,5,10]$. Checking all the possible pairings leads to the conclusion that this also is a complete network. As a next step, one adds a **10** with charge 9, and a $\overline{\mathbf{5}}$ with charge 8. Again one can check, for example, for the **10** with charge 9, that all pairings lead to a gauge invariant coupling:

$i)\ 9-1-8=0,\ \ ii)\ -9+4+5=0,\ \ iii) 9-6-3=0,\ \ iv)\ 9-2-7=0,$
$v)\ 9-6-3=0,\ \ vi)\ 9-7-2=0,\ \ vii)\ -9+4+5=0,\ \ viii)\ -9-1+10=0.$

The same works out for the added $\overline{\mathbf{5}}$ with charge 8, making the network a complete network.

The general extension

The idea now is to extend the network in the same way as in the first step by adding charges that differ by 5 (add 5 for a positive charge, substract 5 for a negative charge) from the charges already present, thereby extending $[1,4,-6,9;-2,3,-7,8;5,10]$ to

$\rightarrow [-1,4,-6,9,-11\ ;\ -2,3,-7,8,-12\ ;\ 5,10,15]$
$\rightarrow [-1,4,-6,9,-11,14\ ;\ -2,3,-7,8,-12,13\ ;\ 5,10,15]$.

As this still is a complete network, it makes sense to assume that this extension always leads to complete networks. This is now proven by induction.

IV.2.2. Proof

n=0

The first step of the proof by induction was already done when it was checked in IV.1.3 that the [2,2,1] network used as the basis of all extensions is a complete network. The next step that concludes the proof is the demonstration of the following theorem:
Starting from a complete network of the type $[-1,4,-6,9,...-n-1,n+4\ ;\ -2,3-7,8...-n-2,n+3\ ;\ 5,10,...n+5]$, *the network* $[-1,4,-6,9,...-n-1,n+4,-n-6,n+9\ ;\ -2,3-7,8...-n-2,n+3,-n-7,n+8\ ;\ 5,10,...n+5,n+10]$ *is also a complete network.*
For proving that this is true, one needs to check that for every pairing of the five newly added charges, a third charge exists that leads to gauge invariance. Therefore, the proof breaks down in 5 parts, checking gauge invariance for each one of the new charges. The main idea behind this proof is that each charge combines with other charges in what can be called a "chain" that moves in differences of 5 upwards in one direction and downwards in the other direction, always guaranteeing the existence of a pairing that leads to gauge invariance of the cubic coupling.

1. -n-6

These pairings need to be worked out for each charge individually: starting with the **10**s, the first charge considered is the $-n-6$. Here, and in the same way for every other charge, it is possible to set up 4 chains in which the charge combines with other charges to form gauge invariant couplings.

- **Type 3 couplings:**
  $-n-6,-1 \rightarrow n+5;\ \ -n-6,-6 \rightarrow n;\ \ -n-6,-11 \rightarrow n-5;\ \ \ldots$
  $-n-6,-n-1 \rightarrow 5$
- **Type 1 couplings:** $-n-6,-1 \rightarrow -n-7;\ \ -n-6,4 \rightarrow -n-2;$
  $-n-6,9 \rightarrow -n+3;\ \ \cdots -n-6,n+4 \rightarrow -2;\ \ -n-6,n+9 \rightarrow -7$
  This chain covers all pairings with extensions of $-2 \rightarrow -n-7$ and of $4 \rightarrow n+9$.
- **Type 2 couplings:** $-n-6,-2 \rightarrow n+8;\ \ -n-6,3 \rightarrow n+3;$
  $-n-6,8 \rightarrow n-2;\ \ \cdots -n-6,n+3 \rightarrow 3$
  This chain covers all pairings with extensions of $3 \rightarrow -n+8$

- From this, one can conclude that this newly added charge can form a gauge invariant coupling with every other charge present, and satisfies the condition for a complete network.

