Implications of Conformal Invariance for Quantum Field Theories in General Dimensions

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Preface

This thesis is an account of my work as research student at the Department of Applied Mathematics and Theoretical Physics between October 1993 and September 1996.

The results I present in this thesis are results of my own original work. Where the work of other authors has been used as a starting point or to complete the discussion, this is clearly marked in the text. The references may be found on page 146.

No part of this thesis has been submitted towards any degree at any other university.

Some of my results presented in this thesis have been included in a joint paper with my research supervisor: J. Erdmenger and H. Osborn, Conserved Currents and the Energy Momentum Tensor in Conformally Invariant Theories for general Dimensions. 43pp., DAMTP 96-7, hep-th/9605009, to be published in Nuclear Physics B.

Abstract

Implications of conformal invariance for quantum field theories in general dimensions $d > 2$ are investigated. A group theoretic formalism for the construction of conformally covariant three point functions for operators of arbitrary spin is presented and applied to conserved vector operators V_μ and the energy momentum tensor $T_{\mu\nu}$. These three point functions are shown to satisfy Ward identities arising from conformal invariance. A method is developed for obtaining three point functions in which the conservation equations for V_μ or for $T_{\mu\nu}$ are trivially satisfied. This procedure reduces the degree of singularity and so also amounts to a form of regularisation. With this method an expression for the three point function involving two vector currents and an axial current is derived, in which vector current conservation is manifest and which is well-defined as a distribution. The divergence of the axial current then gives the standard axial anomaly. By a similar procedure the coefficients of the three linearly independent forms for the energy momentum tensor three point function are related to the coefficients of the two possible terms in the trace anomaly, which is present in conformal field theories on a curved space background in four dimensions. Thus the anomaly coefficients may be determined for general conformal field theories which can also be interacting. These results have encouraging consequences in view of a proof of the Zamolodchikov C-theorem in four dimensions. Furthermore the connections with gravitational effective actions depending on a background metric are described. In particular, actions generating conformal correlation functions on flat space are constructed using conformally covariant differential operators acting on k-forms.

Contents

1 Introduction

Conformally invariant field theories have been extensively studied in two dimensions [1]. One aspect of the interesting mathematical structure of two dimensional conformal field theories is that their n-point functions may be constructed non-perturbatively from symmetry considerations only. The two and three point functions for the energy momentum tensor, which are the starting point for describing these theories, are of a very simple form whose coefficient is the Virasoro central charge c.

A motivation to study conformal field theories also in higher dimensions $d > 2$ is that quantum field theories at renormalisation group fixed points are expected to be conformally invariant. This is of interest for exploring statistical mechanics systems in $d = 3$ at critical points or relativistic quantum field theories in $d = 4$. The interest in studying conformal field theories in four dimensions is stimulated by the recent discovery of non-trivial fixed points in a large class of $N = 1$ and $N = 2$ supersymmetric theories [2], as well as by the well known case of $N = 4$ theories in which the beta function vanishes identically [3]. These infrared fixed points are non-trivial because they arise for non-zero values of the couplings such that the corresponding conformal field theories are interacting. They are present in asymptotically free supersymmetric theories with a single gauge coupling where the presence of additional fields besides the gauge vector multiplet generates the zero of the beta functions which is necessary for a fixed point.

Although in higher dimensions the conformal group is no longer infinite dimensional and there is no algebraic structure equivalent to the Virasoro algebra in two dimensions, conformal invariance still provides very non-trivial symmetry constraints such that it is possible to obtain exact results also for higher dimensional theories. For example, conformal invariance allows for significantly simplified calculations of the dimensions of operators at fixed points than would be allowed in conventional perturbative approaches based on expansions about free theories [4, 5]. In higher dimensions the two and three point functions are also essentially determined by conformal invariance with no arbitrary functions present. For operators with spin however, there may be two or more linearly independent forms compatible with conformal invariance which are possible for the three point functions.

A method for constructing conformally invariant two and three point functions in general dimensions has been developed by Osborn and Petkou [6, 7]. Their group theoretic construction of the conformally invariant two and three point functions for operators of arbitrary spin in general dimensions makes use of the fact that the conformal group is generated by rotations, translations, and inversions. Osborn and Petkou apply their general results in particular to conserved current operators V_μ of scaling dimension $d-1$ and to the energy momentum tensor $T_{\mu\nu}$, which is symmetric and traceless and of scaling dimension d, where the conservation equations

$$\partial_\mu V_\mu(x) = 0 \quad \text{and} \quad \partial_\mu T_{\mu\nu}(x) = 0 \tag{1.1}$$

impose additional constraints. They found that these three point functions are determined in terms of a restricted number of linearly independent forms. In particular, they have shown that the three point function with three energy momentum tensors has three independent parameters in general.

Here we relate these parameters to the coefficients in the conformal anomaly: On a curved space background, the conformal symmetry is broken by anomalous contributions to the trace of the expectation value of the energy momentum tensor. In a semiclassical picture the energy momentum tensor in the Einstein field equations is replaced by its expectation value

$$R_{\mu\nu} - \tfrac{1}{2}R\, g_{\mu\nu} + \Lambda g_{\mu\nu} = \; -8\pi G\, \langle T_{\mu\nu} \rangle \,. \tag{1.2}$$

This energy momentum tensor expectation value needs to be renormalised. The subtraction of the divergent terms within dimensional regularisation leads to finite contributions to the trace. Further trace anomalies arise in a similar way from classical background gauge fields and from sources for scalar fields.

In four dimensions there are two linearly independent purely gravitational contributions to the trace anomaly which are relevant for non-local three point functions. These are dimension four scalars constructed from the Riemann curvature tensor. It is possible to choose one so that it transforms homogeneously under local scale transformations, the other being a scale independent topological term. Their coefficients, called β_a and β_b respectively, have been calculated for the special case of free field theories some time ago [8]. Here, we present a calculation which relates these coefficients to the coefficients in the three point function on flat space for general conformal field theories. With these relations the anomaly coefficients are in principle calculable for any conformal field theory which may also be interacting.

This result is expected to be essential to the proof of the Zamolodchikov C-theorem in four dimensions [9]. Zamolodchikov proved for the two dimensional case [10] that there is a quantity C which decreases along renormalisation group flows from an ultraviolet to an infrared fixed point. At the fixed points, C takes the value of the Virasoro central charge c. The quantity C can be interpreted as a measure of the degrees of freedom and its decrease corresponds to the thinning of degrees of freedom in the large distance scale limit where only massless fields persist. A four dimensional analogue to the C-theorem might be a tool for understanding the structure of QCD.

In this thesis the two coefficients in the conformal anomaly are calculated for general conformal field theories by expressing them in terms of the three parameters in the conformally invariant energy momentum tensor three point function. Similarly we relate the coefficient of the gauge field dependent part of the anomaly to the parameters in the three point function involving the energy momentum tensor and two conserved vector currents. To this effect we construct compact expressions for the independent forms in the conformally covariant three point functions involving the energy momentum tensor. The anomalous terms are also present as additional local terms in the Ward identities linking the two and three point functions. To isolate them, it is necessary to separate anomalous and anomaly-free forms in the energy momentum tensor three point function, as well as to regularise the forms in a convenient way. A convenient procedure for both these tasks is to calculate the three point functions for operators which trivially satisfy the conservation equations (1.1): In general dimensions conserved vector currents and the energy momentum tensor are not uniquely defined. They may be modified by adding terms which satisfy the conservation equations (1.1) by definition and thus lead to the same conserved charges or generators of the conformal group. These terms are of the form

$$\begin{aligned} V'_\mu(x) &= \partial_\nu F_{\nu\mu}(x) \text{ and} \\ T'_{\mu\nu}(x) &= \partial_\alpha \partial_\beta C_{\mu\alpha\beta\nu}(x) \, , \end{aligned} \tag{1.3}$$

where $F_{\nu\mu}(x) = F_{[\nu\mu]}(x)$ is a tensor antisymmetric in $[\nu \leftrightarrow \mu]$ and $C_{\mu\alpha\beta\nu}(x)$ is a tensor with Weyl symmetry, i.e.

$$C_{\mu\alpha\beta\nu}(x) = C_{[\mu\alpha][\beta\nu]}(x), \quad C_{\mu[\alpha\beta\nu]}(x) = 0, \quad C_{\mu\alpha\beta\mu}(x) = 0 \, . \tag{1.4}$$

The operator $T'_{\mu\nu}(x)$ in (1.3) is also automatically symmetric and traceless. If the scaling dimension $d-2$ is assigned both to $F_{\nu\mu}(x)$ and to $C_{\mu\alpha\beta\nu}(x)$, then $V'_\mu(x)$ and $T'_{\mu\nu}(x)$ as defined by (1.3) are quasi-primary operators. It is natural to attempt to pull out derivatives from the expressions for the three point functions in accord with the general relations (1.3), thus using the methods of differential regularisation [11]. In addition to satisfying the conservation equations explicitly, the resulting expressions are then more regular before differentiation because the degree of the singularities in the three spatial variables is reduced. It should be noted that there is an arbitrariness in the definition of $V_\mu(x)$ and $T_{\mu\nu}(x)$ up to terms of the form (1.3) which is ultimately the reason for the presence of several linearly independent forms in the conformally covariant three point functions.

For a consistent treatment of the anomalies we first analyse Ward identities linking the two and three point functions which arise from conformal invariance. Non-anomalous conformal Ward identities may be derived on flat space by using the role of the energy momentum tensor as generator of infinitesimal conformal transformations. To obtain the anomalous contributions it is convenient to introduce classical background fields, i.e. the metric and a gauge field. The energy momentum tensor and conserved currents are then defined as the functional derivatives of a conformally invariant action with respect to the background fields. In this case Ward identities are obtained by functionally differentiating relations arising from the gauge invariance of the action or from its invariance under diffeomorphisms and Weyl rescalings. These relations may have anomalous symmetry-breaking terms, which after functional differentiation and subsequent restriction to flat space and zero background gauge field are present in the Ward identities as additional local terms.

As a first, more simple example for the construction and regularisation of conformally covariant three point functions, we consider the vector operator three point function $\langle V_\mu(x)V_\nu(y)V_\omega(z)\rangle$ and its trivially conserved form obtained by using the relation (1.3a). This form satisfies consistency relations which can be derived from the Ward identities. Subsequently, by calculating the Ward identity for the three point function $\langle V_\mu(x)V_\nu(y)A_\omega(z)\rangle$ with two vector operators and one axial vector operator, we show that using (1.3) to impose vector current conservation it is possible to remove all non-integrable singularities and thus to reproduce the well-known axial anomaly. This suggests that a similar calculation is feasible for the more complicated case of the conformal anomaly which is our focus of interest.

As a further example for a conformal three point function we consider the case of the energy momentum tensor and two vector currents, $\langle T_{\mu\nu}(x)V_\sigma(y)V_\rho(z)\rangle$. There are two independent forms in this three point function as may be shown in a straightforward way after vector current conservation has been imposed. The two independent forms may be characterised by their behaviour in the relevant Ward identity. The Ward identity contains one anomalous term in this case which originates from the background gauge field. One of the two independent forms in the three point function gives rise to this anomalous term while the other is anomaly free. To determine the anomalous form we develop a regularisation procedure combining differential and dimensional regularisation. With this procedure we determine the non-integrable singular contribution to the three point function. The necessary counterterm then generates the trace anomaly. By comparing with the anomalous contribution to the Ward identity we find a relation between the parameters in the three point function and the anomaly coefficients and by this identify the anomalous form in the three point function. We also give an example for a contribution to the anomaly free form.

Furthermore we consider the energy momentum tensor three point function and construct its three independent forms. Subsequently we find linear combinations of these forms which have a definite behaviour with respect to the conformal anomaly. We present an argument within our formalism by which it is shown that the anomaly coefficient β_a of the scale dependent anomalous term is proportional to the scale of the energy momentum tensor two point function. Using the Ward identity we may express β_a in terms of the three coefficients in the three point function. Then the form in $\langle T'_{\mu\nu}(x) T_{\sigma\rho}(y) T_{\alpha\beta}(z)\rangle$ is calculated for which one of the energy momentum tensors is replaced by $T'_{\mu\nu}$ given in (1.3) and thus trivially satisfies the conservation equation. It is shown that the coefficient β_a vanishes for the form regularised in this way such that there remain two independent contributions to this form. We proceed by constructing the form for which all three energy momentum tensors are trivially conserved:

$$\langle T'_{\mu\nu}(x) T'_{\sigma\rho}(y) T'_{\alpha\beta}(z)\rangle \;=\; \partial^x{}_\kappa \partial^x{}_\lambda \partial^y{}_\varepsilon \partial^y{}_\eta \partial^z{}_\gamma \partial^z{}_\delta \langle C_{\mu\kappa\lambda\nu}(x) C_{\sigma\varepsilon\eta\rho}(y) C_{\alpha\gamma\delta\beta}(z)\rangle. \qquad (1.5)$$

This must be anomaly free even in four dimensions. Such an anomaly-free contribution should exist because there are three independent forms in the general conformally covariant energy momentum tensor three point function, but only two independent contributions to the conformal anomaly. Finally, we construct the form in the energy momentum tensor three point function for which two of the three parameters are trivially conserved. We obtain the coefficient β_b of the topological term in the trace anomaly by calculating the counterterms necessary for removing the non-integrable singular contribution to this form using the regularisation procedure developed earlier. Thus we are able to decompose the conformally covariant energy momentum tensor three point function into one anomaly-free form and into two forms generating the two independent anomalous contributions to the trace Ward identity.

On a somewhat different issue we devote the last section to the study of effective gravitational actions $W[g, \mathcal{A}, \mathcal{J}]$ depending on the metric $g_{\mu\nu}$ and background gauge and scalar fields $\mathcal{A}^a{}_\mu$ and $\mathcal{J}$ which generate the trace anomaly under local rescalings of the metric and which upon functional differentiation yield the conformal three point functions discussed in the earlier sections. The motivation for this study is again the generalisation of two dimensional results to higher, in this case four dimensions. The result that the conformal energy momentum tensor three point function on flat space has three independent forms in general indicates that at least to third order in the curvature there should be only three independent forms in the gravitational effective action, two of which give rise to the anomaly.

In two dimensions, where there is only one form in both anomaly and three point function, an elegant non-local expression for the conformal effective action generating the trace anomaly has been found by Polyakov [12]. One of the possible applications of this action beyond more formal interests is the study of the information puzzle in black hole physics. This paradox arises in the context of black hole evolution where infalling matter in a pure initial quantum state appears to evolve into outgoing thermal radiation, i.e. into mixed quantum states. This puzzle has been studied in two dimensions using the Polyakov action by Strominger, Callan and others [13]. The underlying idea is that the conformal anomaly, which relates the trace of the energy momentum tensor expectation value to the curvature, describes the back-reaction of the Hawking radiation on the geometry. It was found that no information leaves the black hole prior to the evaporation endpoint.

Here we show how in two dimensions conformal two and three point functions may be derived from the Polyakov action. A four-dimensional analogue to the Polyakov action which generates the gravitational as well as the background scalar and gauge field terms in the four dimensional trace anomaly has been constructed by Riegert [14]. However this action does not lead to conformal correlation functions on flat space. We discuss in detail this failure of the Riegert action to comply with conformal invariance which is due to its large distance behaviour. Furthermore we propose an alternative way of constructing conformal effective actions in four dimensions using conformal differential operators acting on k-forms. With this approach we construct an action generating the scalar terms in the trace anomaly as well as the conformal three point function involving the energy momentum tensor and two scalar fields. Moreover we find actions generating the anomaly free forms in the conformal three point functions involving the energy momentum tensor and two vector currents or three energy momentum tensors.

The organisation of this thesis is as follows: In chapter 2 we set the scene by discussing the implications of conformal invariance for d dimensional quantum field theories in general terms. In section 2.1 we define conformal transformations and present the formalism for conformal two and three point functions for general operators. Section 2.2 is a discussion of the properties of the conserved vector operators and the energy momentum tensor whose correlation functions we calculate later on, especially of their conformal transformation properties and of how they allow for differential regularisation. In section 2.3 we derive Ward identities arising from conformal invariance. In section 2.4 we discuss the axial and conformal anomalies which break the symmetry and the anomalous terms they generate in the Ward identities.

In chapter 3 we use the general results to construct three point functions involving conserved currents and the energy momentum tensor. We develop a regularisation method for these three point functions and study their Ward identities. As a first explicit example, we consider the vector operator three point function in section 3.1. Subsequently in section 3.2, the axial anomaly is calculated by applying differential regularisation to the three point functions with two vector operators and one axial vector operator. As a first example for the relation between the trace anomaly and the three point functions, we construct the three point function involving the energy momentum tensor and two vector currents in section 3.3 and calculate its non-integrable singular contribution. We develop a regularisation method combining differential and dimensional regularisation and show how renormalisation leads to the trace anomaly. In section 3.4 we discuss the energy momentum three point function. Our compact notation displays that this three point function has three independent forms in general. The three coefficients are given explicitly for free theories. The form in the energy momentum tensor three point function for which one of the energy momentum tensors is trivially conserved is calculated in section 3.5.

In chapter 4 we investigate the structure of the energy momentum tensor three point function in relation to the conformal anomaly. We briefly review the results for the scale dependent part of the anomaly with coefficient β_a which have already emerged in the previous chapters since they are intimately linked to the Ward identity. In section 4.1 the anomaly-free form in the energy momentum tensor three point function is constructed. The essential part of this chapter is devoted to the topological part of the trace anomaly with coefficient β_b. In section 4.2 we construct the three point function involving two tensors with Weyl symmetry and one energy momentum tensor, from which obtain the form in energy momentum tensor three point function in which two of the energy momentum tensors have been replaced by ${T'}_{\mu\nu}(x)$ as in (2.4.2) by taking four derivatives:

$$\langle T_{\mu\nu}(x){T'}_{\sigma\rho}(y){T'}_{\alpha\beta}(z)\rangle = {\partial^y}_\varepsilon {\partial^y}_\eta {\partial^z}_\gamma {\partial^z}_\delta \langle T_{\mu\nu}(x) C_{\sigma\varepsilon\eta\rho}(y) C_{\alpha\gamma\delta\beta}(z)\rangle \, . \tag{1.6}$$

Since this expression is less singular than the general expression for the three point function, it is possible to apply the regularisation method developed earlier in order to relate the anomaly coefficient β_b to the three parameters in the energy momentum tensor three point function. This is the subject of section 4.3. Since the anomaly arises from an interplay between the tensorial structure of the three point function and its scaling behaviour we first discuss the tensorial structure of the anomalous term in 4.3.1. Then we calculate the counterterm to remove the non-integrable singular terms in the three point function in 4.3.2. In section 4.3.3 we attribute a definite quantum field theoretical meaning to the tensor $C_{\mu\kappa\lambda\nu}$ with Weyl symmetry and discuss the anomaly associated with this tensor.

In Chapter 5 we construct and discuss effective actions which give rise to conformal two and three point functions on flat space upon functional differentiation with respect to classical background fields. In section 5.1 we show how in two dimensions conformal two and three point functions may be obtained from the Polyakov action. Section 5.2

is a discussion of the four dimensional Riegert action and why it fails to give conformal correlation functions. In section 5.3 we construct a conformal effective action which is in accord with the scalar contribution to the trace anomaly and which yields the three point function the energy momentum tensor and two scalar operators. Moreover we construct actions for the anomaly free parts of the three point functions involving the energy momentum tensor and two conserved vector operators or three energy momentum tensors. We also construct a conformally covariant differential operator acting on the tensor with Weyl symmetry.

Chapter 6 contains a concluding discussion as well as plans for future calculations.

2 Conformal Invariance in d Dimensions

2.1 Conformal Transformations for Coordinates and Fields

In this section we will first define conformal transformations for general dimensions $d > 2$ acting in coordinate space and discuss some of their properties which will subsequently enable us to define quasi-primary fields transforming covariantly under conformal transformations and to show how conformally invariant two and three point functions may be constructed. The group theoretical approach for the construction of conformal two and three point functions for operators of arbitrary spin presented was developed by Osborn and Petkou in [6] using some earlier results by Mack [15].

Conformal coordinate transformations leave angles invariant. These transformations form a group which on Euclidean space in d dimensions is isomorphic to $O(d+1,1)$. When acting on $\mathbb{R}^d$ with an Euclidean metric, conformal coordinate transformations are defined by

$$x_\mu \to x'_\mu(x), \qquad dx'_\mu dx'_\mu = \Omega(x)^{-2} dx_\mu dx_\mu \; . \tag{2.1.1}$$

This can be written in infinitesimal form as

$$x'_\mu(x) = x_\mu + v_\mu(x) \; , \quad \Omega(x) = 1 - \sigma_v(x), \tag{2.1.2}$$

so that (2.1.1) gives

$$\partial_\mu v_\nu + \partial_\nu v_\mu = 2\sigma_v \delta_{\mu\nu} \; , \quad \sigma_v = \frac{1}{d}\partial \cdot v \; . \tag{2.1.3}$$

When $d \neq 2$ this has the solution

$$v_\mu(x) = a_\mu + \omega_{\mu\nu}x_\nu + \lambda x_\mu + (b_\mu x^2 - 2x_\mu b\cdot x) \; , \omega_{\mu\nu} = -\omega_{\nu\mu} \; , \sigma_v(x) = \lambda - 2b\cdot x. \tag{2.1.4}$$

The four terms in $v_\mu(x)$ correspond to (in this order) infinitesimal translations, rotations, scale transformations, and special conformal transformations. For any of these conformal transformations a local x dependent rotation may be defined by

$$\mathcal{R}_{\mu\nu}(x) = \Omega(x)\frac{\partial x'_\mu}{\partial x_\nu} \; , \qquad \mathcal{R}^T(x)\mathcal{R}(x) = 1 \; , \tag{2.1.5}$$

which in d dimensions is a matrix belonging to $O(d)$.

Special conformal transformations are generated by combining an inversion, a translation, and a second inversion. Inversions are conformal transformations which unlike the transformations listed above are not connected to the identity, and for which

$$x'_\mu = \frac{x_\mu}{x^2} \ , \Omega(x) = x^2 \ , \ \mathcal{R}_{\mu\nu}(x) = I_{\mu\nu}(x) \equiv \delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2} \ , \ \ \det I = -1 \ . \qquad (2.1.6)$$

For an arbitrary conformal transformation, the matrix I acts as a parallel transport matrix between two points x and y:

$$I_{\mu\nu}(x'-y') = \mathcal{R}_{\mu\alpha}(x)\mathcal{R}_{\nu\beta}(y)I_{\alpha\beta}(x-y) \ , \ \ (x'-y')^2 = \frac{(x-y)^2}{\Omega(x)\Omega(y)} \ . \qquad (2.1.7)$$

This will be the essential ingredient for constructing conformal two and three point functions later on. Any conformal transformation may be generated by combining an inversion with a rotation and a translation.

Moreover for three points x, y and z it is possible to define vectors which transform homogeneously under conformal transformations:

$$X_{12\,\mu} = \frac{(x-z)_\mu}{(x-z)^2} - \frac{(y-z)_\mu}{(y-z)^2} \ , \qquad X_{12}^2 = \frac{(x-y)^2}{(x-z)^2(y-z)^2} \ . \qquad (2.1.8)$$

$X_{12\,\mu}$ is a vector at z so that under $\{x_\mu, y_\mu, z_\mu\} \rightarrow \{x'_\mu, y'_\mu, z'_\mu\}$

$$X'_{12\,\mu} = \Omega(z)\mathcal{R}_{\mu\alpha}(z)X_{12\,\alpha} \ . \qquad (2.1.9)$$

$X_{23\,\mu}$ and $X_{31\,\mu}$, which are vectors at x and y respectively, are defined by cyclic permutation:

$$X_{23\,\mu} = \frac{(y-x)_\mu}{(y-x)^2} - \frac{(z-x)_\mu}{(z-x)^2} \ , \ \ X_{31\,\mu} = \frac{(z-y)_\mu}{(z-y)^2} - \frac{(x-y)_\mu}{(x-y)^2} \ . \qquad (2.1.10)$$

Using these properties of conformal coordinate transformations, we are now able to define quasi-primary fields by the transformation property

$$\mathcal{O}^i(x) \rightarrow \mathcal{O}'^i(x') = \Omega(x)^\eta D^i{}_j(\mathcal{R}(x))\mathcal{O}^j(x), \qquad (2.1.11)$$

where η is the scaling dimension of the field and $D^i{}_j(\mathcal{R}(x))$ is an irreducible representation of the local orthogonal transformation $\mathcal{R}(x)$ which has the same dimension as the field

$\mathcal{O}^j(x)$. This is a generalisation of the standard transformation rules for translations and rotations. For infinitesimal transformations as in (2.1.4) we have

$$\delta_v \mathcal{O}^i(x) = -(L_v \mathcal{O})^i(x)\,, \quad L_v = v\cdot\partial + \eta\sigma_v - \tfrac{1}{2}\partial_{[\mu}v_{\nu]}s_{\mu\nu}\,, \tag{2.1.12}$$

where $(s_{\mu\nu})^i{}_j = -(s_{\nu\mu})^i{}_j$ are the generators of $O(d)$ in the representation with the appropriate dimension. L_v satisfies the required Lie algebra $[L_v, L_{v'}] = L_{[v,v']}$ where the commutator is given by $[v,v']_\mu = v\cdot\partial\, v'_\mu - v'\cdot\partial\, v_\mu$.

