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Quantum Fields in Curved Spacetime

Examples VI

To hand in Monday 24th November in the examples class

1. **Mode functions in conformally coupled FRW Universe**

Consider a conformally coupled scalar field in the FRW universe. The metric is given by $ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2)$ and the action by

$$S = \frac{1}{2}\int d^4x\,\sqrt{|g|}\,\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{6}R\phi^2\right). \tag{1}$$

The Ricci scalar corresponding to this metric is

$$R = -6\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right).$$

Find the equation determining the mode functions for the field ϕ. Show that the vacuum state is time-independent, in other words, no ϕ-particles are created by gravity. (The spacetime is conformally flat.)

(5 pts)

2. **Bogolyubov transformation of ground state**

Suppose that the two sets $\hat{a}^{\pm}_{\mathbf{k}}$, $\hat{b}^{\pm}_{\mathbf{k}}$ of creation and annihilation operators for a real scalar field are related by the Bogolyubov transformation

$$\hat{b}^{-}_{\mathbf{k}} = \alpha_{\mathbf{k}}\hat{a}^{-}_{\mathbf{k}} + \beta_{-\mathbf{k}}\hat{a}^{+}_{-\mathbf{k}}, \quad \hat{b}^{+}_{\mathbf{k}} = \alpha^*_{\mathbf{k}}\hat{a}^{+}_{\mathbf{k}} + \beta^*_{-\mathbf{k}}\hat{a}^{-}_{-\mathbf{k}}. \tag{2}$$

The Bogolyubov coefficients $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ in Eq. (2) are known. The b-vacuum state $\left|{}_{(b)}0_{\mathbf{k},-\mathbf{k}}\right\rangle$ of the mode $\chi_{\mathbf{k}}$ is defined by

$$\hat{b}^{-}_{\mathbf{k}}\left|{}_{(b)}0_{\mathbf{k},-\mathbf{k}}\right\rangle = 0, \quad \hat{b}^{-}_{-\mathbf{k}}\left|{}_{(b)}0_{\mathbf{k},-\mathbf{k}}\right\rangle = 0.$$

Show that the b-vacuum is expanded through a-particle states $\left|{}_{(a)}m_{\mathbf{k}}, n_{-\mathbf{k}}\right\rangle$ as

$$\left|{}_{(b)}0_{\mathbf{k},-\mathbf{k}}\right\rangle = \frac{1}{|\alpha_{\mathbf{k}}|}\sum_{n=0}^{\infty}\left(-\frac{\beta^*_{\mathbf{k}}}{\alpha_{\mathbf{k}}}\right)^n \left|{}_{(a)}n_{\mathbf{k}}, n_{-\mathbf{k}}\right\rangle.$$

(5 pts)