2. n+9

Following the same procedure for $n+9$ charge, we find:

- **Type 3 couplings:**
$n+9, -1 \to n+10$; $n+9, 4 \to n+5$; $n+9, 9 \to n$; ...
$n+9, n+4 \to 5$
This chain covers all pairings with extensions of $5 \to -n+10$

- **Type 1 couplings:** $n+9, -1 \to n+8$; $n+9, -6 \to n+3$; ...
$n+9, -n-6 \to 3$;
This chain covers all pairings with extensions of $-1 \to -n-6$ and $3 \to n+8$

- **Type 2 couplings:** $n+9, -2 \to -n-7$; $n+9, -7 \to -n-2$;
$\ldots n+9, -n-7 \to -2$
This chain covers all pairings with extensions of $-2 \to -n-7$

- In the same way as for the n-6, all of the possible pairings are covered in these three chains.

3. -n-7

Continuing with the $\overline{\mathbf{5}}$s, one finds:

- **Type 3 couplings:** $-n-7, 3 \to n+10$; $-n-7, -2 \to n+5$; ...
$-n-7, -n-2 \to 5$;
This chain covers all pairings with extensions of $5 \to -n+10$

- **Type 1 couplings:** $-n-7, -n-6 \to -1$; $-n-7, -n-1 \to -6$; ...
$-n-7, -1 \to -n-6$;
This chain covers all pairings with extensions of $-1 \to -n-6$

- **Type 2 couplings:** $-n-7, -2 \to n+9$; $-n-7, 3 \to n+4$;
$\cdots -n-7, n+3 \to 4$; $-n-7, n+8 \to -1$
This chain covers all pairings with extensions of $3 \to -n+8$ and $4 \to n+9$

4. n+8

As all the pairings from the three charges above included the $n+8$, it is obvious that, for symmetry reasons, the $n+8$ has gauge invariant Yukawa couplings with all charges present, which concludes the proof.

Table IV.1.: *The smallest two-dimensional network with all charges present*

$\mathbf{10}$	$\bar{\mathbf{5}}$	$\pm\mathbf{1}$
$(1,0)$	$(-1,-1)$	$\pm(3,2)$
$(0,1)$	$(1,2)$	
$(-2,-2)$		

IV.2.3. Extending the $[2,3,2]$ network

In a very similar fashion, it is also possible to extend the $[2,3,2]$ network formed by the charges $[3,-2\ ;\ 6,1,-4;\ 5,10]$ to infinity. The extensions are done in the following way: $[3,-2\ ;\ 6,1,-4\ ;\ 5,10] \rightarrow [3,-2,8,-7\ ;\ 11,6,1,-4,-9\ ;\ 5,10,15]\ldots$ Note that, because adding 5 to the $\overline{\mathbf{5}}$ charged by 1 leads to the $\overline{\mathbf{5}}$ charged with 6 that is already present, so the network turns into a $[4,4,3]$, and from this on to a $[6,6,4]$. But as this is the same as a $[6,6,3]$ with an additional (and, for completeness, of the network unnecessary) singlet, the rest of the extension is completely similar as in the $[2,2,1]$ case.

IV.3. Networks with two $U(1)$s

IV.3.1. The simplest two-dimensional complete network

A logical next step was trying to find complete networks with more than one $U(1)$ symmetry. In the case of two $U(1)$ symmetries, every state is denoted by a doublet (q_1, q_2). We began by constructing a smallest complete network with all three representations present. This is fulfilled by the network depicted in table IV.1, which was found by starting with the two independent **10**s and adding charges so that all couplings remained invariant.
This specific [3,2,1] network is not the only possible [3,2,1] network. Rather, one can set the charges in a two-dimensional vector space with (without loss of generality) the first two **10**s $(1,0)$ and $(0,1)$ as basis vectors. Every linear transformation A on this space leaves the complete network structure untouched, because if

$$\vec{q}_1 + \vec{q}_2 + \vec{q}_3 = 0 \qquad \text{then also}$$
$$A(\vec{q}_1 + \vec{q}_2 + \vec{q}_3) = 0 \text{ for all couplings,} \tag{IV.3}$$

and the network remains complete.

IV.3.2. The [3,4,3] network

As in the case of one $U(1)$ symmetry, the network found from Higgsing down E_8 is also a complete network. The charges of the $\langle \mathbf{1}_{12}\mathbf{1}_{13}\rangle$ Higgsing can be read off from the spectrum of states that was found in section III.4.2. In this case, the first $U(1)$ charge is, for example, simply t_3, and the other one t_5. This is possible because the traceless constraint was already enforced in the construction of the states. The charges multiplied in a way that all charges are integer are depicted in table IV.2. By direct computation, it can be checked that this is a complete network.