To give a general expression for the two point functions for quasi-primary fields, it will be convenient to define for each field $\mathcal{O}^i(x)$ a conjugate field $\bar{\mathcal{O}}_i(x)$ transforming as

$$\bar{\mathcal{O}}_i(x) \to \bar{\mathcal{O}}'{}_i(x') = \Omega(x)^\eta \bar{\mathcal{O}}_j(x) D^j{}_i(\mathcal{R}(x))\,. \tag{2.1.13}$$

Using (2.1.7) it is now straightforward to construct conformally covariant two point point functions for quasi-primary operators. The general expression for the two point function of two fields $\mathcal{O}$, $\bar{\mathcal{O}}$ transforming as in (2.1.11), (2.1.13) under the same irreducible representation of $O(d)$ is given by

$$\langle \mathcal{O}^i(x)\bar{\mathcal{O}}_j(y)\rangle = \frac{C_\mathcal{O}}{(x-y)^{2\eta}} D^i{}_j(I(x-y)). \tag{2.1.14}$$

$C_\mathcal{O}$ is an overall scale factor, and η is the scale dimension of both fields. The conformal two point function vanishes if the two fields have different scale dimension or different spin.

Conformal three point functions may be constructed in a similar way using the covariant vectors X_μ:

$$\langle \mathcal{O}_1^i(x)\mathcal{O}_2^j(y)\mathcal{O}_3^k(z)\rangle = \frac{1}{(x-z)^{2\eta_1}(y-z)^{2\eta_2}} D^i_{1i'}(I(x-z))D^j_{2j'}(I(y-z))t_{12,3}^{i'j'k}(X_{12})\,. \tag{2.1.15}$$

In (2.1.15), $t_{12,3}^{ijk}(X)$ is a homogeneous function satisfying

$$D^i_{1i'}(R)D^j_{2j'}(R)D^k_{3k'}(R)t_{12,3}^{i'j'k'}(X) = t_{12,3}^{ijk}(RX) \tag{2.1.16}$$

for all rotations R belonging to $O(d)$ and

$$t_{12,3}^{ijk}(\lambda X) = \lambda^{\eta_3-\eta_1-\eta_2}\, t_{12,3}^{ijk}(X)\,. \tag{2.1.17}$$

The expression (2.1.15) is not explicitly symmetric in x, y, and z but different permutations may be obtained in a similar form, e.g.

$$\langle \mathcal{O}_1^i(x)\mathcal{O}_2^j(y)\mathcal{O}_3^k(z)\rangle = \frac{1}{(y-x)^{2\eta_2}(z-x)^{2\eta_3}} D_{2j'}^{j}(I(y-x))D_{3k'}^{k}(I(z-x))t_{23,1}^{j'k'i}(X_{23})\,, \tag{2.1.18}$$

where

$$t_{23,1}^{jki}(X) = (X^2)^{\eta_1-\eta_3}\, D_{2j'}^{j}(I(X))\, t_{12,3}^{ij'k}(-X)\,. \tag{2.1.19}$$

This demonstrates the symmetry of the expression for the three point function under exchange of the fields. From the properties of the vectors X_μ we also have

$$t_{21,3}^{ijk}(X) = t_{12,3}^{ijk}(-X)\ . \tag{2.1.20}$$

If the three point function is symmetric for all fields $\mathcal{O}^i$ being bosonic and belonging to the same representation, then it is necessary that

$$t_{12,3}^{jik}(X) = t_{12,3}^{ijk}(-X), \qquad D^{i}{}_{i'}(I(X))t_{12,3}^{i'jk}(X) = t_{12,3}^{kij}(-X)\,. \tag{2.1.21}$$

The function $t_{12,3}^{ijk}(X)$ has a direct significance since it represents the leading term in the operator product expansion. According to (2.1.18) and (2.1.19) as $x \to y$ we have since $-X_{23\,\mu} \sim (x-y)_\mu/(x-y)^2$

$$\langle \mathcal{O}_1^i(x)\mathcal{O}_2^j(y)\mathcal{O}_3^k(z)\rangle \sim\ t_{12,3}^{ijk'}(x-y)\frac{D_{3\,k'}^{k}\,(I(z-y))}{(z-y)^{2\eta_3}}\,. \tag{2.1.22}$$

Thus from (2.1.14) the leading contribution of the operator $\bar{\mathcal{O}}_3$ to the operator product of $\mathcal{O}_1(x)\mathcal{O}_2(y)$ as $x \to y$ is given by

$$\mathcal{O}_1^i(x)\mathcal{O}_2^j(y) \sim \frac{1}{C_{\mathcal{O}_3}} t_{12,3}^{ijk}(x-y)\bar{\mathcal{O}}_{3\,k}(y). \tag{2.1.23}$$

We now make use of these general results to discuss some more specific cases. For instance for a general totally symmetric three point function $\langle \mathcal{O}_A(x)\mathcal{O}_B(y)\mathcal{O}_C(z)\rangle$, we can specify the tensor $t^{ijk}(X)$ further. A possible form for the tensor $t_{ABC}(X)$ which satisfies the conditions (2.1.16), (2.1.17) and (2.1.21) is given by

$$t_{ABC}(X) = d_{ABC'}\, I_{C'C}(X)\, \frac{1}{X^\lambda}\,, \tag{2.1.24}$$

where d_{ABC} is a totally symmetric $O(d)$ invariant structure constant and $I_{C'C}(X)$ is a representation of the inversion. Capital letters denote the indices in the representation chosen. In the subsequent we verify that $t_{ABC}(X)$ as defined in (2.1.24) satisfies the conditions (2.1.16), (2.1.17) and (2.1.21): To be in accord with (2.1.16), (2.1.24) must satisfy

$$I_{AA'}(X) I_{BB'}(X) I_{CC'}(X) t_{A'B'C'}(X) = t_{ABC}(IX)\,. \tag{2.1.25}$$

To prove that (2.1.24) satisfies (2.1.25) it is sufficient to check that

$$I_{CD}(X) = I_{CD}(IX)\,. \tag{2.1.26}$$

This holds by virtue of $I_{\mu\nu}(X)X_\nu = -X_\mu$, i.e. $IX = -X$ and $I_{AA'}(X) = I_{AA'}(-X)$. Moreover, to be in accord with (2.1.21), (2.1.24) must satisfy

$$I_{AD}(X) t_{DBC}(X) = t_{CAB}(-X)\,. \tag{2.1.27}$$

Note that by definition, (2.1.24) satisfies $t_{ABC}(X) = t_{BAC}(-X)$ as necessary to agree with (2.1.21). To show that (2.1.24) satisfies (2.1.27) we use the fact that the structure constant d_{ABC} is $O(d)$ invariant:

$$d_{ABC} = I_{AA'}(X) I_{BB'}(X) I_{CC'}(X) d_{A'B'C'}\,. \tag{2.1.28}$$

Using this and also $I_{AC}(X) I_{CB}(X) = \delta_{AB}$ we obtain:

$$\begin{aligned} t_{ABC}(X) = d_{ABD} I_{CD}(X) &= I_{AA'}(X) I_{BB'}(X) d_{A'B'C}\,, \\ I_{AD}(X) t_{DBC}(X) = d_{AB'C} I_{BB'}(X) &= d_{CAB'} I_{BB'}(X) = t_{CAB}(X)\,. \end{aligned} \tag{2.1.29}$$

Furthermore, $t_{ABC}(X)$ as defined in (2.1.24) satisfies (2.1.17) by definition. It thus satisfies all the relevant conditions to be used for the construction of the totally symmetric three point function $\langle \mathcal{O}_A(x) \mathcal{O}_B(y) \mathcal{O}_C(z) \rangle$.

To facilitate the calculations for the three point functions it is convenient to consider a configuration where the three points x,y,z are constrained to lie on a straight line. In this collinear frame, the general expression (2.1.15) reduces to

$$\langle \mathcal{O}^{i_1}(x)\mathcal{O}^{i_2}(y)\mathcal{O}^{i_3}(z)\rangle|_{x_\mu=\hat{x}n_\mu,\, y_\mu=\hat{y}n_\mu, z_\mu=\hat{z}n_\mu}$$
$$= \frac{1}{(\hat{x}-\hat{y})^{\eta_1+\eta_2-\eta_3}(\hat{x}-\hat{z})^{\eta_1+\eta_3-\eta_2}(\hat{y}-\hat{z})^{\eta_2+\eta_3-\eta_1}}\,\mathcal{A}^{i_1i_2i_3}\,, \qquad (2.1.30)$$

where n_μ is a unit vector defining the straight line and we assume $\hat{x} > \hat{y} > \hat{z}$. $\mathcal{A}^{i_1i_2i_3}$ is a constant $O(d-1)$ invariant tensor for $O(d-1)$ rotations leaving n_μ invariant. Comparing with (2.1.15), one finds that

$$\mathcal{A}^{i_1i_2i_3} = D^{i_1}_{1j_1}(I)D^{i_2}_{2j_2}(I)t^{j_1j_2i_3}(n)\,, \qquad (2.1.31)$$

where $t^{j_1j_2i_3}(n)$ is also a constant $O(d-1)$ invariant tensor. In the collinear frame, if $n_1 = 1$, $\underline{n} = \underline{0}$, the non-zero components of the inversion are given by $I_{11} = -1$, $I_{ij} = \delta_{ij}$ where Latin indices denote components perpendicular to the straight line.

Furthermore in the collinear frame it is possible to calculate derivatives of the three point functions in a purely algebraic way. Derivatives will frequently be needed, for example to impose conservation or for differential regularisation. For derivatives with respect to the variables perpendicular to the straight line, we consider an infinitesimal rotation away from the straight line:

$$X_\mu = |X|n_\mu + \delta X_\mu = R_{\mu\nu}(X)n_\nu|X|\,, \qquad \delta X_\mu n_\mu = 0\,, \qquad (2.1.32)$$

$$R_{\mu\nu}(X) = \begin{pmatrix} 1 & -\frac{\delta X_j}{|X|} \\ \frac{\delta X_i}{|X|} & \delta_{ij} \end{pmatrix}. \qquad (2.1.33)$$

From (2.1.16) we have for a second-rank tensor $t_{\mu\nu}(X)$

$$t_{\mu\nu}(X) = R_{\mu\mu'}(X)R_{\nu\nu'}(X)t_{\mu'\nu'}(|X|n)\,. \qquad (2.1.34)$$

From this, derivatives may be obtained by inserting the explicit form for R as given by (2.1.33). Derivatives with respect to X along the line can be calculated in a straightforward way. As an example, the total derivative of one of the components of the tensor

$t_{\mu\nu}(X)$ is, assuming $t_{\mu\nu}(X) \propto X^{-\lambda}$:

$$\begin{aligned}\partial_\mu t_{\mu 1}(X)\Big|_{X=n} &= \frac{\partial}{\partial|X|} t_{11}(|X|n)\,|_{|X|=1} + (d-1)t_{11}(n) - t_{ii}(n) \\ &= (d-1-\lambda)t_{11}(n) - t_{ii}(n)\,. \end{aligned} \tag{2.1.35}$$

2.2 Conserved Currents and the Energy Momentum Tensor

We are mainly interested in applying the results of the previous section for conformal two and three point functions for quasi-primary operators to conserved currents $V_\mu(x)$ and to the energy momentum tensor $T_{\mu\nu}(x)$. These operators satisfy

$$\partial_\mu V_\mu(x) = 0, \quad T_{\mu\nu}(x) = T_{\nu\mu}(x),\ T_{\mu\mu}(x) = 0\,,\ \partial_\mu T_{\mu\nu} = 0\,. \tag{2.2.1}$$

The energy momentum tensor must be symmetric and conserved from its physical interpretation while its tracelessness is implied by conformal invariance: $T_{\mu\nu}(x)$ is the generator for local coordinate transformations and conformal invariance requires that the currents

$$j^v{}_\mu = T_{\mu\nu} v_\nu \tag{2.2.2}$$

with v_ν as in (2.1.4) are conserved, in particular the dilation current $d_\mu = T_{\mu\nu}x_\nu$ for which $v_\mu = \lambda x_\mu$, $j^v{}_\mu = \lambda d_\mu$ in (2.2.2). $\partial_\mu j^v{}_\mu = 0$ implies $T_{\mu\mu} = 0$ since $T_{\mu\nu}$ is conserved and v_μ satisfies (2.1.3). Conformal invariance also dictates that the the scale dimensions of V_μ and $T_{\mu\nu}$ must be $d-1$ and d respectively, as will be shown below. For our subsequent investigations it is important to note that V_μ and $T_{\mu\nu}$ are not uniquely defined. It is possible to add to V_μ or $T_{\mu\nu}$ terms which satisfy the relations (2.2.1) by definition and thus lead to the same conserved charges or the same generators of the conformal group. For the vector current V_μ the arbitrariness consists of the divergence of an antisymmetric second rank tensor,

$$V_\mu(x) \sim V_\mu(x) + V'{}_\mu(x), \quad V'{}_\mu(x) = \partial_\nu F_{\nu\mu}(x),\ F_{\mu\nu}(x) = -F_{\nu\mu}(x)\ . \tag{2.2.3}$$

For the energy momentum tensor we have correspondingly

$$T_{\mu\nu}(x) \sim T_{\mu\nu}(x) + T'{}_{\mu\nu}(x), \quad T'{}_{\mu\nu}(x) = \partial_\alpha\partial_\beta C_{\mu\alpha\beta\nu}(x)\ , \tag{2.2.4}$$

where $C_{\mu\alpha\beta\nu}(x)$ has the symmetries of the Weyl tensor, i.e.

$$C_{\mu\alpha\beta\nu}(x) = C_{[\mu\alpha][\beta\nu]}(x), \quad C_{\mu[\alpha\beta\nu]}(x) = 0, \quad C_{\mu\alpha\beta\mu}(x) = 0 \ , \tag{2.2.5}$$

such that $\partial_\alpha \partial_\beta C_{\mu\alpha\beta\nu}(x)$ is by definition symmetric in $(\mu\nu)$, traceless and conserved. $F_{\mu\nu}$ and $C_{\mu\alpha\beta\nu}$ are each of scale dimension $d-2$. In particular theories operators with the assumed properties of $F_{\mu\nu}$ and $C_{\mu\alpha\beta\nu}$ need not exist. For instance, in $d=2$ or $d=3$ dimensions, $C_{\mu\alpha\beta\nu}$ is necessarily zero. Nevertheless the freedom exhibited in (2.2.3) and (2.2.4) is ultimately the reason for the existence of more than one linearly independent form for the conformal three point functions involving V_μ or $T_{\mu\nu}$. We discuss the question of the existence of the operators ${V'}_\mu$ an ${T'}_{\mu\nu}(x)$ further in sections 3.1 and 3.5.

We will now show how conformal invariance in conjunction with conservation determines the scaling properties of V_μ and $T_{\mu\nu}$. It is clear from the definition in (2.1.11) that the derivative of a quasi-primary field is in general no longer quasi-primary. The infinitesimal transformation law (2.1.12) yields for a vector field of general scale dimension η

$$\delta_v V_\mu = -L_v V_\mu = -(v\cdot\partial + \eta\sigma_v)V_\mu - \partial_{[\mu}v_{\nu]}V_\nu \ , \tag{2.2.6}$$

since for the vector representation the spin generator is $(s_{\mu\nu})_{\alpha\beta}$. Inserting the solution (2.1.4) for the infinitesimal form of conformal transformations which gives $\partial_{[\mu}v_{\nu]} = -\omega_{\mu\nu} + 2(x_\mu b_\nu - x_\nu b_\mu)$, we obtain for the divergence

$$\partial_\mu \delta_v V_\mu = -\left(v\cdot\partial + (\eta+1)\right)\partial_\mu V_\mu + 2(\eta - d + 1) b_\mu V_\mu \ . \tag{2.2.7}$$

Therefore if $\partial_\mu V_\mu$ is to be a conformal scalar, which transforms as $\delta_v(\partial_\mu V_\mu) = -L_v \partial_\mu V_\mu$ with the generator L_v appropriate for a spinless field of dimension $\eta+1$, we must require $\eta = d-1$. For a second rank tensor field we may find in a similar way

$$\partial_\mu \delta_v F_{\mu\nu} = -L_v(\partial_\mu F_{\mu\nu}) - 2(\eta - d + 1) b_\mu F_{\mu\nu} + 2 b_\mu F_{\mu\nu} - 2 b_\nu F_{\mu\mu} \ , \tag{2.2.8}$$

for L_v here acting on a vector field. There are two cases for which the inhomogeneous terms in (2.2.8) vanish: When $F_{\mu\nu} \to T_{\mu\nu}$ with $T_{\mu\nu}$ symmetric and traceless as is the case for the energy momentum tensor, for which $\eta = d$, and secondly when $F_{\mu\nu}$ is antisymmetric and $\eta = d-2$ as required for ${V'}_\mu = \partial_\nu F_{\nu\mu}$ as in (2.2.3). Similarly it is also possible to verify that ${T'}_{\mu\nu} = \partial_\alpha \partial_\beta C_{\mu\alpha\beta\nu}$ is a traceless tensor field if $C_{\mu\alpha\beta\nu}$ has the symmetries of the Weyl tensor (2.2.5) and also has dimension $\eta = d-2$: We have

$$\begin{aligned} \delta_v C_{\mu\sigma\rho\nu} &= -L_v\ C_{\mu\sigma\rho\nu} \\ &= -(v\cdot\partial + \eta\sigma_v) C_{\mu\sigma\rho\nu} - \partial_{[\mu} v_{\gamma]} C_{\gamma\sigma\rho\nu} - \partial_{[\sigma} v_{\gamma]} C_{\mu\gamma\rho\nu} \\ &\quad -\partial_{[\rho} v_{\gamma]} C_{\mu\sigma\gamma\nu} - \partial_{[\nu} v_{\gamma]} C_{\mu\sigma\rho\gamma} \end{aligned} \tag{2.2.9}$$

and therefore

$$\partial_\sigma \delta_v C_{\mu\sigma\rho\nu} = -L_v\ (\partial_\sigma C_{\mu\sigma\rho\nu}) + 2(\eta - d + 1) b_\sigma C_{\mu\sigma\rho\nu} \tag{2.2.10}$$

for a tensor $C_{\mu\sigma\rho\nu}$ with Weyl symmetry as in (2.2.5). For the second derivative we find

$$\begin{aligned} \partial_\sigma \partial_\rho \delta_v C_{\mu\sigma\rho\nu} &= -L_v\ (\partial_\sigma \partial_\rho C_{\mu\sigma\rho\nu}) \\ &\quad +2(\eta - d + 2)(b_\sigma \partial_\rho C_{\mu\sigma\rho\nu} + b_\rho \partial_\sigma C_{\mu\sigma\rho\nu})\,. \end{aligned} \tag{2.2.11}$$

So $\partial_\sigma \partial_\rho C_{\mu\sigma\rho\nu}$ transforms as a tensor only if $C_{\mu\sigma\rho\nu}$ has scale dimension $\eta = d-2$.

It is interesting to note from (2.2.2) that

$$\delta_{v_1} {j^{v_2}}_\mu \equiv (\delta_{v_1} T_{\mu\nu}) v_{2\,\nu} = -L_{v_1} {j^{v_2}}_\mu + {j^{[v_1,v_2]}}_\mu \tag{2.2.12}$$

where L_v is as in (2.2.6) with $\eta = d-1$.

Applying the general result for conformal two point functions (2.1.14) to $V_\mu(x)$ we have

$$\langle V_\mu(x) V_\nu(y) \rangle = \frac{C_V}{(x-y)^{2(d-1)}}\ I_{\mu\nu}(x-y)\ , \tag{2.2.13}$$

while for the energy momentum tensor we obtain

$$\langle T_{\mu\nu}(x) T_{\sigma\rho}(y) \rangle = \frac{C_T}{(x-y)^{2d}}\ \mathcal{I}^T_{\mu\nu,\sigma\rho}(x-y)\ . \tag{2.2.14}$$

$\mathcal{I}^T_{\mu\nu,\sigma\rho}(x)$ is the inversion on the space of traceless symmetric tensors, which is defined with the help of $\mathcal{E}^T_{\mu\nu,\sigma\rho}$, the projection operator onto this space:

$$\mathcal{E}^T_{\mu\nu,\sigma\rho} = \tfrac{1}{2}(\delta_{\mu\sigma}\delta_{\nu\rho} + \delta_{\mu\rho}\delta_{\nu\sigma}) - \tfrac{1}{d}\delta_{\mu\nu}\delta_{\sigma\rho}\ , \tag{2.2.15}$$

$$\mathcal{I}^T_{\mu\nu,\sigma\rho}(x) = \mathcal{E}^T_{\mu\nu,\sigma'\rho'} I_{\sigma'\sigma}(x) I_{\rho'\rho}(x)\ . \tag{2.2.16}$$

Since $\partial_\mu V_\mu$ is a scalar and $\partial_\mu T_{\mu\nu}$ is a vector, the scale dimensions $d-1$ and d of V_μ and $T_{\mu\nu}$ respectively ensure that the two point functions (2.2.13) and (2.2.14) automatically satisfy the required conservation equations as the two point function of $\partial_\mu V_\mu$ and V_μ or $\partial_\mu T_{\mu\nu}$ and $T_{\mu\nu}$ vanish because they involve operators of different spin.

2.3 Ward Identities

Any continuous symmetry present in a field theory which leads to a Noether current reflects itself in the quantum theory in relations between the correlation functions containing this current. These relations are called Ward identities. In particular there are Ward identities for conformal invariance which we are considering here. Subsequently we derive conformal Ward identities relating two and three point functions.

We assume that a functional $W[g,\mathcal{A}]$ depending on a background metric $g_{\mu\nu}$ and on a gauge field $\mathcal{A}^a{}_\mu$ may be defined for general dimensions d such that $T_{\mu\nu}$ and V_μ are given by

$$\begin{aligned} \langle T_{\mu\nu}(x)\rangle &= -\frac{2}{\sqrt{g(x)}}\frac{\delta}{\delta g^{\mu\nu}(x)}W \\ \langle V^{a\mu}(x)\rangle &= -\frac{1}{\sqrt{g(x)}}\frac{\delta}{\delta \mathcal{A}^a{}_\mu(x)}W\,. \end{aligned} \tag{2.3.1}$$

We define the functional derivatives by

$$\frac{\delta}{\delta \mathcal{A}^a{}_\mu(x)}\mathcal{A}^b{}_\nu(y) = \delta_{\mu\nu}\delta^{ab}\delta^d(x-y) \tag{2.3.2}$$

and similarly for the metric.

