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Introduction to Gauge/Gravity Duality

Examples IV

To hand in Thursday 18th November in the examples class

I. Irreducible representations of massless SUSY multiplets

The algebra for $\mathcal{N}$ supercharge operators is given by

$$\{Q^a_\alpha, \bar{Q}_{\dot{\beta}b}\} = -2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^a_b \qquad , \qquad \{Q^a_\alpha, Q^b_\beta\} = 2\epsilon_{\alpha\beta} Z^{ab} = -2\epsilon_{\alpha\beta} Z^{ba},$$

with $\alpha, \beta \in \{1,2\}$, $a, b \in \{1, 2, ...\mathcal{N}\}$ and σ^μ are components of a four vector $(-\mathbf{1}, -\sigma^i)$ of 2×2 matrices with the standard Pauli matrices σ^i.

For studying massless representations, we choose a Lorentz frame with $P^\mu = (E, 0, 0, E)$ and $E > 0$.

a) Show that the SUSY algebra relation reduces to

$$\{Q^a_\alpha, \bar{Q}_{\dot{\beta}b}\} = -2(\sigma^\mu P_\mu)_{\alpha\dot{\beta}} \delta^a_b = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\beta}} \delta^a_b .$$

(1 point)

b) Using the result in (a) and the second SUSY algebra relation given above, show that $Q^a_2 = 0$ and $Z^{ab} = 0$. (1 point)

c) The remaining operators Q^a_1 and $\bar{Q}_{\dot{1}a}$ with $a = 1, 2, ..., \mathcal{N}$ are used to represent field content of SUSY theories with $\mathcal{N}$ supercharges as follows:

- Q^a_1 lowers helicity λ by $\frac{1}{2}$, i.e. $Q^a_1|\lambda\rangle = |\lambda - \frac{1}{2}\rangle$;
- $\bar{Q}_{\dot{1}a}$ raises helicity λ by $\frac{1}{2}$, i.e. $\bar{Q}_{\dot{1}a}|\lambda\rangle = |\lambda + \frac{1}{2}\rangle$.

Determine all the states of a gauge multiplet for $\mathcal{N} = 1, 2, 3, 4$ by starting from the highest helicity states $|\lambda\rangle = |1\rangle$ and applying products of Q^a_1 operators on $|\lambda\rangle$ for all possible values of a.

(Hint: For $\mathcal{N} = D$, there are 2^D states in total.) (3 points)

d) Additional part: Determine all the states of a gauge multiplet for $\mathcal{N} = 1, 2, 3, 4$ by starting from the lowest helicity states $|\lambda\rangle = |-1\rangle$ and applying products of $\bar{Q}_{\dot{1}a}$ operators on $|\lambda\rangle$ for all possible values of a. What is special about the $\mathcal{N} = 4$ case compared to those of $\mathcal{N} = 1, 2, 3$ according to CPT invariance? (2 additional points)

II. **DBI action for a Dp–brane**

The Dirac–Born–Infeld (DBI) action for a Dp–brane reads

$$S_{Dp} = -\tau_p \int d^{p+1}\zeta \, e^{-\Phi} \sqrt{-\det(\mathcal{P}[g]_{ab} + 2\pi\alpha' F_{ab})}\,,$$

where Φ is the dilaton, $g_{\mu\nu}$ the metric of the curved spacetime and $\mathcal{P}$ the Pullback on the worldvolume (given by coordinates ζ^a). Moreover, F_{ab} is the $U(1)$ field strength tensor. det denotes the determinant of the matrix $\mathcal{P}[g]_{ab} + 2\pi\alpha' F_{ab}$.

a) Simplify the action to a flat target spacetime with a vanishing dilaton. Furthermore, the Dp–brane is aligned along the coordinate axis and the embedding functions into the target spacetime (given by coordinates X^m) vanish, i.e. $X^a = \zeta^a$ for $a = 0, \ldots, p$ and $X^m = 0$ for $m = p, \ldots, 9$. (1 points)

b) Show that we can expand $\det(\mathbb{1} + \epsilon M)$ for small ϵ in the form

$$\det(\mathbb{1} + \epsilon M) = 1 + \frac{1}{2}\,\epsilon \,\mathrm{tr}\, M + \epsilon^2 \left(\frac{1}{8} \,(\mathrm{tr}\, M)^2 - \frac{1}{4} \mathrm{tr}\,(M^2) \right) + \mathcal{O}(\epsilon^3)$$

How does the expansion read for antisymmetric M.

Hint: $\det(A) = \exp\,(\mathrm{Tr}\, \ln A)$ and expand the right side of the equation. (2 points)

c) Expand the simplified DBI action of exercise a) using the results of b) up to the first non-trivial order of the field strength tensor F. (2 points)