Table IV.2.: *The two dimensional complete network gained by Higgsing with two singlets* $\langle \mathbf{1}_{12}\mathbf{1}_{13} \rangle$.

$\mathbf{10}$	$\bar{\mathbf{5}}$	$\pm\mathbf{1}$
$(1,0)$	$(2,0)$	$\pm(1,-1)$
$(0,1)$	$(-2,-1)$	$\pm(3,2)$
$(-3,-1)$	$(1,1)$	$\pm(4,1)$
	$(-3,0)$	

Table IV.3.: *The extension that was attempted for the two dimensional complete network.*

$\mathbf{10}$	$\bar{\mathbf{5}}$	$\mathbf{1}$
$(1,0)$	$(2,0)$	$(1,-1)$
$(0,1)$	$(-2,-1)$	$(3,2)$
$(-3,-1)$	$(1,1)$	$(4,1)$
$(0,1+n)$	$(-2,-1-n)$	$(1,-1-n)$
$(-3,-1-n)$	$(1,1+n)$	$(3,2+n)$
$(1,\pm n)$	$(2,\pm n)$	$(0,n)$

IV.3.3. Searching for an extension of the [3,4,3] two dimensional network

Attempting an extension

The complexity of the network increases rapidly by adding more $U(1)$s. The first idea for a possible extension was to separate the different components of each charge, and try to extend them in both dimensions individually. For this, it was helpful to note that every second component of each charge had either a 1, a 0 or a -1 in it, with exception of the (3,2) state, so the hope was that pairings between a 1 and a -1 always had to cancel each other out, meaning that possible extensions would conserve the unique nature of these couplings. The extension we attempted commenced in the following, general way:

- For each state, add an arbitrary integer n to the second component if the charge is positive, and substract n if the charge is negative. For the two states with second component zero, both add and substract n in both cases.
- Add an additional singlet $(0,n)$ which accounts for the difference between the extensions and the states from which they were extended.

This leads to a set of states shown in table IV.3. The idea was that this still forms a complete network, and also does for each layer of extensions corresponding to integers $n, m, l, \ldots$. It is straightforward to see that this would also allow for an extension towards infinity.

Possible Pairings

- As the structure the extension was built upon was a complete network, only the possible pairings between charges of the basis network and the extended network and pairings between charges of the extension had to be tested for gauge invariance.
- All pairings of extensions and the state they were extended from are taken care of by the $(0,n)$ singlet.

- Pairings of the form (x,0) and (y,0) are not affected by the extension and are still gauge invariant.
- All pairings of a state of the form (j,0) with a state of the form (k,1+n) or (k,-1-n) are also uniquely determined and taken care of. This can be made clear by the following consideration: the pairing (j,0) with (k,1) or (k,-1) requires a third state that makes the coupling gauge invariant. This state necessarily exists because the network is complete and has to be of the form (l,-1) or (l,1) for the second component to cancel out. In the extended version of the network, for pairings with (k,1+n) or (k,-1-n), there also exist states (l,-1-n) or (l,1+n), because they were also added in the extension.
- This leaves pairings in which the second components of the states do not cancel out, for example a paring of the states (0,1) with (-3,-1-n) to (3,2+n). This works out for most cases, but leads to a significant problem.

The underlying problem

The fundamental problem with extending the two-dimensional complete network becomes apparent when considering pairings that involve extensions of states of the form (m,0). For this, consider the following example, for which n=2 [3] was plugged in for clarity: Starting off with the pairing

$$(0,1) + (-2,-1) \rightarrow (2,0) \tag{IV.4}$$

from the complete network, we see that, after extending the network, this leads to three different combinations:

$$\begin{aligned} (0,3) + (-2,-3) &\rightarrow (2,0) \\ (0,3) + (-2,-1) &\rightarrow (2,-2) \\ (0,1) + (-2,-3) &\rightarrow (2,2) \end{aligned} \tag{IV.5}$$