The action W, which is a scalar, is assumed to be invariant under diffeomorphisms and local Weyl rescalings of the metric. Restricting to flat space these properties imply conformal invariance. Diffeomorphism invariance requires

$$\int \mathrm{d}^d x \left(\mathcal{L}_v g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}} + \mathcal{L}_v \mathcal{A}_\mu \frac{\delta}{\delta \mathcal{A}_\mu}\right) W = 0 \tag{2.3.3}$$

with the Lie derivative $\mathcal{L}_v$:

$$\begin{aligned}\mathcal{L}_v g^{\mu\nu} &= -(\nabla^\mu v^\nu + \nabla^\nu v^\mu)\,,\\ \mathcal{L}_v \mathcal{A}_\mu &= v^\lambda \partial_\lambda \mathcal{A}_\mu + \partial_\mu v^\lambda \mathcal{A}_\lambda\,.\end{aligned} \tag{2.3.4}$$

Under local scale variations of the metric we require for arbitrary $\sigma(x)$ that

$$\int \mathrm{d}^d x\, \sigma \left(2 g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}}\right) W \;=\; 0\,. \tag{2.3.5}$$

After renormalisation, these relations may have anomalies reflecting the introduction of a renormalisation mass scale. These are studied in detail in the subsequent sections. Here we pursue by explaining the principles of the derivation of Ward identities. In addition to diffeomorphism and Weyl invariance we also assume local non-abelian gauge invariance which implies

$$\int \mathrm{d}^d x \left(\partial_\mu \Lambda^a + f^{abc} \mathcal{A}^b{}_\mu \Lambda^c\right) \frac{\delta}{\delta \mathcal{A}^a{}_\mu} W \;=\; 0\,, \tag{2.3.6}$$

where f^{abc} is the totally antisymmetric structure constant for the algebra of the gauge group and Λ a local gauge transformation. Using (2.3.1) we see that (2.3.6) implies

$$\nabla_\mu \langle V^{a\mu}\rangle + f^{abc} \mathcal{A}^b{}_\mu \langle V^{c\mu}\rangle = 0\,, \tag{2.3.7}$$

for an arbitrary gauge transformation Λ^a. (2.3.3) leads to

$$\nabla^\mu \langle T_{\mu\nu}\rangle + \nabla_\nu \mathcal{A}^a{}_\mu \langle V^{a\mu}\rangle - \nabla_\mu (\mathcal{A}^a{}_\nu \langle V^{a\mu}\rangle) = 0\,, \tag{2.3.8}$$

which using (2.3.7) is equivalent to

$$\nabla^\mu \langle T_{\mu\nu}\rangle + F^c{}_{\nu\mu} \langle V^{c\mu}\rangle = 0 \quad , \tag{2.3.9}$$

with the field strength tensor $F^c{}_{\mu\nu} = \nabla_\mu \mathcal{A}^c{}_\nu - \nabla_\nu \mathcal{A}^c{}_\mu + f^{cab}\, \mathcal{A}^a{}_\mu \mathcal{A}^b{}_\nu$. (2.3.5) implies

$$g^{\mu\nu} \langle T_{\mu\nu}\rangle = 0\,, \tag{2.3.10}$$

but as we describe later there are anomalous contributions to this trace in two and four dimensions. The Ward identities are obtained by functionally differentiating these relations twice and then restricting to flat Euclidean space where $g^{\mu\nu} = \delta^{\mu\nu}$ and to $\mathcal{A}^a{}_\mu = 0$. Three point functions are given by, as here shown in for the example of the energy momentum tensor,

$$\langle T_{\mu\nu}(x) T_{\sigma\rho}(y) T_{\alpha\beta}(z)\rangle = -8 \frac{\delta}{\delta g^{\mu\nu}(x)} \frac{\delta}{\delta g^{\sigma\rho}(y)} \frac{\delta}{\delta g^{\alpha\beta}(z)} W\Big|_{g=\delta,\mathcal{A}=0}, \tag{2.3.11}$$

and analogously for the vector currents. For the three vector current case the Ward identity is obtained by functional differentiation of (2.3.7),

$$\partial_\mu \langle V^p_\mu(x) V^q_\nu(y) V^r_\omega(z)\rangle = f^{pqs}\delta^d(x-y)\langle V^s_\nu(x) V^r_\omega(z)\rangle - f^{prs}\delta^d(x-z)\langle V^s_\omega(x) V^q_\nu(y)\rangle, \tag{2.3.12}$$

while for the trace of the energy momentum tensor we have in this non-anomalous case from (2.3.10)

$$\langle T_{\mu\mu}(x) T_{\sigma\rho}(y) T_{\alpha\beta}(z)\rangle = 2\left(\left(\delta^d(x-y) + \delta^d(x-z)\right)\langle T_{\sigma\rho}(y) T_{\alpha\beta}(z)\rangle\right., \tag{2.3.13}$$

and for the derivative of $T_{\mu\nu}$ from (2.3.9) with $\mathcal{A}^a{}_\mu = 0$

$$\begin{aligned} \partial^x{}_\mu \langle T_{\mu\nu}(x) T_{\sigma\rho}(y) T_{\alpha\beta}(z)\rangle \;&=\; \partial_\nu \delta^d(x-y)\langle T_{\sigma\rho}(x) T_{\alpha\beta}(z)\rangle \\ &\quad + \partial_\sigma \left(\delta^d(x-y)\langle T_{\nu\rho}(x) T_{\alpha\beta}(z)\rangle\right) \\ &\quad + \partial_\rho \left(\delta^d(x-y)\langle T_{\nu\sigma}(x) T_{\alpha\beta}(z)\rangle\right) \\ &\quad + y,\sigma,\rho \leftrightarrow z,\alpha,\beta\,. \end{aligned} \tag{2.3.14}$$

The derivative and the trace Ward identity for the energy momentum tensor are however not unique. It is possible to modify one identity with a compensating change in the other. For the energy momentum tensor, the Ward identities may be derived by a generalisation of techniques used in two-dimensional conformal field theory. The fact that the energy momentum tensor is the generator of conformal transformations and that conformal symmetry requires the conformal current (2.2.2) to be conserved leads to Ward identities of the form [16]

$$\int_S \mathrm{d}S_\mu v_\nu(x)\langle T_{\mu\nu}(x)\mathcal{O}^i(y)\ldots\rangle = \langle \delta_v \mathcal{O}^i(y)\ldots\rangle\,. \tag{2.3.15}$$

S is a surface enclosing the point y (if other local operators are present and S encloses their spatial arguments as well, the right hand side is a sum of terms involving the conformal variation of each such field in turn). For an infinitesimal conformal transformation v as in (2.1.4) the left hand side of (2.3.15) is invariant under smooth changes in S not crossing the points for which local fields are present in the correlation function. If S is restricted to a sphere surrounding the point y with radius tending to zero we may use the operator product expansion [17]

$$T_{\mu\nu}(x)\mathcal{O}(y) \sim A_{\mu\nu}(r)\mathcal{O}(y) + B_{\mu\nu\lambda}(r)\partial_\lambda \mathcal{O}(y)\,, \tag{2.3.16}$$

where $r = x - y$. Then (2.3.15) together with the expression (2.1.12) for $\delta_v \mathcal{O}$, since now $\mathrm{d}S_\mu = |r|^{(d-1)}\hat{r}_\mu \mathrm{d}\Omega_{\hat{r}}$ for $\hat{r}$ the unit vector in r-direction, requires the conditions

$$\int \mathrm{d}\Omega_{\hat{r}}\, \hat{r}_\mu A_{\mu\nu}(\hat{r}) = 0, \qquad \int \mathrm{d}\Omega_{\hat{r}}\, \hat{r}_\mu B_{\mu\nu\lambda}(\hat{r}) = -\delta_{\nu\lambda}\,, \tag{2.3.17}$$

$$\int \mathrm{d}\Omega_{\hat{r}}\, \hat{r}_\mu \hat{r}_\omega A_{\mu\nu}(\hat{r}) = -\frac{\eta}{d}\delta_{\omega\nu} + \tfrac{1}{2}s_{\omega\nu} - \hat{C}_{\omega\nu}\,, \quad \hat{C}_{\omega\nu} = \hat{C}_{\nu\omega}, \quad \hat{C}_{\nu\nu} = 0\,. \tag{2.3.18}$$

Here we have expanded $v_\nu(x)$ around y in (2.3.15) and identified the coefficients of the zeroth order terms proportional to v_ν on both sides of (2.3.15) in (2.3.17) and the first order terms proportional to $\partial_\mu v_\nu$ in (2.3.18) using $\partial_\mu v_\nu = \sigma_v \delta_{\mu\nu} + \partial_{[\mu}v_{\nu]}$. $s_{\mu\nu}$ is the generator of $O(d)$ rotations in the representation defined by $\mathcal{O}$. The term $\hat{C}_{\mu\nu}$ may be present since $\hat{C}_{\mu\nu}\partial_\mu v_\nu = 0$ due to its symmetry and tracelessness. $\hat{C}_{\mu\nu}$ is an $O(d)$ invariant, so that it satisfies $[s_{\mu\nu}, \hat{C}_{\mu\nu}] + (s_{\mu\nu})_{\omega\lambda,\omega'\lambda'}\hat{C}_{\omega'\lambda'} = 0$, but is not determined any further by the Ward identity.

When applying these results to three point functions we may make use of the fact that they are related to the operator product expansion. From (2.1.23) we write

$$\begin{aligned} &\langle T_{\mu\nu}(x)\mathcal{O}^i(y)\bar{\mathcal{O}}_j(z)\rangle \\ &\quad = C_{\mathcal{O}}\frac{1}{(x-z)^{2d}(y-z)^{2\eta}}\,\mathcal{I}^T_{\mu\nu,\sigma\rho}(x-z)D^i{}_{i'}(I(y-z))A_{\sigma\rho}{}^{i'}{}_j(X_{12})\,, \end{aligned} \tag{2.3.19}$$

where $A_{\sigma\rho}(X)$ is $\mathcal{O}(X^{-d})$ and satisfies the conditions (2.1.16) imposed by conformal invariance. The conservation equation $\partial_\mu T_{\mu\nu} = 0$ leads directly to $\partial_\mu A_{\mu\nu}(X) = 0$ for $X \neq 0$. The result (2.3.19) demonstrates how the three point function is fully determined by the leading operator product expansion coefficient $A_{\mu\nu}$. Thus the next to leading order term $B_{\mu\nu\lambda}$ in (2.3.16) may be determined on terms of $A_{\mu\nu}$. We may verify that the condition on $B_{\mu\nu\lambda}$ in (2.3.15) is satisfied so long as the relations for $A_{\mu\nu}$ are satisfied. If $A_{\mu\nu}(s)$ is regarded as a distribution on R^d the relations (2.3.17) and (2.3.18) may be rewritten in the form

$$\partial_\mu A_{\mu\nu}(s) = \left(\frac{\eta}{d}\delta_{\nu\lambda} + C_{\nu\lambda} + \tfrac{1}{2}s_{\nu\lambda}\right)\partial_\lambda \delta^d(s)$$
$$A_{\mu\mu}(s) = C_{\mu\mu}\delta^d(s), \quad \partial_\mu B_{\mu\nu\lambda}(s) = -\delta_{\nu\lambda}\delta^d(s). \tag{2.3.20}$$

$C_{\mu\nu} = C_{(\mu\nu)}$ is an arbitrary constant $O(d)$ invariant tensor which reflects the freedom in representing $A_{\mu\nu}(s)$ as a well-defined distribution up to terms of the form $C_{\mu\nu}\delta^d(s)$. If we define $\partial_\mu A_{\mu\nu}(s) \equiv \lim_{\omega\to 0}\partial_\mu(s^\omega A_{\mu\nu}(s))$ then $C_{\mu\nu} = \hat{C}_{\mu\nu}$ with $\hat{C}_{\mu\nu}$ as in (2.3.15). The ambiguity represented by $C_{\mu\nu}$ arises essentially from the freedom of the three point function (2.3.19) up to changes of the form

$$\begin{aligned}\langle T_{\mu\nu}(x)\mathcal{O}^i(y)\bar{\mathcal{O}}_j(z)\rangle \;\; \to \;\; & \langle T_{\mu\nu}(x)\mathcal{O}^i(y)\bar{\mathcal{O}}_j(z)\rangle \\ & + \delta^d(x-y)\,(C_{\mu\nu})^i{}_k\,\langle\mathcal{O}^k(y)\bar{\mathcal{O}}_j(z)\rangle \\ & + \delta^d(x-z)\langle\mathcal{O}^i(y)\bar{\mathcal{O}}_k(z)\rangle\,(C_{\mu\nu})^k{}_j\;.\end{aligned} \tag{2.3.21}$$

The trace and derivative Ward identities (2.3.13) and (2.3.14) are thus linked to each other via the freedom in choosing $C_{\mu\nu}$. They are in exact agreement with (2.3.20) if we take $(C_{\nu\lambda})_{\sigma\rho,\sigma'\rho'} = \mathcal{E}^T_{\sigma\rho,\nu\varepsilon}\mathcal{E}^T_{\lambda\varepsilon,\sigma'\rho'} + \nu\leftrightarrow\lambda$ such that $(C_{\mu\mu})_{\sigma\rho,\sigma'\rho'} = 2\mathcal{E}^T_{\sigma\rho,\sigma'\rho'}$. Choosing $C_{\mu\nu}$ such that $A_{\mu\mu}(s) = 0$ we may impose tracelessness and the derivative Ward identity is correspondingly modified.

2.4 Anomalies

As discussed in the previous section, the invariance under a symmetry group of the generating functional of a quantum field theory leads to a set of relations between the correlation functions involving the Noether currents of this theory which are called Ward identities. However, after renormalisation the n-point functions may satisfy these relations only up to additional, "anomalous" terms which reflect the breaking of the symmetry by the mass scale introduced for renormalisation. An example for this is the axial anomaly which occurs in the calculation of triangle diagrams involving two vector currents and one axial vector current. The axial anomaly has been discovered twenty-five years ago [18]. In massless QED the global symmetry group $U(1) \times U(1)$ is broken by the axial anomaly to $U(1)_V$. Correspondingly in massless QCD the chiral group $U(N) \times U(N)$ acting in flavour space is broken down to $SU(N) \times SU(N) \times U(1)_V$. The associated $U(1)_A$ currents acquire anomalous divergences. Mass terms do not affect the presence of these anomalous terms. Although anomalies originate from short-distance singularities, they also manifest themselves in low-energy theorems. In the case of gauge fields coupled to vector and axial vector flavour currents, as in the $SU(2) \times U(1)$ model of weak interactions, anomalies must be absent because gauge theories with gauge fields coupled to non-conserved currents are inconsistent. A further discussion of the axial anomaly can be found in textbooks [19].

The axial anomaly manifests itself in the fact that conservation of of the fermion axial current $A_\omega = \langle \bar{\psi} \gamma_\omega \gamma_5 \psi \rangle$ cannot be maintained in a three point function with two conserved vector currents V_μ, V_ν where $V_\mu = \langle \bar{\psi} \gamma_\mu \psi \rangle$. The anomalous divergence the axial current acquires may be expressed in the most succinct way by introducing a classical background gauge field $\mathcal{A}_\mu$. Thus the anomalous divergence for the axial current with gauge field $\mathcal{A}_\mu$ coupled to the vector currents has the form

$$\partial_\omega \langle \bar{\psi} \gamma_\omega \gamma_5 \psi \rangle = \frac{1}{16\pi^2} \varepsilon_{\mu\nu\sigma\rho} F_{\mu\nu} F_{\sigma\rho} \, , \quad F_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu \tag{2.4.1}$$

in four dimensions, if gauge invariance with respect to gauge transformations on $\mathcal{A}_\mu$ is preserved. $\varepsilon_{\mu\nu\sigma\rho}$ is the totally antisymmetric tensor in four dimensions. As will be shown in section 3.2 below, it was possible to reproduce the result for the axial anomaly for the case of conformal, i.e. essentially massless theories by calculating the Ward identity for the conformal three point function $\langle V_\mu(x) V_\nu(y) A_\omega(z) \rangle$ constructed according to the formalism of section 2.1 after applying differential regularisation.

The focus of interest in this thesis is the conformal anomaly, which manifests itself in anomalous contributions to the trace of the energy momentum tensor on a curved space background in four dimensions. In analogy to the axial anomaly we consider the metric $g^{\mu\nu}$ as a classical background field. In this case the gauge symmetry which is assumed to be preserved is diffeomorphism invariance.

The origin of the anomalous contributions to the trace of the energy momentum tensor is briefly explained in the following: In theories with exact conformal invariance the trace of the energy momentum tensor vanishes which follows from the invariance of the action under rescalings of the metric as in equation (2.1.1). In a quantum field theory on flat space with interaction Lagrangian $\mathcal{L}_I = g^i \mathcal{O}_i$, where the $\mathcal{O}_i$ form a basis of scalar operators with couplings g^i, the trace of the energy momentum tensor is $T_{\mu\mu} = \beta^i \mathcal{O}_i$, which vanishes for theories at renormalisation group fixed points where the beta functions are zero. Thus critical theories are conformally invariant on flat space. However, on a curved space background there are contributions to the energy momentum tensor trace even for such critical theories which break conformal invariance, such that (2.3.10) no longer holds. In four dimensions these contributions are given by

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle = \tfrac{1}{2}p\mathcal{J}^2 - \tfrac{1}{4}\kappa F^{\mu\nu}F_{\mu\nu} - \beta_a\, F - \beta_b\, G + h\,\nabla^2 R \tag{2.4.2}$$

where

$$\begin{aligned} F &= R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - \frac{4}{d-2}\,R^{\alpha\beta}R_{\alpha\beta} + \frac{2}{(d-2)(d-1)}\,R^2 \;=\; C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}\,, \\ G &= R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 4\,R^{\alpha\beta}R_{\alpha\beta} + R^2 \;=\; \tfrac{1}{4}\varepsilon^{\mu\nu\sigma\rho}\varepsilon_{\alpha\beta\gamma\delta}R^{\alpha\beta}{}_{\mu\nu}R^{\gamma\delta}{}_{\sigma\rho}\,. \end{aligned} \tag{2.4.3}$$

$C_{\alpha\beta\gamma\delta}$ is the Weyl tensor which is given by

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - 2\left(g_{\alpha[\gamma}K_{\delta]\beta} - g_{\beta[\gamma}K_{\delta]\alpha}\right)\,, \tag{2.4.4}$$

where

$$K_{\alpha\beta} = \frac{1}{d-2}\left(R_{\alpha\beta} - \frac{1}{2(d-1)}g_{\alpha\beta}R\right)\,. \tag{2.4.5}$$

The first two terms in the trace (2.4.2) arise from a scalar dimension two source $\mathcal{J}$ and from the background gauge field $\mathcal{A}_\mu$ in analogy to the axial anomaly. F is the square of the Weyl tensor and G is the Gauß-Bonnet invariant.

These contributions can be calculated in different ways. Here we sketch the calculations according to Buchbinder, Odintsov and Shapiro [20]. The starting point for deriving the conformal anomaly is a conformal field theory coupled to an external classical gravitational field with an action $W[g, \mathcal{A}, \mathcal{J}]$. The energy momentum tensor is defined through functional differentiation as in (2.3.1). $\mathcal{J}$ is a source term for a scalar quantum field, and $\mathcal{A}$ is the background gauge field. For simplicity we consider only the metric and the gauge field in the following discussion. The counterterms which have to be added to the action in order to remove divergences in the effective action $W[g, \mathcal{A}]$ may be calculated using dimensional regularisation. This introduces a mass scale into the theory. These counterterms, which are required to be coordinate invariant, contain scalars involving the Riemann curvature tensor and thus the metric. Inserting the renormalised effective action $W[g, \mathcal{A}]$ which depends on the renormalisation mass scale μ into the renormalisation group equation

$$\left(\mu\frac{\partial}{\partial\mu} + \int \mathrm{d}^d x \left(2g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}}\right)\right) W = 0 \tag{2.4.6}$$

which follows from dimensional analysis, yields the anomalous contributions to the trace of the energy momentum tensor as given in (2.4.2) if $d = 4$.

Although a term proportional to R^2 may be expected in the trace (2.4.2), it must be absent here since such a term cannot be obtained by functionally differentiating an action with respect to the metric. Furthermore the coefficient h in (2.4.2) is undetermined in general since it may be modified by the addition of a local term to the action, for which

$$-\frac{2}{\sqrt{g}}g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}}\int \mathrm{d}^4x\sqrt{g}R^2 = -12\nabla^2 R. \tag{2.4.7}$$

For simplicity we may thus take $h = 0$. The only conformal theories for which β_a, β_b have been computed so far are free field theories, for which [8]

$$\begin{aligned}\beta_a &= -\frac{1}{64\pi^2}\frac{1}{10}(\tfrac{1}{3}n_\phi + 2n_\psi + 4n_V),\\ \beta_b &= \frac{1}{64\pi^2}\frac{1}{90}(n_\phi + 11n_\psi + 62n_V).\end{aligned} \tag{2.4.8}$$

n_ϕ, n_ψ and n_V are the number of free scalar, fermion and vector fields, respectively.

The two anomalous terms F and G arise from different origins [21]: G is a topological invariant which may be present even if the action is dilatation invariant, whereas F is related to the renormalisation mass scale dependence of the quantum effective action. F and G are specific to four dimensions, though corresponding terms exist in all even dimensions. In odd dimensions it is not possible to construct corresponding terms with the required symmetry properties and there are no conformal anomalies.

As the three point functions may be obtained from functionally differentiating the action W, it is clear that the anomalous terms in (2.4.2) give rise to additional local terms in the trace Ward identity for the energy momentum tensor three point functions in four dimensions. Since the anomalous terms F and G given by (2.4.2) are of second order in the metric, local terms persist even on flat space after functionally differentiating (2.4.2) twice with respect to the metric or the background gauge field. In the trace Ward identity for the three point function involving three energy momentum tensors there are now two additional anomalous terms as compared to the Ward identity (2.3.13):

$$\begin{aligned}\langle T_{\mu\mu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle &= 2(\delta^4(x-y)+\delta^4(x-z))\langle T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle \\ &\quad -4\left(\beta_a\mathcal{A}^F_{\sigma\rho,\alpha\beta}(x-y,x-z)+\beta_b\mathcal{A}^G_{\sigma\rho,\alpha\beta}(x-y,x-z)\right) ,\end{aligned} \tag{2.4.9}$$

where we define for general d:

$$\frac{\delta}{\delta g^{\sigma\rho}(y)}\frac{\delta}{\delta g^{\alpha\beta}(z)}F(x)\bigg|_{g=\delta} = \mathcal{A}^F_{\sigma\rho,\alpha\beta}(x-y,x-z) , \tag{2.4.10}$$

$$\frac{\delta}{\delta g^{\sigma\rho}(y)}\frac{\delta}{\delta g^{\alpha\beta}(z)}G(x)\bigg|_{g=\delta} = \mathcal{A}^G_{\sigma\rho,\alpha\beta}(x-y,x-z) . \tag{2.4.11}$$

In four dimensions these expressions are given explicitly by

$$\begin{aligned}\mathcal{A}^F_{\sigma\rho,\alpha\beta}(x-y,x-z) &= 8\,\mathcal{E}^C_{\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}\partial_\varepsilon\partial_\eta\delta^4(x-y)\partial_\gamma\partial_\delta\delta^4(x-z) \\ \mathcal{A}^G_{\sigma\rho,\alpha\beta}(x-y,x-z) &= -\left(\epsilon_{\sigma\alpha\gamma\varepsilon}\epsilon_{\rho\beta\delta\eta}\partial_\varepsilon\partial_\eta(\partial_\gamma\delta^4(x-y)\partial_\delta\delta^4(x-z))+\sigma\leftrightarrow\rho\right) ,\end{aligned} \tag{2.4.12}$$

where $\epsilon_{\sigma\alpha\gamma\kappa}$ is the totally antisymmetric tensor in four dimensions and $\mathcal{E}^C_{\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}$ is the projection operator onto the space of Weyl tensors which has the symmetries of the Weyl tensor in both sets of indices. $\mathcal{E}^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}$ may be defined implicitly for general dimensions, acting on a tensor $P_{\alpha\beta\gamma\delta}$ with $P_{\alpha\beta\gamma\delta} = P_{[\alpha\beta][\gamma\delta]}$, $P_{\alpha\beta} = \frac{1}{2}(P_{\mu\alpha\mu\beta} + P_{\alpha\mu\beta\mu})$, by [6]

$$\begin{aligned}\mathcal{E}^C_{\mu\sigma\rho\nu,\alpha\beta\gamma\delta} P_{\alpha\beta\gamma\delta} &= \frac{1}{3}(P_{\mu\sigma\rho\nu} + P_{\rho\nu\mu\sigma} + P_{\mu[\nu\rho]\sigma} - P_{\sigma[\nu\rho]\mu}) \\ &\quad - \frac{2}{d-2}(\delta_{\rho[\mu}P_{\sigma]\nu} - \delta_{\nu[\mu}P_{\sigma]\rho}) + \frac{2}{(d-2)(d-1)}\delta_{\rho[\mu}\delta_{\sigma]\nu}P_{\lambda\lambda}\,. \end{aligned} \quad (2.4.13)$$

Its explicit form in terms of Kronecker deltas is given in appendix A.1.

It is easy to see that $\mathcal{A}^F$ and $\mathcal{A}^G$ in four dimensions are obtained from functionally differentiating F or G if we take into account that in an expansion $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$ around four dimensional flat space, F and G are given to second order in $h^{\mu\nu}$ by

$$\begin{aligned} F &= 4\,\mathcal{E}^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}\partial_\mu\partial_\nu h_{\sigma\rho}\partial_\gamma\partial_\delta\, h_{\alpha\beta} \\ G &= -2\,\epsilon_{\sigma\alpha\gamma\kappa}\epsilon_{\rho\beta\delta\lambda}\partial_\gamma\partial_\delta h_{\sigma\rho}\partial_\kappa\partial_\lambda h_{\alpha\beta}\,. \end{aligned} \quad (2.4.14)$$

The coefficient β_a of the anomalous term $-\beta_a F$ is related to the scale of the energy momentum tensor two point function C_T. This can be seen by integrating the expression $\partial^x{}_\mu(v_\nu(x)\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle)$ over $\mathbb{R}^d$, dropping the surface term for $|x| \to \infty$. If $v_\nu(x)$ is assumed to satisfy the conditions for infinitesimal conformal transformations as in (2.1.4), the form (2.3.14) for the Ward identity (2.4.9) gives

$$\begin{aligned} &\langle L_v T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle + \langle T_{\sigma\rho}(y)L_v T_{\alpha\beta}(z)\rangle \\ &\qquad = -32\beta_a \mathcal{E}^C_{\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}\partial_\varepsilon\partial_\eta\left(\sigma_v(y)\partial_\gamma\partial_\delta\delta^4(y-z)\right)\,, \end{aligned} \quad (2.4.15)$$

where L_v is the generator of infinitesimal conformal transformations defined in (2.1.12) with scale dimension $\eta = d$. There is no corresponding term involving β_b here because when integrating the β_b-anomaly by parts we obtain a term involving $\partial\partial\sigma_v$ which is zero. To obtain the exact relationship between β_a and C_T requires a careful treatment of the short distance singularities. From (2.2.14) we may write

$$\begin{aligned} \langle T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle &= C_T\frac{\mathcal{I}^T_{\sigma\rho,\alpha\beta}(y-z)}{(y-z)^{2d}} \\ &= \frac{C_T}{(d-3)(d-2)d(d+1)}\mathcal{E}^C_{\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}\partial^y{}_\varepsilon\partial^y{}_\eta\partial^z{}_\gamma\partial^z{}_\delta\frac{1}{(y-z)^{2d}}\,, \end{aligned} \quad (2.4.16)$$

where in the second form the conservation equations are trivially satisfied though it is no longer manifestly conformally covariant. When $d = 4$ the singularity as $y \to z$ is not integrable. We regularise the expression (2.4.16) by replacing

$$\frac{1}{(y-z)^4} \quad \to \quad \mathcal{R}\frac{1}{(y-z)^4}\,, \tag{2.4.17}$$

where we define

$$\mathcal{R}\frac{1}{(y-z)^4} \quad \equiv \quad \left(\frac{\mu^{2\omega}}{(y-z)^{2(2-\omega)}} - \frac{\pi^2}{\omega}\delta^4(y-z)\right)\bigg|_{\omega\to 0}. \tag{2.4.18}$$

μ is the renormalisation mass scale and $(y-z)^{-2(2-\omega)}$ is an analytic function of ω. The limit $\omega \to 0$ may be taken as a distribution after subtracting the pole. Applying differential regularisation to (2.4.18) we find

$$\mathcal{R}\frac{1}{(y-z)^4} = -\partial^2\frac{1}{4(y-z)^2}\left(\ln\mu^2(y-z)^2+1\right)\,. \tag{2.4.19}$$

The conformal variation of the energy momentum tensor two point function may be determined using the result obtained earlier, $L_v(\partial_\beta\partial_\gamma C_{\alpha\beta\gamma\delta}) = \partial_\beta\partial_\gamma(L_v C_{\alpha\beta\gamma\delta})$ if $C_{\alpha\beta\gamma\delta}$ has Weyl symmetry and scale dimension $d-2$. In four dimensions the conformal variation given by L_v applied to the two point function (2.4.16) with the regularisation (2.4.18) then reduces to a term involving

$$\Big(v(y)\cdot\partial^y + v(z)\cdot\partial^z + 2\sigma_v(y) + 2\sigma_v(z)\Big)\mathcal{R}\frac{1}{(y-z)^4} = 2\pi^2\sigma_v(y)\delta^4(y-z) \tag{2.4.20}$$

for the part depending on y, z. Moreover there are terms associated with the conformal variation of the tensor $\mathcal{E}^C_{\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}$ which are given by the action of the rotation generator corresponding to $\partial_{[\mu}v_{\nu]}(y)$ on the indices $\sigma\varepsilon\eta\rho$ and of $\partial_{[\mu}v_{\nu]}(z)$ on $\alpha\gamma\delta\beta$. Using the fact that $\mathcal{E}^C_{\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}$ is an invariant tensor we then obtain

$$\begin{aligned}&\langle L_v T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle + \langle T_{\sigma\rho}(y)L_v T_{\alpha\beta}(z)\rangle\\ &\quad= \frac{C_T\pi^2}{20}\mathcal{E}^C_{\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}\partial^y{}_\varepsilon\partial^y{}_\eta\partial^z{}_\gamma\partial^z{}_\delta\left(\sigma_v(y)\delta^4(y-z)\right)\\ &\qquad+\frac{3C_T}{40}b_\lambda\partial^y{}_\varepsilon\partial^y{}_\eta\partial^z{}_\gamma\partial^z{}_\delta\partial^y{}_{[\lambda}(\mathcal{E}^C_{\sigma\varepsilon]\eta\rho,\alpha\gamma\delta\beta}+\mathcal{E}^C_{\rho\eta]\sigma\varepsilon,\alpha\gamma\delta\beta})\frac{1}{(y-z)^2}\,.\end{aligned} \tag{2.4.21}$$

The symmetries of $\mathcal{E}^C_{\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}$ ensure that the second line, depending on b_λ, vanishes. Comparing with (2.4.15) gives finally

$$\beta_a = -\frac{\pi^2}{640} C_T \,. \tag{2.4.22}$$

This result may alternatively be obtained by functionally differentiating the renormalisation group equation (2.4.6) twice with respect to the metric, which gives

$$\begin{aligned} \mu\frac{\partial}{\partial\mu}\langle T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle &= -4\,\frac{\delta}{\delta g^{\sigma\rho}(y)}\frac{\delta}{\delta g^{\alpha\beta}(z)}\int \mathrm{d}^d x\,(\beta_a F + \beta_b G) \\ &= -32\,\beta_a\,\mathcal{E}^C_{\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}\partial_\varepsilon\partial_\eta\partial_\gamma\partial_\delta\delta^4(y-z)\,. \end{aligned} \tag{2.4.23}$$

There is no contribution involving β_b since $\int \mathrm{d}^d x\, G$ is a topological invariant such that its functional derivative vanishes. Inserting the two point function (2.4.16) regularised using (2.4.17) into this equation yields the relation (2.4.22) if we use

$$\lim_{\omega\to 0}\ \mu\frac{\partial}{\partial\mu}\frac{\mu^{2\omega}}{(y-z)^{2(2-\omega)}} = 2\pi^2\delta^4(y-z)\,. \tag{2.4.24}$$

This alternative derivation of the proportionality between β_a and C_T emphasises the importance of the scale dependence of the anomaly term $-\beta_a F$ for the relation (2.4.22).

3 Conformal Three Point Functions Involving Conserved Currents and the Energy Momentum Tensor

3.1 Three Point Function for Three Vector Operators

As a first example for a conformally covariant three point function we consider the three point function for three vector operators. In the general formalism developed in section 2.1 this reads

$$\langle V^p{}_\mu(x) V^q{}_\nu(y) V^r{}_\omega(z)\rangle = \frac{f^{pqr}}{(x-z)^{2d-2}(y-z)^{2d-2}} I_{\mu\alpha}(x-z) I_{\nu\beta}(y-z) t_{\alpha\beta\omega}(X_{12})\,, \quad (3.1.1)$$

where p,q,r are group indices with f^{pqr} the corresponding totally antisymmetric structure constant. $t_{\mu\nu\omega}(X)$ is $\mathcal{O}(X^{-d+1})$. The condition (2.1.21) is in this special case

$$I_{\mu\alpha}(X) t_{\alpha\nu\omega}(X) = -t_{\omega\mu\nu}(X), \qquad t_{\mu\nu\omega}(X) = -t_{\nu\mu\omega}(-X), \quad (3.1.2)$$

whereas the conservation equation $\partial_\mu V_\mu(x) = 0$ imposes the additional condition

$$\partial_\mu t_{\mu\nu\omega}(X) = 0\,. \quad (3.1.3)$$

The general solution for $t_{\mu\nu\omega}(X)$ satisfying these constraints is

$$t_{\mu\nu\omega}(X) = a\,\frac{X_\mu X_\nu X_\omega}{X^{d+2}} + b\,\frac{1}{X^d}\left(X_\mu\delta_{\nu\omega} + X_\nu\delta_{\mu\omega} - X_\omega\delta_{\mu\nu}\right), \quad (3.1.4)$$

for independent constants a, b. Throughout this thesis we use the abbreviation

$$X^{-d} \equiv (X^2)^{-\frac{d}{2}} \quad (3.1.5)$$

for powers of X in the denominator.

Furthermore, by analysing the short distance behaviour of (3.1.1) and comparing with the Ward identity (2.3.12) it is possible to relate the coefficients a, b in (3.1.4) to the scale of the two point function C_V [6]. When $s \equiv x - y$ tends to zero then (3.1.1) reduces to

$$\langle V^p{}_\mu(x) V^q{}_\nu(y) V^r{}_\omega(z)\rangle \sim f^{abc} t_{\mu\nu\rho}(s) \frac{I_{\rho\omega}(y-z)}{(y-z)^{2(d-1)}}\,. \quad (3.1.6)$$

To differentiate this with respect to ∂_μ in order to compare to the Ward identity (2.3.12) it is necessary to use relations like

$$\partial_\mu \frac{s_\nu}{s^d} = S_d \frac{\delta_{\mu\nu}}{d} \delta^d(s) + \left(\delta_{\mu\nu} - d \frac{s_\mu s_\nu}{s^2} \right) \frac{1}{s^d} \,, \tag{3.1.7}$$

where

$$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \,. \tag{3.1.8}$$

Comparing the derivative of (3.1.6) to the Ward identity (2.3.12) yields

$$S_d \left(\frac{1}{d} a + b \right) = C_V \,, \tag{3.1.9}$$

which relates the coefficients a and b to the overall scale of the two point function C_V. generalisations of (3.1.7) are essential to calculations for the energy momentum tensor in subsequent sections where the differentiation of singular functions is treated in detail.

We now determine the form in $\langle V^p{}_\mu(x) V^q{}_\nu(y) V^r{}_\omega(z) \rangle$ for which the conservation equations for the first two vector operators are trivially satisfied. This form can be written as

$$\begin{aligned} \langle V'^p{}_\mu(x) V'^q{}_\nu(y) V^r{}_\omega(z) \rangle &= \partial^x{}_\sigma \partial^y{}_\rho \langle F^p{}_{\mu\sigma}(x) F^q{}_{\nu\rho}(y) V^r{}_\omega(z) \rangle \\ &= f^{pqr} \frac{I_{\mu\mu'}(x-z) I_{\nu\nu'}(y-z)}{(x-z)^{2(d-1)} (y-z)^{2(d-1)}} t^{V'V'V}_{\mu'\nu'\omega}(X_{12}) \,, \end{aligned} \tag{3.1.10}$$

where $\langle F^p{}_{\mu\sigma}(x) F^q{}_{\nu\rho}(y) V^r{}_\omega(z) \rangle$ is given by

$$\langle F^p{}_{\mu\sigma}(x) F^q{}_{\nu\rho}(y) V^r{}_\omega(z) \rangle = f^{pqr} \frac{\mathcal{I}^F_{\mu\sigma,\mu'\sigma'}(x-z) \mathcal{I}^F_{\nu\rho,\nu'\rho'}(y-z)}{(x-z)^{2(d-2)} (y-z)^{2(d-2)}} t^{FFV}_{\mu'\sigma'\nu'\rho'\omega}(X_{12}) \,. \tag{3.1.11}$$

$t^{FFV}_{\mu\sigma\nu\rho\omega}(X)$ is a tensor antisymmetric in $[\mu\sigma]$ and in $[\nu\rho]$ and is $\mathcal{O}(X^{-d+3})$. $\mathcal{I}^F_{\mu\sigma,\mu'\sigma'}$ is the representation of the inversion acting on the space of antisymmetric tensors:

$$\mathcal{I}^F_{\mu\sigma,\nu\rho}(x) = \mathcal{E}^F_{\mu\sigma,\mu'\sigma'} I_{\mu'\nu}(x) I_{\sigma'\rho}(x) \,, \tag{3.1.12}$$

where

$$\mathcal{E}^F_{\mu\sigma,\mu'\sigma'} = \tfrac{1}{2}(\delta_{\mu\mu'}\delta_{\sigma\sigma'} - \delta_{\mu\sigma'}\delta_{\mu'\sigma}) \tag{3.1.13}$$

is the projection operator onto the space of antisymmetric second rank tensors. By virtue of

$$\partial_\mu \left(\frac{1}{x^{2(d-2)}} \mathcal{I}^F_{\mu\sigma,\mu'\sigma'}(x) \right) = 0 \quad \text{and} \quad \frac{\partial}{\partial x_\mu} f(X_{12}) = \frac{I_{\mu\sigma}(x-z)}{(x-z)^2} \frac{\partial}{\partial X_{12\,\sigma}} f(X_{12}) \qquad (3.1.14)$$

we may identify

$$\partial_\sigma \partial_\rho t^{FFV}_{\mu\sigma\nu\rho\omega}(X) = t^{V'V'V}_{\mu\nu\omega}(X) \,. \qquad (3.1.15)$$

The most general possible form for $t^{FFV}_{\mu\sigma\nu\rho\omega}(X)$ is

$$\begin{aligned} t_{\mu\sigma\nu\rho\omega}(X) \quad = \quad & A \, \frac{X_\omega}{X^{d-2}} \left(\delta_{\mu\nu}\delta_{\sigma\rho} - \delta_{\sigma\nu}\delta_{\mu\rho} \right) \\ & + B \, \frac{X_\omega}{X^d} \left(\delta_{\mu\nu} X_\sigma X_\rho + \delta_{\sigma\rho} X_\mu X_\nu - \delta_{\nu\sigma} X_\mu X_\rho - \delta_{\mu\rho} X_\sigma X_\nu \right) \qquad (3.1.16) \\ & + C \, \frac{1}{X^{d-2}} (\delta_{\omega\mu}\delta_{\nu\sigma} X_\rho - \delta_{\omega\sigma}\delta_{\nu\mu} X_\rho - \delta_{\omega\mu}\delta_{\rho\sigma} X_\nu + \delta_{\omega\sigma}\delta_{\mu\rho} X_\nu \\ & \qquad + \delta_{\omega\nu}\delta_{\mu\rho} X_\sigma - \delta_{\omega\nu}\delta_{\sigma\rho} X_\mu - \delta_{\omega\rho}\delta_{\mu\nu} X_\sigma + \delta_{\omega\rho}\delta_{\sigma\nu} X_\mu) \,, \end{aligned}$$

where A, B and C are constants.

$\langle F^p{}_{\mu\sigma}(x) F^q{}_{\nu\rho}(y) V^r{}_\omega(z) \rangle$ is more regular than $\langle V^p{}_\mu(x) V^q{}_\nu(y) V^r{}_\omega(z) \rangle$ because the powers in the denominator have been reduced in the sense that the derivatives $\partial^x{}_\mu$ or $\partial^y{}_\nu$ acting on $\langle F^p{}_{\mu\sigma}(x) F^q{}_{\nu\rho}(y) V^r{}_\omega(z) \rangle$ give just zero, while for $\langle V^p{}_\mu(x) V^q{}_\nu(y) V^r{}_\omega(z) \rangle$ they give rise to a delta distribution as in (3.1.7). This feature of differential regularisation of reducing the degree of singularities will have much more dramatic consequences for more involved calculations in later sections where the original three point functions have a higher degree of singularity before regularisation than $\langle V^p{}_\mu(x) V^q{}_\nu(y) V^r{}_\omega(z) \rangle$ has here.

Taking the derivatives ∂_σ, ∂_ρ of (3.1.16) and comparing with (3.1.4) gives

$$\begin{aligned} a &= -d(d-2)A + 2d(d-2)C \\ b &= (d-2)A - 2(d-2)C \,, \end{aligned} \qquad (3.1.17)$$

The second term with coefficient B vanishes when differentiated twice. (3.1.17) yields

$$a + db = 0 \,. \qquad (3.1.18)$$

There is only one free parameter left in the form $\langle V'^p{}_\mu(x)V'^q{}_\nu(y)V^r{}_\omega(z)\rangle$. (3.1.18) is consistent with the Ward identity (2.3.12) because $\partial^x{}_\mu\langle V'^p{}_\mu(x)V'^q{}_\nu(y)V^r{}_\omega(z)\rangle$ is trivially zero and therefore the right hand side of the Ward identity and thus C_V must also vanish.

The form in $\langle F^p{}_{\mu\sigma}(x)F^q{}_{\nu\rho}(y)V^r{}_\omega(z)\rangle$ for which all three vector operators are trivially conserved can be calculated by taking one derivative of the totally symmetric three point function $\langle F^p{}_{\mu\sigma}(x)F^q{}_{\nu\rho}(y)F^r{}_{\omega\alpha}(z)\rangle$:

$$\langle F^p{}_{\mu\sigma}(x)F^q{}_{\nu\rho}(y)V'^r{}_\omega(z)\rangle = \partial^z{}_\alpha\langle F^p{}_{\mu\sigma}(x)F^q{}_{\nu\rho}(y)F^r{}_{\omega\alpha}(z)\rangle\,. \tag{3.1.19}$$

Using the results for totally symmetric three point functions of section (2.1) we can now construct the three point function $\langle F^p{}_{\mu\sigma}(x)F^q{}_{\nu\rho}(y)F^r{}_{\omega\alpha}(z)\rangle$. The totally symmetric structure constant $d^{pqr}_{\mu\sigma\nu\rho\omega\alpha}$ as in (2.1.24) can be constructed in this case from $\mathcal{E}^F$, the projection operator onto the space of antisymmetric tensors, in the following way:

$$d^{pqr}_{\mu\sigma\nu\rho\omega\alpha} = f^{pqr}\,\mathcal{E}^F_{\mu\sigma,\beta\gamma}\mathcal{E}^F_{\nu\rho,\beta\eta}\mathcal{E}^F_{\omega\alpha,\eta\gamma}\,, \tag{3.1.20}$$

where the combination $\mathcal{E}^A_{\mu\sigma,\beta\gamma}\mathcal{E}^F_{\nu\rho,\beta\eta}\mathcal{E}^F_{\omega\alpha,\eta\gamma}$ of the three $\mathcal{E}^A$ tensors is totally antisymmetric, as is the structure constant f^{pqr}. Their combination in (3.1.20) is thus totally symmetric.

With this structure constant, $\langle F^p{}_{\mu\sigma}(x)F^q{}_{\nu\rho}(y)F^r{}_{\omega\alpha}(z)\rangle$ is given by

$$\langle F^p{}_{\mu\sigma}(x)F^q{}_{\nu\rho}(y)F^r{}_{\omega\alpha}(z)\rangle = f^{pqr}\,\frac{\mathcal{I}^F_{\mu\sigma,\mu'\sigma'}(x-z)\mathcal{I}^F_{\nu\rho,\nu'\rho'}(y-z)}{(x-z)^{2(d-2)}(y-z)^{2(d-2)}}t^{FFF}_{\mu'\sigma'\nu'\rho'\omega\alpha}(X_{12})\,, \tag{3.1.21}$$

where

$$t^{FFF}_{\mu\sigma\nu\rho\omega\alpha}(X) = \mathcal{E}^F_{\mu\sigma,\beta\gamma}\mathcal{E}^F_{\nu\rho,\beta\eta}\mathcal{E}^F_{\omega\alpha,\eta\gamma}\,\frac{I_{\alpha'\alpha}(X)I_{\omega'\omega}(X)}{X^{(d-2)}}\,. \tag{3.1.22}$$

$t^{FFF}_{\mu\sigma\nu\rho\omega\alpha}(X)$ satisfies the conditions (2.1.16), (2.1.17) and (2.1.21). Taking the derivative ∂_α after a change of variables according to (2.1.19) gives the values

$$A = \frac{1}{2},\quad B = \frac{d-4}{2},\quad C = \frac{d-4}{8} \tag{3.1.23}$$

for the coefficients in (3.1.16).

To conclude this section we comment on the existence of an $F_{\nu\mu}$ such that we can write $V_\mu = \partial_\nu F_{\nu\mu}$. We consider the three point function

$$\langle V_\mu(x)\mathcal{O}_1{}^i(y)\mathcal{O}_2{}^j(z)\rangle = \frac{1}{(x-z)^{2(d-1)}(y-z)^{2\eta_1}} I_{\mu\mu'}(x-z) D_1{}^i{}_{i'}(I(y-z)) t^V{}_{\mu'}{}^{i'j}(X_{12}) \tag{3.1.24}$$

for two general operators $\mathcal{O}_1, \mathcal{O}_2$ with scale dimensions η_1, η_2. We consider only $\eta_1 \neq \eta_2$ such that the corresponding two point function $\langle \mathcal{O}_1{}^i(x)\mathcal{O}_2{}^j(z)\rangle$ vanishes and there are no non-trivial Ward identities. $t^V{}_\mu{}^{ij}(X)$ is $\mathcal{O}(X^{-(d-1+\eta_1-\eta_2)})$ and from the conservation equation $\partial_\mu t^V{}_\mu{}^{ij}(X) = 0$ it is easy to see that

$$(\eta_1 - \eta_2) t^V{}_\mu{}^{ij}(X) = \partial_\nu \Big(X_\mu t^V{}_\nu{}^{ij}(X) - X_\nu t^V{}_\mu{}^{ij}(X) \Big)\,, \tag{3.1.25}$$

so that we may always write

$$\begin{aligned} t^V{}_\mu{}^{ij}(X) &= \partial_\nu t^F{}_{\mu\nu}{}^{ij}(X)\,, \\ t^F{}_{\mu\nu}{}^{ij}(X) &= \frac{1}{\eta_1 - \eta_2} \Big(X_\mu t^V{}_\nu{}^{ij}(X) - X_\nu t^V{}_\mu{}^{ij}(X) \Big)\,. \end{aligned} \tag{3.1.26}$$

Thus using (3.1.14) in this case, we find

$$\begin{aligned} &\langle V_\mu(x)\mathcal{O}_1{}^i(y)\mathcal{O}_2{}^j(z)\rangle \\ &= \partial^x{}_\nu \left(\frac{1}{(x-z)^{2(d-2)}(y-z)^{2\eta_1}} \mathcal{I}^F_{\mu\nu,\mu'\nu'}(x-z) D_1{}^i{}_{i'}(I(y-z)) t^F{}_{\mu'\nu'}{}^{i'j}(X_{12}) \right). \end{aligned} \tag{3.1.27}$$

When $\eta_1 = \eta_2$ there are non-trivial Ward identities so that writing $V_\mu = \partial_\nu F_{\nu\mu}$ is no longer possible in general, though it may still hold for some of the forms in the three point function considered. An example for $\eta_1 = \eta_2$ is given by (3.1.10).

3.2 Axial Anomaly

To demonstrate the applicability of the regularisation method presented, we use it in this section to calculate the axial anomaly for massless fields present in the conformal three point function involving to vector and one axial vector current. Imposing conservation of the two vector currents by writing them as the divergence of an antisymmetric rank two tensor allows for applying differential regularisation and calculating the anomalous Ward identity reflecting the non-conservation of the axial current in an unambiguous way.