Therefore, two new charges have to be added to account for one coupling, which has critical consequences, because the coupling

$$(0,1) + (2,-2) \rightarrow (-2,1) \text{ or } (2,-3) \tag{IV.6}$$

would require the presence of on of the two states $(-2,1)$ (a $\overline{\mathbf{5}}$) or $(2,-3)$ (a $\mathbf{10}$), which are both not present in the way the extension was set up, and would have to be added manually. But this, in turn, would require further additions to the network, because, assuming we added the state $(-2,1)$, the pairing

$$\begin{aligned} (-2,1) + (1,0) &\rightarrow (1,-1) \quad (\text{a } \overline{\mathbf{5}}) \\ \text{or } (1,0) - (-2,1) &\rightarrow (3,-1) \quad (\text{a } \mathbf{10}) \end{aligned} \tag{IV.7}$$

would, in change, require one of the two states on the right as a further addition. Because of the growing complexity of the network, it becomes increasingly difficult to see

[3] In the case of n=1, this already leads to the problem that there is both a $\overline{\mathbf{5}}$ charged as (2,1) and one charged as (-2,-1), which cancel each other out, and therefore do not allow for a gauge invariant cubic coupling.

whether, at some point, this chain would stop, and an extension would be found. This was attempted, but without success. Nevertheless, the conclusion can be drawn from these results that a general extension towards infinity as in the one-dimensional case is no longer possible for any two dimensional complete network, and is either generally impossible, or requires special relations between the charges (e.g. no charge with a zero in one of the components).

IV.4. The four-dimensional complete network

IV.4.1. The network

It is interesting to note that, for a network in which 4 $U(1)$ symmetries are present, it is also possible to find a complete network, which is exactly the network consisting of the following charges:

- The 5 $\mathbf{10}$s charged as t_i.
- The 10 $\mathbf{5}$s charged as $t_i + t_j$.
- The 10 $\mathbf{1}s$ charged as $t_i - t_j$.
- The 15 $\boldsymbol{X}s$ charged as $t_i + t_j - t_k - t_l$.

This [5,10,25] network is complete. This explains that the networks found by Higgsing down from this complete network and that were used as examples for the one-and two-dimensional case are also complete networks. [4] It was attempted without success to both find an extension for this network and to construct a smallest possible four-dimensional complete network. But as was already seen in the search for an extension of the two-dimensional complete network, these are very complex tasks probably requiring a different approach, for example the use of a computer.

IV.4.2. Open questions

The relation between complete networks and the group E_8 leads to interesting questions leaving space for possible research in the future:

- Are there other 4 dimensional complete networks, or is the one coming from E_8 the only possibility?
- If so, then why is this the only possibility and how are these structures related on a fundamental level?

[4] The explanation for this is that, by Higgsing a network, the number of possible couplings increases, because, after the breaking of a symmetry, combinations that could not form any gauge invariant couplings are able to afterwards. For example, a coupling $t_1 - t_3 + t_3 + t_2 = (t_1 - t_2)$ would be forbidden before, but is allowed after Higgsing with the singlet $\langle \mathbf{1}_{12} \rangle$.

- Because the up-type Yukawa coupling is connected to E_6, another exceptional Lie group (see [Maharana & Palti, 2012, chapter 5]), the question arises as to how exceptional groups are connected to couplings between matter curves. As E_8 is the largest possible exceptional Lie group and cannot be extended any further, this would put constrains on the possible matter spectrums and directly connect them to the structure of the exceptional groups.

V. Conclusion

V.1. Summary

In this thesis, we studied the Standard Model of particle physics and the Higgs mechanism and their connections to group theory. After that, we showed how theories further unifying the Standard Model could be set up and examined the example of the SU(5) Georgi-Glashow Model. Advancing farther, complete networks were introduced that play a role in String Theory, which has the aim of unifying all of physics under general principles. In the research part of this thesis, complete networks were looked at and extensions of complete networks gained from Higgsing down spectrums coming from E_8 were attempted.

V.2. Results

An infinite extension for the one-dimensional complete network was found, but it turned out that already for the two-dimensional case, finding an extension was a much more difficult task. It was seen that the complete networks coming from E_8 are not the only possible complete networks, but there is for example a smaller network in two dimensions and an infinite extension in one dimension. Nevertheless, considering the rapidly growing complexity of the networks with growing dimension, it is not trivial that a four-dimensional complete network is gained from Higgsing E_8, and one can even consider the question wether this network might be the smallest or even the only possible complete network in four dimensions. This would lead to questions about underlying connections between exceptional groups and intersecting matter curves in String Theory leaving space for research in the future.