We begin by considering the conformally invariant three point function for two vector operators V_μ, V_ν and one axial vector operator A_ω in four dimensions, for which the general expression (2.1.15) gives:

$$\langle V_\mu(x)V_\nu(y)A_\omega(z)\rangle \;=\; \frac{I_{\mu\mu'}(x-z)I_{\nu\nu'}(y-z)}{(x-z)^6(y-z)^6}t_{\mu'\nu'\omega}(X_{12}). \tag{3.2.1}$$

All three operators have the scaling dimension $d-1=3$. The conditions (2.1.16) and (2.1.17) require $t_{\mu\nu\omega}(X)$ to be a homogeneous function of degree -3 in X transforming covariantly under $O(4)$ rotations. From (2.1.21) there is a symmetry condition

$$t_{\mu\nu\omega}(X) = t_{\nu\mu\omega}(-X)\,. \tag{3.2.2}$$

Note the crucial difference to the three vector case (3.1.2) where there is a minus sign on the right hand side. The conservation equations for the two vector operators for non-coincident points impose the condition

$$\partial_\mu t_{\mu\nu\omega}(X) = \partial_\nu t_{\mu\nu\omega}(X) = 0\ . \tag{3.2.3}$$

The most general solution for $t_{\mu\nu\omega}(X)$ with these constraints and with the required parity properties is

$$t_{\mu\nu\omega}(X) = C\ \varepsilon_{\mu\nu\omega\lambda}\frac{X_\lambda}{X^4}\ . \tag{3.2.4}$$

Under a parity transformation

$$\mathcal{P}V_\mu(x) = \Theta_{\mu\nu}V_\nu(x_1,-\mathbf{x})\,,\;\; \mathcal{P}A_\mu(x) = -\Theta_{\mu\nu}A_\nu(x_1,-\mathbf{x}) \tag{3.2.5}$$

where

$$\Theta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbf{1}_3 \end{pmatrix}\,, \tag{3.2.6}$$

we have - assuming parity invariance -

$$-\left\langle \Theta_{\mu\mu'} V_{\mu'}(\Theta x) \Theta_{\nu\nu'} V_{\nu'}(\Theta y) \Theta_{\omega\omega'} A_{\omega'}(\Theta z) \right\rangle \;=\; \left\langle V_\mu(x) V_\nu(y) A_\omega(z) \right\rangle . \tag{3.2.7}$$

This is indeed satisfied by the expression (3.2.1) with (3.2.4) since $X_{12\,\mu}$ transforms like a vector under parity, i. e.

$$X_{12\,\mu} \to \Theta_{\mu\nu} X_{12\,\nu} \;\;\text{for}\;\; x_\mu, y_\mu, z_\mu \to \Theta_{\mu\nu} x_\nu, \Theta_{\mu\nu} y_\nu, \Theta_{\mu\nu} z_\nu \,, \tag{3.2.8}$$

and

$$\Theta_{\mu\mu'} \Theta_{\nu\nu'} \Theta_{\omega\omega'} \Theta_{\lambda\lambda'} \varepsilon_{\mu'\nu'\omega'\lambda'} \;=\; -\,\varepsilon_{\mu\nu\omega\lambda} \tag{3.2.9}$$

since the left hand side is a representation of the determinant and $\det\Theta = -1$. Furthermore we have for the inversions

$$I\,(\Theta x) = \Theta_{\mu\mu'} \Theta_{\nu\nu'} I_{\mu'\nu'}(x) \,. \tag{3.2.10}$$

In order to calculate the anomaly we impose conservation of the two vector currents for all points by considering the expression

$$\begin{aligned} \langle V_\mu(x) V_\nu(y) A_\omega(z) \rangle \;&=\; \partial^x{}_\sigma \partial^y{}_\rho \langle F_{\mu\sigma}(x) F_{\nu\rho}(y) A_\omega(z) \rangle \\ &= -\partial^x{}_\sigma \partial^y{}_\rho \left(\frac{\mathcal{I}^F_{\mu\sigma,\mu'\sigma'}(x-z) \mathcal{I}^F_{\nu\rho,\nu'\rho'}(y-z)}{(x-z)^4 (y-z)^4} t^{FFA}_{\mu'\sigma'\nu'\rho'\omega}(X_{12}) \right) , \end{aligned} \tag{3.2.11}$$

where $t^{FFA}_{[\mu\sigma][\nu\rho]\omega}(X)$ is antisymmetric in $[\mu\sigma]$ and in $[\nu\rho]$. Using (3.1.14) we may identify (3.2.11) with (3.2.1) if

$$t_{\mu\nu\omega}(X) = \partial_\sigma \partial_\rho t^{FFA}_{\mu\sigma\nu\rho\omega}(X) \,. \tag{3.2.12}$$

The most general possible form for $t^{FFA}_{\mu\sigma\nu\rho\omega}(X)$ is

$$t^{FFA}_{\mu\sigma\nu\rho\omega}(X) = A \frac{1}{X^2} \left(\varepsilon_{\mu\sigma\omega[\nu} X_{\rho]} - \varepsilon_{\nu\rho\omega[\mu} X_{\sigma]} \right) + B \frac{X_\lambda}{X^4} X_{[\mu} \varepsilon_{\sigma]\lambda\omega[\rho} X_{\nu]} \,, \tag{3.2.13}$$

where (3.2.12) gives

$$C = \, 4\,A \, + \, \tfrac{1}{2}\,B \,. \tag{3.2.14}$$

Requiring the axial vector operator to be conserved for $x \neq y \neq z$, one obtains the relation $4A + B = 0$ which leaves $C = 2A = -B/2$. Performing a change of variables according to (2.1.19) and multiplying out the terms involving z leads to

$$\begin{aligned} &\langle V_\mu(x) V_\nu(y) A_\omega(z) \rangle \\ &= -\frac{C}{4} \partial^x{}_\sigma \partial^y{}_\rho \left(\frac{\mathcal{I}^F_{\mu\sigma,\mu'\sigma'}(x-y)}{(x-y)^2} \, \varepsilon_{\mu'\sigma'\nu\rho} \left(\frac{(z-y)_\omega}{(z-x)^2 (z-y)^4} - \frac{(z-x)_\omega}{(z-x)^4 (z-y)^2} \right) \right) . \end{aligned} \tag{3.2.15}$$

The expression given by (3.2.1) with (3.2.4) is non-integrable as a function on $\mathbb{R}^8$, with coordinates $x - z$, $y - z$, due to the singular behaviour as $x - z \sim y - z \to 0$ so that the Fourier transform is ill-defined without regularisation. However $\langle F_{\mu\sigma}(x) F_{\nu\rho}(y) A_\omega(z) \rangle$ as in (3.2.11) or equivalently in (3.2.15) has no non-integrable short distance singularities and represents a well defined distribution on $\mathbb{R}^8$. Thus the divergence of the axial vector operator may be unambiguously calculated in (3.2.15) which yields

$$\begin{aligned} &\partial^z{}_\omega \langle V_\mu(x) V_\nu(y) A_\omega(z) \rangle \\ &\quad = -C \frac{\pi^2}{2} \partial^x{}_\sigma \partial^y{}_\rho \left(\frac{\mathcal{I}^F_{\mu\sigma,\mu'\sigma'}(x-y)}{(x-y)^4} \, \varepsilon_{\mu'\sigma'\nu\rho} \, \left(\delta^4(z-y) - \delta^4(z-x) \right) \right) . \end{aligned} \tag{3.2.16}$$

The remaining derivatives $\partial^x{}_\sigma$ and $\partial^y{}_\rho$ can be calculated with the aid of

$$\begin{aligned} \partial_\sigma \frac{\mathcal{I}^F_{\mu\sigma,\nu\rho}(x)}{x^{2\lambda}} &= \frac{2-\lambda}{2\lambda} (\delta_{\mu\nu}\delta_{\sigma\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}) \partial_\sigma \frac{1}{x^{2\lambda}} \\ &\sim \frac{\pi^2}{4} (\delta_{\mu\nu}\delta_{\sigma\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}) \partial_\sigma \delta^4(x) \text{ as } \lambda \to 2 , \end{aligned} \tag{3.2.17}$$

which depends on the result, in general dimension d,

$$\frac{1}{(x^2)^{\frac{1}{2}(d+\omega)}} \sim -\frac{1}{\omega} S_d \delta^d(x) \text{ for } \omega \to 0, \quad S_d = \frac{2\pi^{d/2}}{\Gamma(\frac{1}{2}d)} , \tag{3.2.18}$$

which will be discussed in detail in section 3.3. Taking the remaining two derivatives in (3.2.16) using (3.2.17) yields

$$\partial^z{}_\omega \langle V_\mu(x) V_\nu(y) A_\omega(z) \rangle = -C \frac{\pi^4}{2} \, \varepsilon_{\mu\sigma\nu\rho} \, \partial^x{}_\sigma \partial^y{}_\rho \left(\delta^4(x-z) \delta^4(y-z) \right) \tag{3.2.19}$$

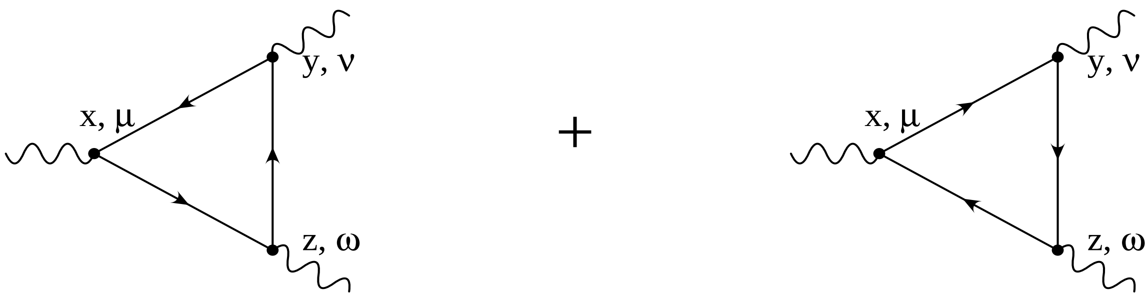
Up to the factor C, this has the expected form of the Ward identity for the axial anomaly. The parameter C can be calculated by comparing with the three point function for two vector currents $V_\mu = \bar{\psi}\gamma_\mu\psi$ and one axial vector current $A_\omega = \bar{\psi}\gamma_\omega\gamma_5\psi$, where $\psi(x)$ is a free fermion field with the propagator

$$\langle\psi(x)\bar{\psi}(0)\rangle = \frac{\gamma\cdot x}{2\pi^2(x^2)^2}\,. \tag{3.2.20}$$

The one loop result is then

$$\begin{aligned}\langle V_\mu(x)V_\nu(y)A_\omega(z)\rangle_\psi \quad = \quad & -\frac{1}{(2\pi^2)^3}\mathrm{tr}\left[\gamma_\mu\frac{\gamma\cdot(x-y)}{(x-y)^4}\gamma_\nu\frac{\gamma\cdot(y-z)}{(y-z)^4}\gamma_\omega\gamma_5\frac{\gamma\cdot(z-x)}{(z-x)^4}\right] \\ & -\frac{1}{(2\pi^2)^3}\mathrm{tr}\left[\gamma_\mu\frac{\gamma\cdot(x-z)}{(x-z)^4}\gamma_\omega\gamma_5\frac{\gamma\cdot(z-y)}{(z-y)^4}\gamma_\nu\frac{\gamma\cdot(y-x)}{(y-x)^4}\right]\,,\end{aligned} \tag{3.2.21}$$

with the two terms corresponding to the two Feynman diagrams



where the solid lines represent fermion propagators. This anomaly is crucial for calculating $\pi^0 \to \gamma\gamma$ decays.

To compare this to the conformal expression for the three point function, we consider (3.2.1) with (3.2.4) in a configuration where the three points x, y, z are constrained to lie on a straight line:

$$\langle V_i(x)V_j(y)A_k(z)\rangle = -\,C\,\frac{1}{(\hat{x}-\hat{z})^3(\hat{y}-\hat{z})^3(\hat{x}-\hat{y})^3}\varepsilon_{ijk1}\,, \tag{3.2.22}$$

where $\mathbf{x} = \hat{x}\mathbf{n}$ with $\mathbf{n}$ the unit vector along the line and the index 1 denoting components along the line, Latin indices the components perpendicular to it. In this collinear frame,

$$(X_{12})_1 = -\frac{\hat{x}-\hat{y}}{(\hat{x}-\hat{z})(\hat{y}-\hat{z})} \tag{3.2.23}$$

and $I_{11} = -1,\ I_{ij} = \delta_{ij}\ ,\ I_{1i} = 0$. In the same configuration the expression (3.2.21) reads

$$\langle V_i(x)V_j(y)A_k(z)\rangle_\psi = \frac{1}{(\hat{x}-\hat{z})^3(\hat{y}-\hat{z})^3(\hat{x}-\hat{y})^3}\,\frac{2}{(2\pi^2)^3}\,\mathrm{tr}(\gamma_i\gamma_j\gamma_k\gamma_1\gamma_5) \qquad (3.2.24)$$

such that we find

$$C = -\frac{1}{\pi^6}, \qquad (3.2.25)$$

by comparing (3.2.22) and (3.2.24) and using $\mathrm{tr}(\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta\gamma_5) = 4\varepsilon_{\alpha\beta\gamma\delta}$.

This agrees with the known result for the axial anomaly calculated from the usual triangle graph, as given for example in [19]. The derivation for the axial anomaly presented here was primarily performed as a test for the methods to be used later on, but it also emphasises the independence of the result for the axial anomaly from any regularisation scheme once vector current conservation has been imposed.

3.3 First Example for the Trace Anomaly

Before we turn to the construction of the energy momentum tensor three point function and the calculation of its non-integrable singular terms, we first perform a similar calculation in the case of the three point function $\langle T_{\mu\nu}(x)V_\sigma(y)V_\rho(z)\rangle$ involving one energy momentum tensor and two conserved vector currents. For this three point function the calculations are somewhat easier because the number of indices is smaller. This example displays how the regularisation procedure works which will ultimately allow us to calculate the non-integrable singular terms in the energy momentum tensor three point function $\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle$ for general conformal field theories. Furthermore this example demonstrates the relationship between the non-integrable singularities in the three point function, the Ward identities calculated on flat space and those derived from functionally differentiating the action W. The example also elucidates the relation between regularisation and the anomalies.

Since conservation of the vector currents leads to no non-trivial Ward identities we consider the form

$$\langle T_{\mu\nu}(x)V_\sigma(y)V_\rho(z)\rangle = \partial^y{}_\alpha\partial^z{}_\beta\langle T_{\mu\nu}(x)F_{\sigma\alpha}(y)F_{\rho\beta}(z)\rangle \tag{3.3.1}$$

in which the vector currents are conserved by definition. This leads to no loss of generality. According to the general formalism the three point function $\langle T_{\mu\nu}(x)F_{\mu\alpha}(y)F_{\nu\beta}(z)\rangle$ is given by

$$\langle T_{\mu\nu}(x)F_{\sigma\alpha}(y)F_{\rho\beta}(z)\rangle = \frac{\mathcal{I}^F_{\sigma\alpha,\sigma'\alpha'}(y-x)\mathcal{I}^F_{\rho\beta,\rho'\beta'}(z-x)}{(y-x)^{2(d-2)}(z-x)^{2(d-2)}}t^{TFF}_{\mu\nu\sigma'\alpha'\rho'\beta'}(X_{23})\,, \tag{3.3.2}$$

where X_{23} is given by (2.1.10). $t_{\mu\nu\sigma\alpha\rho\beta}(X)$ is symmetric and traceless in $\mu\nu$ but antisymmetric in both $\sigma\alpha$ and $\rho\beta$. It can be expanded in general in the form

$$\begin{aligned} t^{TFF}_{\mu\nu\sigma\alpha\rho\beta}(X) &= A\,\mathcal{E}^F_{\sigma\alpha,\lambda\varepsilon}\mathcal{E}^F_{\rho\beta,\lambda\eta}\mathcal{E}^T_{\varepsilon\eta,\mu\nu}\frac{1}{X^{d-4}} \\ &+ B\,\mathcal{E}^F_{\sigma\alpha,\kappa\varepsilon}\mathcal{E}^F_{\rho\beta,\lambda\eta}\mathcal{E}^T_{\kappa\lambda,\mu\nu}\frac{X_\varepsilon X_\eta}{X^{d-2}} \\ &+ C\,\mathcal{E}^F_{\sigma\alpha,\lambda\varepsilon}\mathcal{E}^F_{\rho\beta,\lambda\eta}\left(\mathcal{E}^T_{\varepsilon\kappa,\mu\nu}X_\eta + \mathcal{E}^T_{\eta\kappa,\mu\nu}X_\varepsilon\right)\frac{X_\kappa}{X^{d-2}} \\ &+ D\,\mathcal{E}^F_{\sigma\alpha,\rho\beta}\left(\frac{X_\mu X_\nu}{X^2}-\frac{1}{d}\delta_{\mu\nu}\right)\frac{1}{X^{d-4}} \\ &+ E\,\mathcal{E}^F_{\sigma\alpha,\lambda\varepsilon}\mathcal{E}^F_{\rho\beta,\lambda\eta}X_\varepsilon X_\eta\left(\frac{X_\mu X_\nu}{X^2}-\frac{1}{d}\delta_{\mu\nu}\right)\frac{1}{X^{d-2}}\,. \end{aligned} \tag{3.3.3}$$

Alternatively, the three point function (3.3.2) may be rewritten as

$$\langle T_{\mu\nu}(x)F_{\sigma\alpha}(y)F_{\rho\beta}(z)\rangle = \frac{\mathcal{I}^T_{\mu\nu,\mu'\nu'}(x-z)\mathcal{I}^F_{\sigma\alpha,\sigma'\alpha'}(y-z)}{(x-z)^{2d}(y-z)^{2(d-2)}}\hat{t}^{TFF}_{\mu'\nu'\sigma'\alpha'\rho\beta}(X_{12})\,, \tag{3.3.4}$$

where

$$\hat{t}^{TFF}_{\mu\nu\sigma\alpha\rho\beta}(X) = \frac{1}{(X^2)^2}\,\mathcal{I}^F_{\sigma\alpha,\sigma'\alpha'}(X)t^{TFF}_{\mu\nu\sigma'\alpha'\rho\beta}(-X)\,. \tag{3.3.5}$$

Imposing the conservation equation for the energy momentum tensor

$$\partial^x{}_\mu\langle T_{\mu\nu}(x)F_{\sigma\alpha}(y)F_{\rho\beta}(z)\rangle = 0 \tag{3.3.6}$$

leads to

$$\partial_\mu t^{TFF}_{\mu\nu\sigma\alpha\rho\beta}(X) = 0\,. \tag{3.3.7}$$

This gives rise to the single condition

$$K \equiv 2(2C+4D+E)+(d+2)B-(d+2)(d-4)A = 0\,. \tag{3.3.8}$$

By explicitly calculating the two derivatives in (3.3.1) we find an expression for the three point function involving the energy momentum tensor and two vector currents

$$\langle T_{\mu\nu}(x)V_\sigma(y)V_\rho(z)\rangle = \frac{I_{\sigma\sigma'}(y-x)I_{\rho\rho'}(z-x)}{(y-x)^{2(d-1)}(z-x)^{2(d-1)}}t^{TVV}_{\mu\nu\sigma'\rho'}(X_{23})\,, \tag{3.3.9}$$

where

$$\begin{aligned} t^{TVV}_{\mu\nu\sigma\rho}(X) &\equiv -\partial_\alpha\partial_\beta t^{TFF}_{\mu\nu\sigma\alpha\rho\beta}(X) \\ &= I\left(\mathcal{E}^T_{\mu\nu,\sigma\rho} + \tfrac{1}{2}d(d-2)\frac{X_\sigma X_\rho}{X^2}\left(\frac{X_\mu X_\nu}{X^2} - \frac{1}{d}\delta_{\mu\nu}\right)\right)\frac{1}{X^{d-2}} \\ &\quad + J\left(\mathcal{E}^T_{\mu\nu,\lambda\rho}X_\sigma + \mathcal{E}^T_{\mu\nu,\lambda\sigma}X_\rho\right)\frac{X_\lambda}{X^d} \\ &\quad - \left(J - \tfrac{1}{2}d(d-2)\,I\right)\delta_{\sigma\rho}\left(\frac{X_\mu X_\nu}{X^2} - \frac{1}{d}\delta_{\mu\nu}\right)\frac{1}{X^{d-2}}\,, \\ I = C + D - \tfrac{1}{2}B\,, &\qquad J = \tfrac{1}{4}\left(2E + (d-4)(d-2)A - (d-2)(B+2C+4D)\right)\,. \end{aligned} \tag{3.3.10}$$

Thus it is clear that there are just two linearly independent forms for this conformally invariant three point function.

For the short distance behaviour when $x \to y$ we have for $\langle T_{\mu\nu}(x)V_\sigma(y)V_\rho(z)\rangle$ with $s \equiv x - y$

$$\langle T_{\mu\nu}(x)V_\sigma(y)V_\rho(z)\rangle \sim \hat{t}^{TVV}_{\mu\nu\sigma\rho'}(s)\frac{I_{\rho'\rho}(z-y)}{(z-y)^{2(d-1)}}\,, \tag{3.3.11}$$

where

$$\hat{t}^{TVV}_{\mu\nu\sigma\rho}(X) = \frac{1}{X^2}I_{\sigma\sigma'}(X)\,t^{TVV}_{\mu\nu\sigma'\rho}(-X) \ = \mathcal{O}(X^{-d})\,. \tag{3.3.12}$$

Therefore we are able to determine the coefficient C_V of the two point function in terms of I and J by calculating the Ward identity (2.3.18) for the three point function (3.3.9). For the symmetric and antisymmetric contributions to this Ward identity we have

$$\begin{aligned} \int \mathrm{d}\Omega_{\hat{x}}\,\hat{x}_\mu\hat{x}_\nu\hat{t}^{TVV}_{\mu\nu\sigma\rho}(\hat{x}) &= -(d-1)C_V\delta_{\sigma\rho}\,, \\ \int \mathrm{d}\Omega_{\hat{x}}\,\hat{x}_\nu\hat{x}_{[\omega}\hat{t}^{TVV}_{\mu]\nu\sigma\rho}(\hat{x}) &= -2C_V\mathcal{E}^F_{\omega\mu,\sigma\rho} \end{aligned} \tag{3.3.13}$$

for $\hat{t}^{TVV}$ given by (3.3.12) and (3.3.10) with

$$\begin{aligned} C_V &= \frac{S_d}{d}\left(I + J\right) \\ &= \frac{S_d}{2d}\Big(d(d-4)A - (d+1)B - (d-2)C - 2(d-1)D\Big), \end{aligned} \tag{3.3.14}$$

where we have used the conservation condition (3.3.8) to eliminate E.

It is also interesting to consider the three point function (3.3.2) with (3.3.3) subject to the condition (3.3.8) as appropriate for the energy momentum tensor and two antisymmetric tensor operators of dimension $(d-2)$. In this case the short distance limit $x \to y$ with $s \equiv x - y$ is given by

$$\langle T_{\mu\nu}(x) F_{\sigma\alpha}(y) F_{\rho\beta}(z) \rangle \sim \hat{t}^{TFF}_{\mu\nu\sigma\alpha\rho'\beta'}(s) \frac{\mathcal{I}^F_{\rho\beta,\rho'\beta'}(z-y)}{(z-y)^{2(d-2)}} , \tag{3.3.15}$$

where $\hat{t}^{TFF}$ is given by (3.3.5). If we calculate the symmetric and antisymmetric parts of the Ward identity (2.3.18),

$$\begin{aligned} \int \mathrm{d}\Omega_{\hat{x}}\, \hat{x}_\mu \hat{x}_\nu \hat{t}^{TFF}_{\mu\nu\sigma\alpha\rho\beta}(\hat{x}) &= -(d-2) C_{F,s} \mathcal{E}^F_{\sigma\alpha,\rho\rho} , \\ \int \mathrm{d}\Omega_{\hat{x}}\, \hat{x}_\nu \hat{x}_{[\omega} \hat{t}^{TFF}_{\mu]\nu\sigma\alpha\rho\beta}(\hat{x}) &= -2 C_{F,a} \mathcal{E}^F_{\sigma\alpha,\lambda[\omega} \mathcal{E}^F_{\mu]\lambda,\rho\beta} , \end{aligned} \tag{3.3.16}$$

then we find

$$C_{F,s} = S_d \frac{1}{4d(d-2)} \Big[(d^2 - 3d - 2) A - (d+1) B - 2(d-1) D \Big] \tag{3.3.17}$$

$$C_{F,a} = -S_d \frac{1}{4d} \left[A + C \right] . \tag{3.3.18}$$

According to the Ward identity (2.3.18) we should expect

$$C_{F,s} = C_{F,a} = C_F , \tag{3.3.19}$$

where C_F is the scale of the two point function

$$\langle F_{\sigma\alpha}(x) F_{\rho\beta}(y) \rangle = C_F \frac{\mathcal{I}^F_{\sigma\alpha,\rho\beta}(x-y)}{(x-y)^{2(d-2)}} . \tag{3.3.20}$$

Thus in this case we must require

$$\begin{aligned} 0 = C_{F,s} - C_{F,a} &= \frac{S_d}{4d(d-2)} \left[d(d-4) A - (d+1) B - (d-2) C - 2(d-1) D \right] \\ &= \frac{1}{2(d-2)} C_V \end{aligned} \tag{3.3.21}$$

for C_V as in (3.3.14). Hence $C_V = 0$ is a necessary extra condition in this case. Note that we have

$$\langle V'_\sigma(x) V'_\rho(y) \rangle = \partial^x{}_\alpha \partial^y{}_\beta \langle F_{\sigma\alpha}(x) F_{\rho\beta}(y) \rangle = 0 . \tag{3.3.22}$$

This behaviour is expected since any conformal two point function involving two operators of different spin is zero (cf. the general formalism in (2.1.14)). Indeed we have

$$\langle V'_\sigma(x) F_{\rho\beta}(y)\rangle \;=\; \partial^x{}_\alpha \langle F_{\sigma\alpha}(x) F_{\rho\beta}(y)\rangle = 0 \tag{3.3.23}$$

for the two point function given by (3.3.20). This result shows that for three point functions involving the energy momentum tensor it is necessary to impose extra conditions in order to satisfy Ward identities beyond the conservation condition $\partial_\mu T_{\mu\nu} = 0$.

From the term $g^{\mu\nu}\langle T_{\mu\nu}\rangle = -\frac{1}{4}\kappa F^{\mu\nu}F_{\mu\nu}$ in the anomalous trace of the renormalised energy momentum tensor (2.4.2) in four dimensions originating from the background gauge field $\mathcal{A}_\mu$ with the field strength $F_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$ we expect an anomaly in the trace $\langle T_{\mu\mu}(x) V_\sigma(y) V_\rho(z)\rangle$ of the three point function even when $\mathcal{A}_\mu = 0$. We obtain this anomalous contribution to the Ward identity, which is a local term for x, y, z all coincident, from functionally differentiating $\langle T_{\mu\mu}(x)\rangle$ with respect to $\mathcal{A}_\sigma(y)$ and to $\mathcal{A}_\rho(z)$ conjugate to $V_\sigma(y)$ and to $V_\rho(z)$:

$$\begin{aligned}\langle T_{\mu\mu}(x) V_\sigma(y) V_\rho(z)\rangle &= \kappa\,(\partial_\sigma\partial_\rho - \delta_{\sigma\rho}\partial^2)\delta^4(y-x)\delta^4(z-x) \\ &= -2\,\kappa\,\mathcal{E}^F_{\sigma\alpha,\rho\beta}\partial_\alpha\partial_\beta\delta^4(y-x)\delta^4(z-x)\end{aligned} \tag{3.3.24}$$

in an equivalent notation. We expect to find a similar term in the trace of the three point function $\langle T_{\mu\nu}(x) V_\sigma(y) V_\rho(z)\rangle$ constructed using the general rules of section 2.1, (3.3.1) with (3.3.2) and (3.3.3). As we will show, the trace of the counterterm necessary for removing non-integrable singular terms in the three point function is exactly of the form (3.3.24) with an overall coefficient which is a function of the parameters A, B, C, D, E. By comparing the trace $\langle T_{\mu\mu}(x) V_\sigma(y) V_\rho(z)\rangle$ to (3.3.24) we are are thus able to express the anomaly coefficient κ as a linear function of the parameters A, B, C, D, E. In the following we therefore first explain in detail how to determine the non-integrable singular terms in the three point function defined in (3.3.1).

The three point functions constructed according to the formalism of section 2.1 may be ill-defined even as a distribution in the sense that the short distance singularity for all three points coincident, $x \to y, x \to z$, is non-integrable. In particular when $d = 4$ the three point functions may have a non-integrable singularity as a function on $\mathbb{R}^8$, as we have already seen in the case of the axial anomaly. They may also contain subdivergences when only two of the three points are coincident.

To analyse the structure of these singularities we first discuss some general results from the theory of generalised functions [23]. For a homogeneous function $f(X)$ of degree $-q$, where X^i are coordinates in $\mathbb{R}^p$, we expect a singularity for $q \sim p + 2n$, which reflects the divergence for $X \sim 0$, of the form

$$f(X) \sim \frac{1}{q-p-2n} c^{i_1 \cdots i_{2n}} \partial_{i_1} \cdots \partial_{i_{2n}} \delta^{(p)}(X) \,. \tag{3.3.25}$$

This formula may be defined rigorously as a distribution. As a function of q it has a pole when $q = p + 2n$. We assume that in this formula (3.3.25) there are no subdivergences arising from singularities in some proper subspace of $\mathbb{R}^p$. The coefficient c for $n = 0$ is calculated later in specific cases.

As an illustration we consider first the case of simple singular scalar functions on $\mathbb{R}^d$. For $q = 2\lambda$ and $p = d$ we have [23]

$$\frac{1}{(x-y)^{2\lambda}} \sim \frac{1}{d+2n-2\lambda} \frac{1}{2^{2n} n!} \frac{\Gamma(d/2)}{\Gamma(d/2+n)} S_d (\partial^2)^n \delta^d(x-y) \tag{3.3.26}$$

with S_d given by (3.1.8). This has poles at $\lambda = \frac{1}{2}d + n$, $n = 0, 1, \ldots$ in accord with what is expected from (3.3.25).

For the case of a function on $\mathbb{R}^{2d}$ with coordinates $x - y, x - z$, a detailed derivation in appendix A.2 which follows Jack and Osborn [24] gives the following useful formula:

$$\frac{1}{(x-y)^{2\lambda_1}(x-z)^{2\lambda_2}(y-z)^{2\lambda_3}} \sim \frac{1}{d - \sum_1^3 \lambda_i} \frac{\pi^d}{\Gamma(d/2)} \prod_1^3 \frac{\Gamma(d/2-\lambda_i)}{\Gamma(\lambda_i)} \, \delta^d(x-y)\delta^d(x-z) \,. \tag{3.3.27}$$

Here $q = 2\sum \lambda_i$ and $p = 2\,d$. This has additional singularities when $\lambda_i = \frac{1}{2}d + n$, $n = 0, 1, 2, \ldots$ due to subdivergences as exhibited in (3.3.26) when $x \to y$ or $x \to z$ or $y \to z$. When $\sum\limits_i \lambda_i = d + 2n$, $n = 1, 2, \ldots$ this expression may be extended to terms involving $2n$ derivatives acting on the delta distributions. However, for all the three point functions considered in this thesis we can restrict to the expression (3.3.27) after applying techniques of differential regularisation, which reduces the degree of the singularities, and contracting tensorial indices. Thus for the three point functions considered in the subsequent discussions the singularities will just involve the product $\delta^d(x-y)\delta^d(x-z)$ without derivatives as far as the dependence on the spatial variables is concerned.

We now discuss how the results (3.3.26) and (3.3.27) may be extended to correlation functions involving tensors. For this purpose let us consider the tensorial structure of forms involving spatial vectors like $(x-y)_\mu$ when $x \to y$. In this limit, terms depending on spatial vectors must be absent. Thus we have an obvious generalisation of (3.3.26) when $\lambda = \frac{1}{2}(d+\omega)$ in the easiest case:

$$\frac{(x-y)_\mu (x-y)_\nu}{(x-y)^{d+2+\omega}} \sim -\frac{S_d}{\omega}\delta^d(x-y)\frac{\delta_{\mu\nu}}{d} \ . \tag{3.3.28}$$

Similarly we have

$$\frac{x_\mu x_\nu x_\sigma x_\rho}{x^{d+4+\omega}} \sim -\frac{S_d}{\omega}\delta^d(x)\frac{\delta_{\mu\nu}\delta_{\sigma\rho}+\delta_{\mu\sigma}\delta_{\nu\rho}+\delta_{\mu\rho}\delta_{\nu\sigma}}{d(d+2)} \ . \tag{3.3.29}$$

This generalises to

$$\frac{x_{\mu_1}x_{\mu_2}\cdots x_{\mu_{2n}}}{x^{d+2n+\omega}} \sim -\frac{S_d}{\omega}\delta^d(x)\frac{1}{C_n}\,\delta_{(\mu_1\mu_2}\delta_{\mu_3\mu_4}\cdots\delta_{\mu_{2n-1}\mu_{2n})} \ , \tag{3.3.30}$$

where the brackets () denote total symmetrisation and C_n is the contraction in pairs of the tensorial expression in the numerator:

$$\begin{aligned} C_n &= \delta_{\mu_1\mu_2}\delta_{\mu_3\mu_4}\cdots\delta_{\mu_{2n-1}\mu_{2n}} \times \left(\delta_{(\mu_1\mu_2}\delta_{\mu_3\mu_4}\cdots\delta_{\mu_{2n-1}\mu_{2n})}\right) \\ &= 2^n n!\, d(d+2)\cdots(d+2n-2)/(2n)! \ . \end{aligned} \tag{3.3.31}$$

For a composite tensor of scaling dimension d which is a linear combination of the terms discussed above or similar ones, such that the different terms have a different number of spatial vectors, we obtain the short-distance limit by contracting the tensor to a scalar in a suitable way. The limit of two or three points coincident is then given by the expression for scalars (3.3.26) or (3.3.27) times a coordinate independent tensor of the same symmetry structure as the original one, divided by its contraction.

With these considerations it is now straightforward to determine the singular behaviour of $\langle T_{\mu\nu}(x)F_{\sigma\alpha}(y)F_{\rho\beta}(z)\rangle$ when $x \to y$. In this case we apply (3.3.28) and (3.3.29) directly to the short distance expression for $\langle T_{\mu\nu}(x)F_{\sigma\alpha}(y)F_{\rho\beta}(z)\rangle$ when $x \to y$ given in (3.3.15). Introducing a regularising factor $s^{-\omega}$ to remove the logarithmic divergence which would arise on integration on $\mathbb{R}^d$ since $\hat{t}^{TFF}_{\mu\nu\sigma\alpha\rho\beta}(s)$ is $\mathcal{O}(s^{-d})$, we obtain

$$s^{-\omega}\hat{t}^{TFF}_{\mu\nu\sigma\alpha\rho\beta}(s) \sim \frac{S_d}{\omega}\frac{K}{d(d+2)}\mathcal{E}^F_{\sigma\alpha,\lambda\varepsilon}\mathcal{E}^F_{\rho\beta,\lambda\eta}\mathcal{E}^T_{\varepsilon\eta,\mu\nu}\delta^d(s) \tag{3.3.32}$$

with K as in (3.3.8). Hence imposing the condition $K = 0$ is sufficient to ensure that all two-point subdivergences vanish. Note that assuming $K = 0$ we have shown that there are no subdivergences arising for $x \to y$ or $x \to z$. For $y \to z$, the three point function is $\mathcal{O}((y-z)^{d-4})$ which does not lead to non-integrable singularities for $d \to 4$.

We now consider the case of all three points coincident. As may be seen from the definition (3.3.2), $\langle T_{\mu\nu}(x)F_{\sigma\alpha}(y)F_{\rho\beta}(z)\rangle$ is a homogeneous function of degree $-(3d-4)$, i.e.

$$\langle T_{\mu\nu}(ax)F_{\sigma\alpha}(ay)F_{\rho\beta}(az)\rangle = \frac{1}{a^{3d-4}}\langle T_{\mu\nu}(x)F_{\sigma\alpha}(y)F_{\rho\beta}(z)\rangle \tag{3.3.33}$$

for a dimensionless scaling factor a. It follows from (3.3.25), taking $n = 0$, that as a function on $\mathbb{R}^{2d}$ with variables $y-x$ and $z-x$, $\langle T_{\mu\nu}(x)F_{\sigma\alpha}(y)F_{\rho\beta}(z)\rangle$ is singular in a distributional sense when $d \to 4$. In this case since $q = 3d-4$, $p = 2d$ and therefore $-(q-p) = 4-d \equiv \varepsilon$ in (3.3.25) the singularity manifests itself in a simple pole in ε whose residue involves just delta functions without derivatives. The tensorial structure of the residue of this pole is then uniquely determined by symmetry requirements and tracelessness in $\mu\nu$. Hence we may write in general for $d \approx 4$

$$\langle T_{\mu\nu}(x)F_{\sigma\alpha}(y)F_{\rho\beta}(z)\rangle \sim \frac{R}{\varepsilon}\,\mathcal{E}^F_{\sigma\alpha,\lambda\varepsilon}\mathcal{E}^F_{\rho\beta,\lambda\eta}\mathcal{E}^T_{\varepsilon\eta,\mu\nu}\delta^d(y-x)\delta^d(z-x)\,. \tag{3.3.34}$$

In order to determine the coefficient R it is convenient to contract the tensorial indices of the three point function,

$$\delta_{\mu\sigma}\delta_{\nu\rho}\delta_{\alpha\beta}\langle T_{\mu\nu}(x)F_{\sigma\alpha}(y)F_{\rho\beta}(z)\rangle\,, \tag{3.3.35}$$

in order to be able to apply (3.3.27) and thus to avoid complicated tensorial expressions. Inserting the general expression (3.3.3) into (3.3.35) yields a sum of five scalar three point functions with coefficients A, B, C, D, E respectively. These are listed in appendix A.3. Each of the terms is of the form of the left hand side of (3.3.27) with $\sum_i \lambda_i = \frac{1}{2}(3d-4)$ where in the individual terms $\lambda_i = \frac{1}{2}d + n_i$ for n_i an integer. When $n_i = 0, 1, 2\ldots$ for some i the formula (3.3.27) is singular. In the following, we regularise potential singularities due to subdivergences when $\lambda_i = \frac{1}{2}d + n_i$ by replacing $\lambda_i \to \lambda_i + \frac{1}{2}\omega_i$ where the limit $\omega_i \to 0$ can be taken at the end of the calculation. Hence we introduce an additional factor

$$\frac{1}{(x-y)^{\omega_1}(x-z)^{\omega_2}(y-z)^{\omega_3}} \tag{3.3.36}$$

into the three point function such that the subdivergences now correspond to poles in ω_i. The poles in ω_3 cancel when terms corresponding to each coefficient A, B, C, D, E are combined, so long as the residues given by (3.3.27) are evaluated at the pole $d = 4 - \sum \omega_i$. After taking the limit $\omega_3 \to 0$ the poles in ω_1 and ω_2 cancel as well if the condition (3.3.8) is applied which eliminates the subdivergences for either $y \to x$ or $z \to x$. We then obtain

$$\begin{aligned}
(x-y)^{-\omega_1}(x-z)^{-\omega_2}\langle T_{\mu\nu}(x)F_{\mu\alpha}(y)F_{\nu\alpha}(z)\rangle &\sim \frac{\pi^d}{2(\omega_1+\omega_2-4)(\omega_1-4)(\omega_2-4)} \\
\times\Big[\,(\omega_1+\omega_2-2)\,(\omega_1+\omega_2-3)\,(\omega_2+\omega_1)\,\Big({\omega_2}^3 - 14\,{\omega_2}^2 + 3\,\omega_1\,{\omega_2}^2 \\
+76\,\omega_2 - 28\,\omega_1\,\omega_2 + 3\,{\omega_1}^2\omega_2 - 14\,{\omega_1}^2 + 76\,\omega_1 + {\omega_1}^3 - 152\Big)\,A \\
-4(\omega_1+\omega_2-4)(2\,\omega_1+2\,\omega_2-10)\,(\omega_1+\omega_2-2)\,(\omega_1+\omega_2-3)\,B \\
-(\omega_1+\omega_2-4)(\omega_1+\omega_2+2)\,(\omega_1+\omega_2-8)\,(\omega_1+\omega_2-2)\,(\omega_1+\omega_2-3)\,C \\
-4(\omega_1+\omega_2-2)\,(\omega_1+\omega_2-3)\,(\omega_1+\omega_2-6)\,(\omega_1+\omega_2-8)\,D\Big]\,,
\end{aligned} \tag{3.3.37}$$

where we have used (3.3.8) to eliminate E and divided by the contraction of the invariant tensor,

$$\mathcal{E}^F_{\mu\alpha,\lambda\varepsilon}\mathcal{E}^F_{\nu\alpha,\lambda\eta}\mathcal{E}^T_{\varepsilon\eta,\mu\nu} = \tfrac{1}{8}(d-1)(d-2)(d+2)\,, \tag{3.3.38}$$

evaluated at the pole $d = 4 - \sum \omega_i$. Taking the limit $\omega_1, \omega_2 \to 0$ of (3.3.37) , which implies $d \to 4$, yields for R in (3.3.34)

$$R = -\frac{\pi^4}{3}\,(5B + 2C + 6D)\ . \tag{3.3.39}$$

With this result we may now write down a regularised expression for the three point function (3.3.2) which is integrable on $\mathbb{R}^8$ by subtracting the pole in ε according to the standard procedure of dimensional regularisation:

$$\begin{aligned}\langle T_{\mu\nu}(x)F_{\sigma\alpha}(y)F_{\rho\beta}(z)\rangle_r &= \langle T_{\mu\nu}(x)F_{\sigma\alpha}(y)F_{\rho\beta}(z)\rangle \\ &- \mu^{-\varepsilon}\frac{R}{\varepsilon}\left(\mathcal{E}^F_{\sigma\alpha,\lambda(\mu}\mathcal{E}^F_{\rho\beta,\lambda|\nu)} - \tfrac{1}{4}\delta_{\mu\nu}\mathcal{E}^F_{\sigma\alpha,\rho\beta}\right)\delta^d(y-x)\delta^d(z-x)\,,\end{aligned} \tag{3.3.40}$$

where the invariant tensor in (3.3.40) is equivalent to the one in (3.3.34) up to terms $\mathcal{O}(\varepsilon)$ which vanish if present in the residue of the pole in ε. The counterterm in (3.3.40) contains an arbitrary scale μ which reflects the short distance singularity present for $d = 4$. The regularised expression (3.3.40) has now a trace anomaly since, after taking the trace in d dimensions, we have

$$\delta_{\mu\nu}\ \left(\mathcal{E}^F_{\sigma\alpha,\lambda(\mu}\mathcal{E}^F_{\rho\beta,\lambda|\nu)} - \tfrac{1}{4}\delta_{\mu\nu}\mathcal{E}^F_{\sigma\alpha,\rho\beta}\right) = \tfrac{1}{4}\varepsilon\,\mathcal{E}^F_{\sigma\alpha,\rho\beta} \tag{3.3.41}$$

such that the ε factor cancels the pole. Reducing to four dimensions this then gives

$$\langle T_{\mu\mu}(x)F_{\sigma\alpha}(y)F_{\rho\beta}(z)\rangle_r = -\frac{R}{4}\mathcal{E}^F_{\sigma\alpha,\rho\beta}\delta^4(y-x)\delta^4(z-x) \tag{3.3.42}$$

or equivalently

$$\langle T_{\mu\mu}(x)V_\sigma(y)V_\rho(z)\rangle_r = -\frac{R}{4}\mathcal{E}^F_{\sigma\alpha,\rho\beta}\partial^y{}_\alpha\partial^z{}_\beta\delta^4(y-x)\delta^4(z-x)\,. \tag{3.3.43}$$

If we compare this to the anomalous Ward identity (3.3.24) obtained from a background gauge field we may identify

$$\kappa = \frac{1}{8}R = -\frac{\pi^4}{24}(5B + 2C + 6D)\,. \tag{3.3.44}$$

Thus we have related the anomaly coefficient κ to the parameters in the three point function constructed according to the general formalism of section 2.1. Since these parameters are in principle calculable for any explicit conformal field theory which may also be interacting, this is now also possible for κ.

We may also derive a relation between κ and C_V in the same way as the relation between β_a and C_T in section 2.4. In four dimensions the Ward identity (2.3.15) with the anomalous term (3.3.24) gives now in analogy to (2.4.15)

$$\langle L_v V_\sigma(y) V_\rho(z)\rangle + \langle V_\sigma(y) L_v V_\rho(z)\rangle = -2\kappa\, \mathcal{E}^F_{\sigma\alpha,\rho\beta}\partial_\alpha \Big(\sigma_v(y)\partial_\beta \delta^4(y-z)\Big)\,. \tag{3.3.45}$$

A regularised expression for the vector current two point function is

$$\langle V_\sigma(y) V_\rho(z)\rangle_r = -\frac{C_V}{(d-2)(d-1)}\mathcal{E}^F_{\sigma\alpha,\rho\beta}\partial_\alpha\partial_\beta \mathcal{R}\frac{1}{(y-z)^4} \tag{3.3.46}$$

after applying differential regularisation as in (2.4.18). Finally, calculating the conformal variation of this regularised two point function in analogy to the procedure for $T_{\mu\nu}$ in section 2.4 yields

$$\begin{aligned}\langle L_v V_\sigma(y) V_\rho(z)\rangle + \langle V_\sigma(y) L_v V_\rho(z)\rangle \;\; = \;\; & -\frac{C_V\pi^2}{3}\mathcal{E}^F_{\sigma\alpha,\rho\beta}\partial^x{}_\alpha \partial^y{}_\beta(\sigma_v(x)\delta^4(x-y)) \\ & -\frac{C_V}{2} b_\lambda \partial^x{}_\alpha\partial^y{}_\beta\partial^x{}_{[\lambda}\mathcal{E}^F_{\sigma\alpha],\rho\beta}\frac{1}{(x-y)^2}\,.\end{aligned} \tag{3.3.47}$$

The second term vanishes identically from the symmetries of $\mathcal{E}^F$, such that in four dimensions we have

$$\kappa = \frac{\pi^2}{6}C_V\,. \tag{3.3.48}$$

We may check that C_V as given by (3.3.14) agrees with (3.3.48) in four dimensions.

The additional freedom in the definition of the energy momentum tensor $T'_{\mu\nu} = \partial_\kappa\partial_\lambda C_{\mu\kappa\lambda\nu}$ allows the construction of an independent form in $\langle T_{\mu\mu}(x)F_{\sigma\alpha}(y)F_{\rho\beta}(z)\rangle$ which is a trivial solution to the Ward identity without anomaly term. This form may thus be called anomaly free. It is given by

$$\langle T'_{\mu\nu}(x)F_{\sigma\alpha}(y)F_{\rho\beta}(z)\rangle = \partial^x{}_\kappa\partial^x{}_\lambda\langle C_{\mu\kappa\lambda\nu}(x)F_{\sigma\alpha}(y)F_{\rho\beta}(z)\rangle\,. \tag{3.3.49}$$

The three point function involving the Weyl tensor may be found according to the general rules for obtaining conformally invariant three point functions set out in section 2.1.

In general the resulting expression is less singular for coincident points since the scale dimension of the Weyl tensor is $d-2$ while it is d for the energy momentum tensor. Thus according to (3.3.27) there will be no pole for $d \to 4$ in this three point function. A particular contribution to (3.3.49) is

$$
\begin{aligned}
&\langle C_{\mu\kappa\lambda\nu}(x) F_{\sigma\alpha}(y) F_{\rho\beta}(z)\rangle \\
&\qquad = \frac{\mathcal{I}^F{}_{\sigma\alpha,\sigma'\alpha'}(y-x)\mathcal{I}^F{}_{\rho\beta,\rho'\beta'}(z-x)}{(y-x)^{2(d-2)}(z-x)^{2(d-2)}}\,\mathcal{E}^C{}_{\sigma'\alpha'\rho'\beta',\mu\kappa\lambda\nu}\frac{S}{X_{23}{}^{d-2}}\,,
\end{aligned}
\tag{3.3.50}
$$

where $\mathcal{E}^C$ is the projection operator onto tensors with Weyl symmetry. The expression for the three point function obtained from (3.3.49) with (3.3.50) can be related to the previous forms given by (3.3.2) with (3.3.3) by considering

$$
\hat{t}^{FFT}_{\mu\nu,\sigma\alpha,\rho\beta}(X)_C = \partial_\kappa\partial_\lambda\left(\mathcal{I}^F{}_{\rho\beta,\rho'\beta'}(X)\,\mathcal{E}^C{}_{\sigma\alpha\rho'\beta',\mu\kappa\lambda\nu}\frac{S}{X^{d-2}}\right). \tag{3.3.51}
$$

Comparing the result of this calculation with (3.3.4) we may then find the coefficients A, B, C, D, E in terms of S, or equivalently

$$
J = -I = \frac{d^2(d-3)}{2(d-1)}S\,,\quad K = E = 0\,,\quad D = -\frac{d^2}{(d-1)(d-2)}S\,. \tag{3.3.52}
$$

The result of using (3.3.49) with (3.3.50) in (3.3.1) gives an expression for the three point function $\langle T_{\mu\nu}(z)V_\sigma(y)V_\rho(z)\rangle$ for which $C_T = 0$ since $I = -J$ and thus also $\kappa = 0$. This confirms that this form is anomaly free.

The results of this section show how the the non-integrable singular terms of conformally covariant three point functions may be calculated using differential and dimensional regularisation. Thus the parameters of the three point function may be related to the coefficient of the anomalous term in the Ward identity as derived from a classical background field. In the specific example considered here this anomaly coefficient is related to the scale of the two point function. Furthermore we have indicated how to construct the anomaly free form for which the anomaly coefficient vanishes.