VI. Appendix

VI.1. Important notes on group theory

As much of modern theoretical physics is deeply rooted in group theory and the theory of semi-simple Lie algebras, a few mathematical definitions are made for completeness. There are many books on group theory of which some are fitted to the needs of physicists. [von Steinkirch, M., 2011],[Nattermann, T., 2001], [Luedeling, C., 2010], [Cahn, 2014] and [Georgi, 1999] were consulted throughout this thesis, of which the last two are recommended most.

Lie groups

A Lie group is a group that also has the structure of a differentiable manifold. Because of that, around the origin, one can consider every element as a Taylor expansion around the trivial transformation, so that every element of a Lie group G can be written in the following way[Weigand, 2011b]:

$$G = e^{i\alpha_i T^i} \tag{VI.1}$$

where the T^is are called the generators of the groups as mentioned earlier.

Lie algebras

The generators of the group form the so-called Lie algebra which is entirely specified by the commutators of the generators:

$$[T^a, T^b] = if^{abc}T^c \tag{VI.2}$$

The f^{abc} are the so-called structure constants which define the whole structure of the algebra and can be directly derived from the properties of the group. For $SU(2)$, they are given by the ϵ-tensor $f^{ijk} = \epsilon^{ijk}$.
To see why they arise, one asks the question how the action of two group elements α and β can be written in terms of one γ.

$$e^{i\alpha_i T_i} e^{i\beta_i T_i} = e^{i\gamma_i T_i} \tag{VI.3}$$

A helpful mathematical formula, the **Baker–Campbell–Hausdorff Formula**[1] [Cahn, 2014], gives a more detailed insight into this topic. The important conclusion is that

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]} \tag{VI.4}$$

so that the question is entirely answered by knowing the commutators of the generators.

[1]This formula is valid if the double commutator $[X, [X, Y]] = 0$, but this is always true for the generators because $[T^a, [T^a, T^b]] = [T^a, if^{abc}T^c] = -f^{acd}f^{abc}T^d = 0$ due to the antisymmetry of the structure constants.

Representations

A representation of an algebra is a set of matrices T^i that satisfy the commutation relations

$$[T^a, T^b] = f^{abc} T_c. \tag{VI.5}$$

A representation is said to be **irreducible** if there is no subspace V of the vector space spanned by the matrices except for V itself and the trivial subspace which is mapped on itself by the T matrices. One can form a reducible representation by taking two spaces with two irreducible representations and forming the tensor product $V = V_1 \otimes V_2$. In the same way, a reducible representation of a space can be decomposed in a sum of irreducible representations.

The adjoint representation

The adjoint representation is defined in the following way:

$$ad(X)Y = [X, Y] \tag{VI.6}$$

So the vector space of the representation is the Lie algebra itself because the adjoint action is the action of the algebra, the commutator. Therefore, the dimension of the representations corresponds to the dimension of the algebra. On a more intuitive level, this means that the generators of the algebra are the basis vectors of the linear space. In the same way that every element of the algebra is a linear combination of the generators of the algebra, in the vector space of the representation, every element is a linear combination of the basis vectors. The adjoint representation exists for every algebra.
The matrix elements of every operator in that space is given by the structure constants, because

$$(adT^i)^j_k = ad(T^i \dot{T}^j)_k = [T^i, T^j]_k = f^{ij}_k. \tag{VI.7}$$

For this to be a represention, for any element $[x, y] = z$, $[adx, ady] = adz$ has to hold.
Plugging in the definition of the adjoint representation, and writing elements of the representation as $ad\,x$ for elements x of the algebra, we can calculate

$$[ad\,x, ad\,y]w = [x, [y, w]] - [y, [x, w]] \tag{VI.8}$$
$$= -[w[x, y]] = [[x, y], w] = [z, w] = ad\,zw \tag{VI.9}$$

SU(N) basics

The SU(N)-groups are easily defined. They consist of all NxN unitary matrices with determinant 1. These are called "special" and therefore consequently give rise to the name "special unitary group of dimension N". The algebra correponding to each SU(N) contains $N^2 - 1$ generators, which are traceless and hermitian. This is seen by looking at the definition of the group elements $G = e^{i\alpha_i T_i}$. In order for them to have determinant 1, the generators T_i have to be traceless.

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Erklärung

Ich versichere, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

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