3.4 Energy Momentum Tensor Three Point Function

In this section the techniques developed and tested in the previous sections are applied to the energy momentum tensor three point function, which is essential to the main aim of this work of calculating the coefficients of the conformal anomaly. Due to the larger number of indices, the calculations are more involved in this case.

Applied to the three point function for three energy momentum tensors, the general expression (2.1.15) gives:

$$\begin{aligned}&\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle \\ &\quad = \frac{1}{(x-z)^{2d}(y-z)^{2d}}\mathcal{I}^T_{\mu\nu,\mu'\nu'}(x-z)\mathcal{I}^T_{\sigma\rho,\sigma'\rho'}(y-z)t_{\mu'\nu'\sigma'\rho'\alpha\beta}(X_{12})\ . \qquad (3.4.1)\end{aligned}$$

$\mathcal{I}^T_{\mu\nu,\sigma\rho}(x)$ is the inversion on the space of traceless symmetric tensors defined in (2.2.16). $t_{\mu\nu\sigma\rho\alpha\beta}(X)$ is homogeneous of degree $-d$ in X, symmetric and traceless in the three pairs of indices $\mu\nu$, $\sigma\rho$ and $\alpha\beta$, and from (2.1.16) must satisfy

$$t_{\mu\nu\sigma\rho\alpha\beta}(X) = t_{\sigma\rho\mu\nu\alpha\beta}(X)\,, \qquad (3.4.2)$$

$$\mathcal{I}^T_{\mu\nu,\mu'\nu'}(X)t_{\mu'\nu'\sigma\rho\alpha\beta}(X) = t_{\alpha\beta\mu\nu\sigma\rho}(X)\,. \qquad (3.4.3)$$

The conservation equation $\partial^x{}_\mu\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle = 0$ requires

$$\partial_\mu t_{\mu\nu\sigma\rho\alpha\beta}(X) = 0\ . \qquad (3.4.4)$$

Osborn and Petkou have shown in [6] that the three point function (3.4.1) has three independent forms in general. In three dimensions, the number of independent forms is reduced to two and in two dimensions to one. Here we rederive this result by constructing an expression for $\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle$ in which the Bose symmetry of the three point function is manifest such that the symmetry conditions (3.4.2) and (3.4.3) are automatically satisfied. It is convenient to write

$$\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle = \frac{S_{\mu\nu\sigma\rho\alpha\beta}(x,y,z)}{(x-z)^d(y-z)^d(x-y)^d}\ . \qquad (3.4.5)$$

Using the results of section 2.1, especially the existence of the covariant vectors $X_{12\,\mu}$, $X_{23\,\mu}$, $X_{31\,\mu}$ given by (2.1.8) and (2.1.10), as well as the relations

$$I_{\mu\alpha}(x-z)X_{12\,\alpha} = -\frac{(x-y)^2}{(z-y)^2}X_{23\,\mu}\,,$$
$$I_{\mu\alpha}(x-z)I_{\alpha\nu}(z-y) = I_{\mu\nu}(x-y) + 2(x-y)^2 X_{23\,\mu}X_{31\,\nu}\,, \tag{3.4.6}$$

we may find in general five possible completely symmetric expressions with the required properties under conformal transformations so that in general

$$\begin{aligned}
S_{\mu\nu\sigma\rho\alpha\beta}(x,y,z) &= \mathcal{E}^T_{\mu\nu,\mu'\nu'}\mathcal{E}^T_{\sigma\rho,\sigma'\rho'}\mathcal{E}^T_{\alpha\beta,\alpha'\beta'}\Big[\mathcal{A}\; I_{\nu'\sigma'}(x-y)I_{\rho'\alpha'}(y-z)I_{\beta'\mu'}(z-x) \\
&\qquad + \mathcal{B}\; I_{\mu'\sigma'}(x-y)I_{\nu'\alpha'}(x-z)X_{31\,\rho'}X_{12\,\beta'}(y-z)^2 + \text{cycl. perm.}\Big] \\
&+ \mathcal{C}\; \mathcal{I}^T_{\mu\nu,\sigma\rho}(x-y)\left(\frac{X_{12\,\alpha}X_{12\,\beta}}{X_{12}^2} - \frac{1}{d}\delta_{\alpha\beta}\right) + \text{cyclic permutations} \\
&+ \mathcal{D}\; \mathcal{E}^T_{\mu\nu,\mu'\nu'}\mathcal{E}^T_{\sigma\rho,\sigma'\rho'}X_{23\,\mu'}X_{31\,\sigma}(x-y)^2 I_{\nu'\rho'}(x-y)\left(\frac{X_{12\,\alpha}X_{12\,\beta}}{X_{12}^2} - \frac{1}{d}\delta_{\alpha\beta}\right) \\
&\qquad + \text{cyclic permutations} \\
&+ \mathcal{E}\left(\frac{X_{23\,\mu}X_{23\,\nu}}{X_{23}^2} - \frac{1}{d}\delta_{\mu\nu}\right)\left(\frac{X_{31\,\sigma}X_{31\,\rho}}{X_{31}^2} - \frac{1}{d}\delta_{\sigma\rho}\right)\left(\frac{X_{12\,\alpha}X_{12\,\beta}}{X_{12}^2} - \frac{1}{d}\delta_{\alpha\beta}\right).
\end{aligned} \tag{3.4.7}$$

This agrees with the general form for conformal three point functions (2.1.15) since (3.4.5) with (3.4.7) can be rewritten identically in the form (3.4.1) with

$$\begin{aligned}
t_{\mu\nu\sigma\rho\alpha\beta}(X) &= \mathcal{A}\; \mathcal{E}^T_{\mu\nu,\varepsilon\eta}\mathcal{E}^T_{\sigma\rho,\eta\lambda}\mathcal{E}^T_{\alpha\beta,\lambda\varepsilon}\frac{1}{X^d} \\
&+ (\mathcal{B}-2\mathcal{A})\; \mathcal{E}^T_{\alpha\beta,\varepsilon\eta}\mathcal{E}^T_{\sigma\rho,\eta\kappa}\mathcal{E}^T_{\mu\nu,\lambda\varepsilon}\frac{X_\kappa X_\lambda}{X^{d+2}} \\
&- \mathcal{B}\left(\mathcal{E}^T_{\mu\nu,\varepsilon\eta}\mathcal{E}^T_{\sigma\rho,\eta\kappa}\mathcal{E}^T_{\alpha\beta,\lambda\varepsilon} + (\mu\nu)\leftrightarrow(\sigma\rho)\right)\frac{X_\kappa X_\lambda}{X^{d+2}} \\
&+ \mathcal{C}\left(\mathcal{E}^T_{\mu\nu,\sigma\rho}\left(\frac{X_\alpha X_\beta}{X^2} - \frac{1}{d}\delta_{\alpha\beta}\right)\right. \\
&\qquad \left. + \mathcal{E}^T_{\sigma\rho,\alpha\beta}\left(\frac{X_\mu X_\nu}{X^2} - \frac{1}{d}\delta_{\mu\nu}\right) + \mathcal{E}^T_{\alpha\beta,\mu\nu}\left(\frac{X_\sigma X_\rho}{X^2} - \frac{1}{d}\delta_{\sigma\rho}\right)\right)\frac{1}{X^d} \\
&+ (\mathcal{D}-4\mathcal{C})\; \mathcal{E}^T_{\mu\nu,\varepsilon\kappa}\mathcal{E}^T_{\sigma\rho,\varepsilon\lambda}\left(\frac{X_\alpha X_\beta}{X^2} - \frac{1}{d}\delta_{\alpha\beta}\right)\frac{X_\kappa X_\lambda}{X^{d+2}} \\
&- (\mathcal{D}-2\mathcal{B})\left(\mathcal{E}^T_{\sigma\rho,\eta\kappa}\mathcal{E}^T_{\alpha\beta,\varepsilon\lambda}\left(\frac{X_\mu X_\nu}{X^2} - \frac{1}{d}\delta_{\mu\nu}\right) + (\mu\nu)\leftrightarrow(\sigma\rho)\right)\frac{X_\kappa X_\lambda}{X^{d+2}} \\
&+ (\mathcal{E}+4\mathcal{C}-2\mathcal{D}) \\
&\quad \times\left(\frac{X_\mu X_\nu}{X^2} - \frac{1}{d}\delta_{\mu\nu}\right)\left(\frac{X_\sigma X_\rho}{X^2} - \frac{1}{d}\delta_{\sigma\rho}\right)\left(\frac{X_\alpha X_\beta}{X^2} - \frac{1}{d}\delta_{\alpha\beta}\right)\frac{1}{X^d}\,,
\end{aligned} \tag{3.4.8}$$

where $X^d \equiv (X^2)^{\frac{1}{2}d}$. Since (3.4.1) with (3.4.8) is totally symmetric in the three energy momentum tensors by definition, it only remains to impose conservation as in (3.4.4). The calculation of $\partial_\mu t_{\mu\nu\sigma\rho\alpha\beta}(X)$ is performed using the algebraic computing program FORM [22], and $\partial_\mu t_{\mu\nu\sigma\rho\alpha\beta}(X) = 0$ implies two relations between the five coefficients $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$:

$$\begin{aligned} R_1 &\equiv (d-2)(d+2)\mathcal{A} + (d+2)\mathcal{B} - 4d\,\mathcal{C} - 2\mathcal{D} = 0\,, \\ R_2 &\equiv (d-2)(d+4)\mathcal{B} - 2d(d+2)\mathcal{C} + 8\mathcal{D} - 4\mathcal{E} = 0\,. \end{aligned} \tag{3.4.9}$$

Thus there remain three independent coefficients in general which may be taken to be $\mathcal{A}, \mathcal{B}, \mathcal{C}$ by using (3.4.9) to eliminate $\mathcal{D}, \mathcal{E}$.

Using expression (3.4.8) it is straightforward to verify the Ward identities

$$\begin{aligned} \int \mathrm{d}\Omega_{\hat{x}}\, \hat{x}_\mu \hat{x}_\nu t_{\mu\nu\sigma\rho\alpha\beta}(\hat{x}) &= -d\, C_T\, \mathcal{E}^T_{\sigma\rho,\alpha\beta}\,, \\ \int \mathrm{d}\Omega_{\hat{x}}\, \hat{x}_\mu \hat{x}_{[\omega} t_{\mu|\nu]\sigma\rho\alpha\beta}(\hat{x}) &= -2\, C_T\, \mathcal{E}^T_{\sigma\rho,\lambda[\omega}\mathcal{E}^T_{\nu]\lambda,\alpha\beta}\,, \end{aligned} \tag{3.4.10}$$

where, with the conditions (3.4.9), we find

$$C_T = \frac{S_d}{d(d+2)}\Big(\tfrac{1}{2}(d+2)(d-1)\mathcal{A} - \mathcal{B} - 2(d+1)\mathcal{C}\Big)\,. \tag{3.4.11}$$

The results (3.4.10) are in exact agreement with the forms imposed by the Ward identities (2.3.17) and (2.3.18) with C_T the coefficient of the energy momentum tensor two point function as shown in (2.2.14). Using the relation between C_T and the anomaly coefficient β_a derived earlier (2.4.22), we obtain in four dimensions

$$\beta_a = \frac{\pi^4}{64 \times 120}\,(9\mathcal{A} - \mathcal{B} - 10\mathcal{C})\ . \tag{3.4.12}$$

This result holds for any conformally invariant theory which may also be interacting.

In subsequent calculations we will frequently use the representation of the energy momentum tensor in the collinear frame defined in section 2.1. In this frame the energy momentum tensor three point function is of the form [6]

$$\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle = \frac{1}{(\hat{x}-\hat{y})^d(\hat{x}-\hat{z})^d(\hat{y}-\hat{z})^d}\,\mathcal{A}^{TTT}_{\mu\nu\sigma\rho\alpha\beta}\,, \tag{3.4.13}$$

$$\mathcal{A}^{TTT}_{\mu\nu\sigma\rho\alpha\beta} = \mathcal{A}^{TTT}_{\sigma\rho\mu\nu\alpha\beta} = \mathcal{A}^{TTT}_{\alpha\beta\mu\nu\sigma\rho}\,, \tag{3.4.14}$$

with $\mathcal{A}^{TTT}_{\mu\nu\sigma\rho\alpha\beta}$ symmetric and traceless on each pair of indices. With these constraints, the non-zero components of $\mathcal{A}^{TTT}_{\mu\nu\sigma\rho\alpha\beta}$ are

$$\begin{aligned}
\mathcal{A}^{TTT}_{111111} &= \alpha\,,\ \mathcal{A}^{TTT}_{ij1111} = \beta\,\delta_{ij}\,,\ \mathcal{A}^{TTT}_{i1k111} = \gamma\,\delta_{ik}\,,\\
\mathcal{A}^{TTT}_{ijkl11} &= \delta\ \delta_{ij}\delta_{kl} + \epsilon\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,,\ \mathcal{A}^{TTT}_{ijk1m1} = \rho\,\delta_{ij}\delta_{km} + \tau(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})\,,\\
\mathcal{A}^{TTT}_{ijklmn} &= r\,\delta_{ij}\delta_{kl}\delta_{mn}\\
&+ s\ (\delta_{ij}(\delta_{mk}\delta_{ln} + \delta_{kn}\delta_{lm}) + \delta_{kl}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + \delta_{mn}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}))\\
&+ t\,(\delta_{ik}\delta_{jm}\delta_{ln} + \delta_{jk}\delta_{im}\delta_{ln} + \delta_{il}\delta_{jm}\delta_{kn} + \delta_{jl}\delta_{im}\delta_{kn}\\
&\quad +\delta_{ik}\delta_{jn}\delta_{lm} + \delta_{jk}\delta_{in}\delta_{lm} + \delta_{il}\delta_{jn}\delta_{km} + \delta_{jl}\delta_{in}\delta_{km})\,,
\end{aligned} \tag{3.4.15}$$

where the tracelessness gives five and the conservation of $T_{\mu\nu}$ two linear relations between the ten coefficients $\alpha, \beta, \gamma, \delta, \epsilon, \rho, \tau, r, s$ and t, such that there are three independent parameters. Tracelessness implies

$$\begin{aligned}
&\alpha + (d-1)\beta = 0, \quad \beta + (d-1)\delta + 2\epsilon = 0, \quad \gamma + (d-1)\rho + 2\tau = 0,\\
&\delta + (d-1)r + 4s = 0, \quad \epsilon + (d-1)s + 4t = 0,
\end{aligned} \tag{3.4.16}$$

whereas the conservation equation leads to

$$\begin{aligned}
2\tau + dr + (d+4)s + 2t &= 0\,,\\
2\rho - dr + (d-2)s + 2(d+4)t &= 0\,.
\end{aligned} \tag{3.4.17}$$

r, s, t are chosen as free parameters for convenience here. If the form (3.4.1) with (3.4.8) is expressed in the collinear frame, we find the following relations between the coefficients $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and r, s, t:

$$\mathcal{A} = 8t, \ \ \mathcal{B} = -4(dr + (d+4)s), \ \mathcal{C} = -2(ds + 4t). \tag{3.4.18}$$

The scale of the two point function C_T as given by (3.4.11) may then alternatively be written in terms of the parameters in the collinear frame as

$$C_T = \frac{4S_d}{d(d+2)}\left(dr + (d^2+2d+4)s + (d^2+5d+2)t\right), \tag{3.4.19}$$

with S_d as in (3.1.8).

In the collinear frame, it may be seen from the basis (3.4.15) together with the tracelessness and conservation conditions (3.4.16) and (3.4.17) that the number of independent forms in the energy momentum tensor three point function is reduced when $d = 2$ or $d = 3$. In two dimensions there is only one transverse direction, and the three point function depends only on the combinations $\delta + 2\,\varepsilon$, $\rho + 2\,\tau$ and $r + 6\,s + 8\,t$. Therefore there is only one independent form in two dimensions. In three dimensions, $\mathcal{A}^{TTT}_{ijklmn}$ depends only on $r - 4\,t$ and $s + 2\,t$ rather than on r, s, t separately such that the number of independent forms is two in this case.

The coefficients $\mathcal{A}, \mathcal{B}, \mathcal{C}$ or alternatively r, s, t are in principle calculable for any conformal field theory in general dimensions. The simplest explicit example are free field theories, for which the energy momentum tensor is given by

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{4}\frac{1}{d-1}\left((d-2)\partial_\mu\partial_\nu - \delta_{\mu\nu}\partial^2\right)\varphi^2 \tag{3.4.20}$$

in the scalar case or

$$T_{\mu\nu} = \tfrac{1}{2}\bar{\psi}(\gamma_\mu \overleftrightarrow{\partial}_\nu + \gamma_\nu \overleftrightarrow{\partial}_\mu)\psi\,, \quad \overleftrightarrow{\partial}_\mu = \tfrac{1}{2}(\partial_\mu - \overleftarrow{\partial}_\mu) \tag{3.4.21}$$

in the fermion case. $(\gamma_\mu)^\dagger = \gamma_\mu$ are the Euclidean gamma matrices with $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. The two point functions for two scalar or two fermion fields are

$$\langle\varphi(x)\varphi(0)\rangle = \frac{1}{(d-2)S_d}\frac{1}{x^{d-2}}\,, \quad \langle\psi(x)\bar{\psi}(0)\rangle = \frac{1}{S_d}\frac{\gamma\cdot x}{x^d}\,, \tag{3.4.22}$$

with S_d as in (3.1.8). With these results Osborn and Petkou determined the energy momentum tensor two point function which is compatible with the conformally covariant form (2.2.14) [6]. In the scalar case, if φ has n_φ components, then

$$C_T = n_\varphi \frac{d}{d-1}\frac{1}{S_d^2}\,, \tag{3.4.23}$$

while in the fermion case with n_ψ Dirac fields

$$C_T = \tilde{n}_\psi 2d \frac{1}{S_d^2} \tag{3.4.24}$$

where $\tilde{n}_\psi = \frac{1}{4}\mathrm{tr}_D(1) n_\psi$ with tr_D the Dirac trace. Determining the parameters of the three point function is considerably easier in the collinear frame, where a similar calculation yields

$$r = n_\varphi \frac{1}{8}\frac{d^3+28d-16}{(d-1)^3}\frac{1}{S_d^3}\,, \quad s = n_\varphi \frac{1}{8}\frac{d(d^2-8d+4)}{(d-1)^3}\frac{1}{S_d^3}\,, \quad t = n_\varphi \frac{1}{8}\frac{d^3}{(d-1)^3}\frac{1}{S_d^3} \tag{3.4.25}$$

for scalar fields and

$$r = -2\tilde{n}_\psi \frac{1}{S_d^3}\,, \quad s = \frac{d}{2}\tilde{n}_\psi \frac{1}{S_d^3}\,, \quad t = 0 \tag{3.4.26}$$

for fermion fields. The corresponding values for $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are listed in the appendix A.4. In four dimensions there is an additional conformal field theory described in terms of free spin one abelian vector fields, for which the energy momentum tensor is

$$T_{\mu\nu} = F_{\mu\lambda}F_{\nu\lambda} - \tfrac{1}{4}\delta_{\mu\nu}F_{\alpha\beta}F_{\alpha\beta}\,, \tag{3.4.27}$$

neglecting terms dependent on gauge-fixing and ghost fields which do not contribute to physical correlation functions involving gauge invariant operators. The two point function for the field strength tensor is

$$\langle F_{\mu\nu}(x)F_{\sigma\rho}(0)\rangle = \frac{2}{S_4}\frac{1}{x^4}\mathcal{I}^F_{\mu\sigma,\nu\rho}(x) \tag{3.4.28}$$

with $\mathcal{I}^F_{\mu\sigma,\nu\rho}(x)$ the inversion as in (3.1.12). For the energy momentum tensor two point function a comparison with the general expression (2.2.14) gives

$$C_T = 16\frac{1}{S_4^2} \tag{3.4.29}$$

and for the three point function

$$r = -48\frac{1}{S_4^3}\,, \quad s = 32\frac{1}{S_4^3}\,, \quad t = -16\frac{1}{S_4^3}\,. \tag{3.4.30}$$

Using these results we obtain for the parameters $\mathcal{A}, \mathcal{B}, \mathcal{C}$

$$\begin{aligned}
\mathcal{A} &= \frac{1}{\pi^6}\left(\tfrac{8}{27}n_\varphi - 16 n_V\right)\,, \\
\mathcal{B} &= -\frac{1}{\pi^6}\left(\tfrac{16}{27}n_\varphi + 4n_\psi + 32 n_V\right)\,, \\
\mathcal{C} &= -\frac{1}{\pi^6}\left(\tfrac{2}{27}n_\varphi + 2n_\psi + 16 n_V\right)
\end{aligned} \tag{3.4.31}$$

in four dimensions, where n_V is the number of vector fields. Substituting into the expression for C_T (3.4.11) gives standard results for C_T in free field theories.

Furthermore, we may calculate the anomaly coefficients in terms of r, s, t using the result for free field theories in four dimensions (2.4.8)

$$\begin{aligned} \beta_a &= -\frac{\pi^4}{64 \times 15}(2r + 14s + 19t) \\ \beta_b &= \frac{\pi^4}{64 \times 90}(4r + 48s + 53t) . \end{aligned} \tag{3.4.32}$$

Equivalently, we may find the anomaly coefficients for free field theories in terms of the parameters $\mathcal{A}, \mathcal{B}, \mathcal{C}$ using the relations (3.4.18):

$$\begin{aligned} \beta_a &= \frac{\pi^4}{64 \times 120}\left(9\mathcal{A} - \mathcal{B} - 10\mathcal{C}\right) , \\ \beta_b &= \frac{\pi^4}{512 \times 90}(13\mathcal{A} - 2\mathcal{B} - 40\mathcal{C}) . \end{aligned} \tag{3.4.33}$$

The expression for β_a is equivalent to the one found for general theories (3.4.12), which provides a consistency check for our calculations. Moreover (3.4.19) and (3.4.33a) agree with the proportionality between β_a and C_T (2.4.22) in four dimensions. Chapter 4 will be devoted to proving that the expression for β_b in (3.4.33) also holds for general theories.

3.5 Application of $T'_{\mu\nu} = \partial_\kappa \partial_\lambda C_{\mu\kappa\lambda\nu}$ to the Energy Momentum Tensor Three Point Function

In this section we use differential regularisation to calculate the form in the energy momentum tensor three point function for which one of the energy momentum tensors is trivially conserved. We show that the anomaly coefficient β_a vanishes for this form. Moreover some of the results will be needed for subsequent calculations in chapter 4.

The form in the energy momentum tensor three point function in which one of the tensors is replaced by $T'_{\mu\nu} = \partial_\kappa \partial_\lambda C_{\mu\kappa\lambda\nu}$ is given by

$$\langle T'_{\mu\nu}(x) T_{\sigma\rho}(y) T_{\alpha\beta}(z) \rangle = \partial_\kappa \partial_\lambda \langle C_{\mu\kappa\lambda\nu}(x) T_{\sigma\rho}(y) T_{\alpha\beta}(z) \rangle , \tag{3.5.1}$$

where

$$\partial^x{}_\kappa\partial^x{}_\lambda\langle C_{\mu\kappa\lambda\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle$$
$$= \partial^x{}_\kappa\partial^x{}_\lambda\left(\frac{1}{(x-z)^{2(d-2)}(y-z)^{2d}}\,\mathcal{I}^C_{\mu\kappa\lambda\nu,\mu'\kappa'\lambda'\nu'}(x-z)\mathcal{I}^T_{\sigma\rho,\sigma'\rho'}(y-z)t^{CTT}_{\mu'\kappa'\lambda'\nu',\sigma'\rho'\alpha\beta}(X_{12})\right)$$
$$= \frac{1}{(x-z)^{2d}(y-z)^{2d}}\,\mathcal{I}^T_{\mu\nu,\mu'\nu'}(x-z)\mathcal{I}^T_{\sigma\rho,\sigma'\rho'}(y-z)\partial_\kappa\partial_\lambda t^{CTT}_{\mu'\kappa\lambda\nu',\sigma'\rho'\alpha\beta}(X_{12})\,. \tag{3.5.2}$$

$\mathcal{I}^C_{\alpha\beta\gamma\delta,\mu\sigma\rho\nu}(x)$ is the inversion on the space of Weyl tensors,

$$\mathcal{I}^C_{\mu\sigma\rho\nu,\alpha\beta\gamma\delta}(x) = \mathcal{E}^C_{\mu\sigma\rho\nu,\alpha'\beta'\gamma'\delta'}\,I_{\alpha'\alpha}(x)I_{\beta'\beta}(x)I_{\gamma'\gamma}(x)I_{\delta\delta'}(x)\,. \tag{3.5.3}$$

In (3.5.2) we have used

$$\partial_\kappa\left(\frac{1}{x^{2(d-2)}}\mathcal{I}^C_{\mu\kappa\lambda\nu,\mu'\kappa'\lambda'\nu'}(x)\right) = 0 \quad\text{and}\quad \frac{\partial}{\partial x_\mu}f(X_{12}) = \frac{I_{\mu\sigma}(x-z)}{(x-z)^2}\frac{\partial}{\partial X_{12\,\sigma}}f(X_{12})\,. \tag{3.5.4}$$

$t^{CTT}_{\mu\kappa\lambda\nu,\sigma\rho\alpha\beta}(X)$ is $\mathcal{O}(X^{-d+2})$. It has Weyl symmetry in the first four indices $\mu\kappa\lambda\nu$ and is symmetric and traceless in each of the two pairs of indices $(\sigma\rho)$ and $(\alpha\beta)$. We may now identify

$$t^{T'TT}_{\mu\nu\sigma\rho\alpha\beta}(X) = \partial_\kappa\partial_\lambda t^{CTT}_{\mu\kappa\lambda\nu,\sigma\rho\alpha\beta}(X)\,. \tag{3.5.5}$$

The components of $\mathcal{A}^{TTT}$ in the collinear frame can be calculated from

$$\begin{aligned}\mathcal{A}^{TTT}_{\mu\nu\sigma\rho\alpha\beta} &= I_{\mu\mu'}I_{\nu\nu'}I_{\sigma\sigma'}I_{\rho\rho'}\,\hat{t}^{TTT}_{\mu'\nu'\sigma'\rho'\alpha\beta}\\ \mathcal{A}^{CTT}_{\mu\kappa\lambda\nu,\sigma\rho\alpha\beta} &= I_{\mu\mu'}I_{\kappa\kappa'}I_{\lambda\lambda'}I_{\nu\nu'}I_{\sigma\sigma'}I_{\rho\rho'}\,\hat{t}^{CTT}_{\mu'\kappa'\lambda'\nu',\sigma'\rho'\alpha\beta}\,.\end{aligned} \tag{3.5.6}$$

$\mathcal{A}^{TTT\,1}_{\mu\nu\sigma\rho\alpha\beta}$ satisfies the symmetry conditions (3.4.14) and for $\mathcal{A}^{CTT}$ we have

$$\mathcal{A}^{CTT}_{\mu\kappa\lambda\nu,\sigma\rho\alpha\beta} = \mathcal{A}^{CTT}_{\mu\kappa\lambda\nu,\alpha\beta\sigma\rho} \tag{3.5.7}$$

in addition to Weyl symmetry in the first for indices and symmetry and tracelessness in each of the last two pairs of indices.

Our aim is now to calculate the number of independent quantities in $\mathcal{A}^{CTT}$ and to relate them to the three independent forms in $\mathcal{A}^{TTT}$ according to (3.5.5) and (3.5.6). From the requirement of rotational invariance, there are eight non-zero components of $\mathcal{A}^{CTT}_{\mu\kappa\lambda\nu,\sigma\rho\alpha\beta}$ with the required index symmetries:

$$
\begin{aligned}
\mathcal{A}^1_{ijkl,mnpq} &= H\ (\delta_{mp}\,\mathcal{A}_{ijklnq} + \delta_{mq}\,\mathcal{A}_{ijklnp} \\
&\qquad\qquad + \delta_{np}\,\mathcal{A}_{ijklmq} + \delta_{nq}\,\mathcal{A}_{ijklmp}) \\
\mathcal{A}^2_{ijkl,mnpq} &= I\ \delta_{mn}\delta_{pq}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \\
\mathcal{A}^3_{ijkl,mnpq} &= J\ (\delta_{pm}\delta_{nq} + \delta_{np}\delta_{mq})\,(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \\
\mathcal{A}^4_{ijkl,mnpq} &= K\ (\delta_{mn}\mathcal{A}_{ijklpq} + \delta_{pq}\mathcal{A}_{ijklmn})
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^5_{ijkl,mnpq} &= L\ \Big(\delta_{im}\delta_{jp}\delta_{kq}\delta_{ln} - \delta_{jm}\delta_{ip}\delta_{kq}\delta_{ln} - \delta_{im}\delta_{jp}\delta_{lq}\delta_{kn} + \delta_{jm}\delta_{ip}\delta_{lq}\delta_{kn} \\
&\qquad + [12 \text{ terms to symmetrise in } (mn\Big) \text{ and in } (pq)]\ ) \\
\mathcal{A}^6_{ijkl,1n1q} &= M\ \delta_{nq}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \\
\mathcal{A}^7_{ijkl,1n1q} &= N\ \mathcal{A}_{ijklnq} \\
\mathcal{A}^8_{1jkl,1npq} &= P\ (\ (d-2)(-\delta_{pl}\delta_{qj}\delta_{nk} - \delta_{ql}\delta_{pj}\delta_{nk} + \delta_{pk}\delta_{qj}\delta_{nl} + \delta_{qk}\delta_{pj}\delta_{nl}) \\
&\qquad - (-\delta_{jk}\delta_{pn}\delta_{ql} - \delta_{jk}\delta_{pl}\delta_{qn} + \delta_{jl}\delta_{pn}\delta_{qk} + \delta_{jl}\delta_{pk}\delta_{qn}) \\
&\qquad - 2\,\delta_{pq}\,(\delta_{jk}\delta_{nl} - \delta_{jl}\delta_{nk})\ )\,,
\end{aligned} \tag{3.5.8}
$$

where $\mathcal{A}_{ijklmn} = \delta_{im}\delta_{kn}\delta_{jl} - \delta_{im}\delta_{ln}\delta_{kj} - \delta_{jm}\delta_{kn}\delta_{il} + \delta_{jm}\delta_{ln}\delta_{ik}\ + (m \leftrightarrow n)\,.$

Using the algebraic computing programming language FORM, it is easy to check that the expressions (3.5.8) indeed satisfy the relation $\mathcal{A}^{(n)}_{\mu[\kappa\lambda\nu],\sigma\rho\alpha\beta} = 0$ required by Weyl symmetry. By taking the derivatives on the right hand side of (3.5.5) in the collinear frame according to (2.1.35) and using (3.5.6), linear relations between the components of $\mathcal{A}^{T'TT}$ and $\mathcal{A}^{CTT}$ are obtained. The imposition of the required symmetries (3.4.14) for the components of $\mathcal{A}^{T'TT}$,

$$
\begin{aligned}
\mathcal{A}^{TTT}_{ijklmn} = \mathcal{A}^{TTT}_{klijmn}, \quad \mathcal{A}^{TTT}_{1ijk1m} = \mathcal{A}^{TTT}_{jk1i1m}, \\
\mathcal{A}^{TTT}_{ij1111} = \mathcal{A}^{TTT}_{11ij11}, \quad \mathcal{A}^{TTT}_{11k1m1} = \mathcal{A}^{TTT}_{k111m1},
\end{aligned} \tag{3.5.9}
$$

expressed in the components of $\mathcal{A}^{CTT}$, leads to five independent linear equations for the coefficients H, I, J, K, L, M, N, P:

$$
\begin{aligned}
&dI + (8-d)J + 4dK + 2(d-4)L + 4M + 8N + 4H(d+2) = 0, \\
&2d^2 I + 2(3d+1)J + 4d(d+3)K - 8(d+1)L + (d+4)M \\
&\qquad\qquad +2(3d-2)N + 4(d-3)(d+2)P + 2H(d^2+7d+10) = 0, \\
&-d^2 I - (2d+3)J - 2d(3+d)K + (d^2+d+6)L - 3M \\
&\qquad\qquad -2dN - 4d(d-3)P - 4H(2d+3) = 0, \qquad (3.5.10) \\
&(d-1)^2 I + 2(d-1)J + 8(d-1)K - 8L + 16H = 0, \\
&-d^2 I - (3d+2)J - 2d(d+3)K + (3d+5)L - 3M \\
&\qquad\qquad -3(d-1)N - 2(d-3)(d+1)P - H(d^2+8d+11) = 0.
\end{aligned}
$$

The symmetry conditions

$$
\mathcal{A}^{TTT}_{ijkl11} = \mathcal{A}^{TTT}_{klij11}, \qquad \mathcal{A}^{TTT}_{i1k111} = \mathcal{A}^{TTT}_{k1i111}, \qquad \mathcal{A}^{TTT}_{i1k1mn} = \mathcal{A}^{TTT}_{k1i1mn}, \quad (3.5.11)
$$

are automatically satisfied by the components of $\mathcal{A}^{CTT}$ and give no further equations. Requiring the two energy momentum tensors $T_{\sigma\rho}(y)$ and $T_{\alpha\beta}(z)$ in $\langle C_{\mu\kappa\lambda\nu}(x) T_{\sigma\rho}(y) T_{\alpha\beta}(z)\rangle$ to be conserved leads to the following conditions for the components of t^{CTT}:

$$
\partial_\sigma t^{CTT}_{1jkl,\sigma npq} = 0, \qquad \partial_\sigma t^{CTT}_{ijkl,\sigma np1} = 0, \qquad \partial_\sigma t^{CTT}_{ijkl,\sigma 1pq} = 0. \qquad (3.5.12)
$$

These relations give rise to one further linearly independent equation for the coefficients H, I, J, K, L, M, N, P:

$$
(d-1)I + 2J + 4K + M - 4P = 0. \qquad (3.5.13)
$$

Thus there are six relations between the the eight coefficients H, I, J, K, L, M, N, P. Therefore there are two independent parameters in $\langle C_{\mu\gamma\lambda\nu}(x) T_{\sigma\rho}(y) T_{\alpha\beta}(z)\rangle$ which are taken to be I, K for convenience. The other coefficients are then given by:

$$
\begin{aligned}
&H = \tfrac{1}{16}(d-1)(d-4)I + \tfrac{1}{2}(d-2)K, \qquad J = -\tfrac{1}{2}(d-3)I - 4K, \\
&L = \tfrac{1}{8}(d-2)(d-1)I + (d-2)K, \qquad M = \tfrac{1}{4}(d^2-d-8)I + 2(d+2)K, \\
&N = -\tfrac{1}{4}(d-1)M, \qquad P = \tfrac{1}{2}\tfrac{d}{(d-2)}L. \qquad (3.5.14)
\end{aligned}
$$

Inserting (3.4.15) and (3.5.8) into the relations between $\mathcal{A}^{T'TT}$ and $\mathcal{A}^{CTT}$ and using (3.5.14), we obtain the following expressions for the parameters in $\mathcal{A}^{T'TT}$ in terms of the parameters in $\mathcal{A}^{CTT}$:

$$\begin{aligned} r &= -\tfrac{1}{2}(d^3 - 3d^2 - 24)I - 4(d^2 - d + 6)K, \\ s &= \tfrac{1}{2}d(d-7)I + 8dK, \qquad t = dI - 4dK. \end{aligned} \tag{3.5.15}$$

For the scale of the two point function C_T, (3.4.19) gives

$$C_T = 0, \tag{3.5.16}$$

and for the anomaly coefficients β_a, β_b we obtain for free field theories from (3.4.33) using (3.5.15) in four dimensions:

$$\beta_a = 0, \qquad \beta_b = \frac{\pi^4}{32 \times 9}(-3I + 20K). \tag{3.5.17}$$

By replacing $T_{\mu\nu}(x)$ by $\partial_\kappa \partial_\lambda C_{\mu\kappa\lambda\nu}(x)$ in (3.5.1), we have thus singled out the component in the energy momentum tensor three point function which does not contain the β_a-anomaly.

As for $F_{\nu\mu}$ in section 3.1, we conclude this section by commenting on the existence of $C_{\mu\kappa\lambda\nu}$ such that $T_{\mu\nu} = \partial_\kappa \partial_\lambda C_{\mu\kappa\lambda\nu}$. For the three point function involving $T_{\mu\nu}$ and two general operators $\mathcal{O}_1, \mathcal{O}_2$ we have

$$\begin{aligned} &\langle T_{\mu\nu}(x) \mathcal{O}_1{}^i(y) \mathcal{O}_2{}^j(z) \rangle \\ &\quad = \frac{1}{(x-z)^{2d}(y-z)^{2\eta_1}} \mathcal{I}^T_{\mu\nu,\mu'\nu'}(x-z) D_1{}^i{}_{i'}(I(y-z)) t^T{}_{\mu'\nu'}{}^{i'j}(X_{12}) \, , \end{aligned} \tag{3.5.18}$$

with $t^T{}_{\mu\nu}{}^{ij}(X) = \mathcal{O}(X^{-(d+\eta_1-\eta_2)})$. In this case we may write

$$t^T{}_{\mu\nu}{}^{ij}(X) = \partial_\kappa \partial_\lambda t^C{}_{\mu\kappa\lambda\nu}{}^{ij}(X) \tag{3.5.19}$$

where $t^C{}_{\mu\kappa\lambda\nu}{}^{ij}(X)$ has Weyl symmetry in $(\mu\kappa\lambda\nu)$. To demonstrate the existence of the tensor $t^C{}_{\mu\kappa\lambda\nu}{}^{ij}(X)$, we consider the Fourier transform $\tilde{t}^T{}_{\mu\nu}{}^{ij}(k)$ which is $\mathcal{O}(k^{(\eta_1-\eta_2)})$. When $\eta_1 \neq \eta_2$ the Fourier transform is unambiguous and satisfies $k_\mu \tilde{t}^T{}_{\mu\nu}{}^{ij}(k) = 0$, $\tilde{t}^T{}_{\mu\mu}{}^{ij}(k) = 0$. For $\eta_1 = \eta_2$ the Fourier transform is ambiguous up to a constant $C_{\mu\nu}$ and we have

$k_\mu \tilde{t}^T{}_{\mu\nu}{}^{ij}(k) = k_\mu a_{\mu\nu},\, \tilde{t}^T{}_{\mu\mu}{}^{ij}(k) = b$ with $a_{\mu\nu},\ b$ constrained by Ward identities as in (2.3.20). Subject to these results we may define

$$\tilde{t}^C{}_{\mu\kappa\lambda\nu}{}^{ij}(k) = -\frac{1}{k^4}\frac{4(d-2)}{d-3}\,\mathcal{E}^C_{\mu\kappa\lambda\nu,\alpha\gamma\delta\beta} k_\gamma k_\delta \tilde{t}^T{}_{\alpha\beta}{}^{ij}(k)\,, \tag{3.5.20}$$

since from the definition of the projection operator $\mathcal{E}^C$ in appendix A.1 it is clear that

$$-k_\kappa k_\lambda \tilde{t}^C{}_{\mu\kappa\lambda\nu}{}^{ij}(k) = \tilde{t}^T{}_{\alpha\beta}{}^{ij}(k)\,. \tag{3.5.21}$$

Therefore we may conclude that a non-trivial form $\langle T_{\mu\nu}(x)\mathcal{O}_1{}^i(y)\mathcal{O}_2{}^j(z)\rangle$ exists for given operators $\mathcal{O}_1, \mathcal{O}_2$ with a given $t^T{}_{\mu\nu}{}^{ij}(X)$ if $\tilde{t}^C{}_{\mu\kappa\lambda\nu}{}^{ij}(k)$ defined in (3.5.20) is non-zero.

4 Conformal Anomaly and Energy Momentum Tensor Three Point Function

In this chapter we find linear combinations of the three independent forms in the energy momentum tensor three point function with definite properties with respect to the conformal anomaly. First we discuss the anomaly-free form which may be obtained from the Weyl tensor three point function. The largest part of this chapter is devoted to relating the coefficient β_b of the topological contribution to the trace anomaly to the three parameters in the energy momentum tensor three point function. For this purpose we construct the three point function involving two tensors with Weyl symmetry and the energy momentum tensor and calculate its non-integrable singular terms. Then we construct a suitable counterterm for the contribution to the energy momentum tensor three point function arising from this form which generates the anomalous trace expected from the Ward identity.

As far as the scale dependent part of the anomaly, $-\beta_a F$, is concerned, β_a has already been related to the parameters in the energy momentum tensor three point function within the previous chapters. In section 2.4 β_a is related to the scale of the two point function C_T. To this effect we have derived from the anomalous Ward identity (2.4.9) that the conformal variation of the two point function is equal to the scale dependent anomalous term in the Ward identity (2.4.15). Inserting the explicit regularised expression (2.4.16) with (2.4.17) for the two point function into this relation gives the proportionality between β_a and C_T (2.4.22) in four dimensions. Furthermore in section 3.4 we have used the Ward identity in the form (2.3.18) to express C_T in term of the parameters in the three point function which yields the desired relation for β_a, (3.4.12).

4.1 Anomaly Free Form in the Energy Momentum Tensor Three Point Function

In this section we construct a symmetric expression for the three point function $\langle C_{\mu\kappa\lambda\nu}(x)C_{\sigma\varepsilon\eta\rho}(y)C_{\alpha\gamma\delta\beta}(z)\rangle$ involving three tensors with Weyl symmetry and subsequently calculate

$$\langle T'_{\mu\nu}(x)T'_{\sigma\rho}(y)T'_{\alpha\beta}(z)\rangle = \partial^x{}_\kappa\partial^x{}_\lambda\partial^y{}_\varepsilon\partial^y{}_\eta\partial^z{}_\gamma\partial^z{}_\delta\langle C_{\mu\kappa\lambda\nu}(x)C_{\sigma\varepsilon\eta\rho}(y)C_{\alpha\gamma\delta\beta}(z)\rangle, \qquad (4.1.1)$$

using results like (3.5.15) of section 3.5. This contribution to the energy momentum tensor three point function $\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle$ trivially satisfies all six conservation equations and all three tracelessness conditions. Even in four dimensions $\langle T'_{\mu\nu}(x)T'_{\sigma\rho}(y)T'_{\alpha\beta}(z)\rangle$ as in (4.1.1) is anomaly free. This can be seen by inserting it into the trace identity (2.4.9): $\langle T'_{\mu\mu}(x)T'_{\sigma\rho}(y)T'_{\alpha\beta}(z)\rangle$ vanishes by virtue of the tracelessness property $C_{\mu\kappa\lambda\mu}(x) = 0$ of the Weyl tensor, and therefore the anomaly coefficients β_a and β_b on the right hand side of the trace identity (2.4.9) must be zero for this form, as well as the scale of the two point function in (2.4.9) which in four dimensions is proportional to β_a by (2.4.22). In four dimensions there should be only one expression of the form (4.1.1) since this form does not give rise to any anomaly in the energy momentum tensor three point function. The energy momentum tensor three point function contains three independent forms in general, two of which are anomalous in four dimensions such that there may be only one anomaly free form.

In the following we construct in general two possible completely symmetric expressions for $\langle C_{\mu\kappa\lambda\nu}(x)C_{\sigma\varepsilon\eta\rho}(y)C_{\alpha\gamma\delta\beta}(z)\rangle$. We can then apply the results obtained in section 2.1 (equation 2.1.24) for general totally symmetric three point functions.

For the Weyl tensor three point function we may write according to (2.1.15) with (2.1.24)

$$
\begin{aligned}
&\langle C_{\mu\kappa\lambda\nu}(x) C_{\sigma\varepsilon\eta\rho}(y) C_{\alpha\gamma\delta\beta}(z) \rangle \\
&\quad = \frac{\mathcal{I}^C_{\mu\kappa\lambda\nu,\mu'\kappa'\lambda'\rho'}(x-z)\mathcal{I}^C_{\sigma\varepsilon\eta\rho,\sigma'\varepsilon'\eta'\rho'}(y-z)}{(x-z)^{2(d-2)}(y-z)^{2(d-2)}}\; t^{(i)}_{\mu'\kappa'\lambda'\nu',\sigma'\varepsilon'\eta'\rho',\alpha\gamma\delta\beta}(X)\,,
\end{aligned}
\tag{4.1.2}
$$

where

$$
t^{(i)}_{\mu\kappa\lambda\nu,\sigma\varepsilon\eta\kappa\rho,\alpha\gamma\delta\beta} = d^{(i)}_{\mu\kappa\lambda\nu,\sigma\varepsilon\eta\rho,\alpha'\gamma'\delta'\beta'} \frac{I_{\alpha\alpha'}(X) I_{\gamma\gamma'}(X) I_{\delta\delta'}(X) I_{\beta\beta'}(X)}{X^{d-2}}\,. \tag{4.1.3}
$$

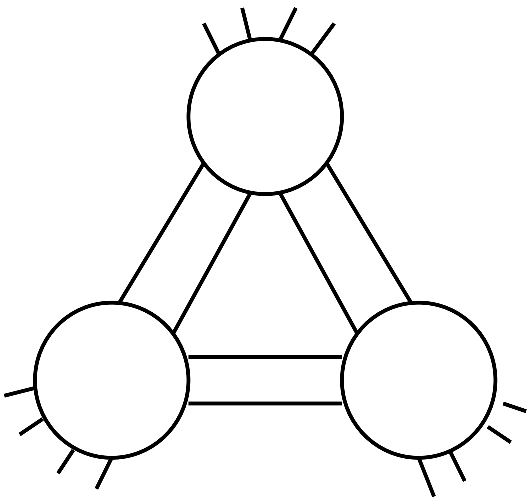
There are two possibilities to construct a totally symmetric structure constant using $\mathcal{E}^C_{\mu\sigma\rho\nu,\alpha\beta\gamma\delta}$, the projection operator onto the space of Weyl tensors:

$$
d^{(1)}_{\mu\kappa\lambda\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta} = \mathcal{E}^C_{\mu\kappa\lambda\nu,\mu'\kappa'\lambda'\nu'} \mathcal{E}^C_{\sigma\varepsilon\eta\rho,\mu'\kappa'\eta'\rho'} \mathcal{E}^C_{\alpha\gamma\delta\beta,\lambda'\nu'\eta'\rho'} \tag{4.1.4}
$$

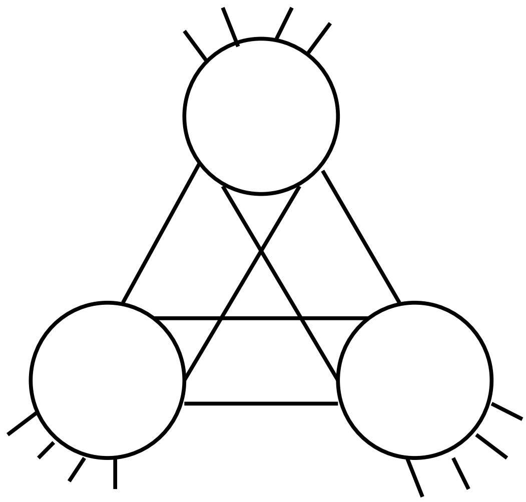
and

$$
d^{(2)}_{\mu\kappa\lambda\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta} = \mathcal{E}^C_{\mu\kappa\lambda\nu,\mu'\kappa'\lambda'\nu'} \mathcal{E}^C_{\sigma\varepsilon\eta\rho,\mu'\varepsilon'\lambda'\rho'} \mathcal{E}^C_{\alpha\gamma\delta\beta,\kappa'\rho'\nu'\varepsilon'}\,. \tag{4.1.5}
$$

The totally symmetric invariant tensors $d^{(1)}$ and $d^{(2)}$ differ in the way the indices are combined for summation. They may be represented graphically as follows:



$d^{(1)}$



$d^{(2)}$

Each circle represents an $\mathcal{E}^C$ tensor, each line being a symbol for one index. Each $\mathcal{E}^C$ tensor has four free indices, the others being coupled to the other two tensors in a completely symmetric way. In $d^{(1)}$ we join antisymmetric pairs of indices, in $d^{(2)}$ pairs of indices with mixed symmetry.

The two structure constants may be thought of as functional derivatives, with respect to a tensor $\mathcal{C}_{\mu\kappa\lambda\nu}$ with Weyl symmetry as in (2.2.5), of two of the invariants of third order in $\mathcal{C}_{\mu\kappa\lambda\nu}$ constructed by Barvinsky et al [25]. The relevant totally symmetric invariants are

$$\mathcal{Q}^1 = \mathcal{C}_{\mu'\kappa'\lambda'\nu'}\mathcal{C}^{\mu'\kappa'}{}_{\eta'\rho'}\mathcal{C}^{\lambda'\nu'\eta'\rho'} \tag{4.1.6}$$

$$\mathcal{Q}^2 = \mathcal{C}_{\mu'\kappa'\lambda'\nu'}\mathcal{C}^{\mu'}{}_{\varepsilon'}{}^{\lambda'}{}_{\rho'}\mathcal{C}^{\kappa'\rho'\nu'\varepsilon'} . \tag{4.1.7}$$

Assuming

$$\frac{\delta}{\delta\mathcal{C}^{\mu\kappa\lambda\nu}}\mathcal{C}^{\sigma\varepsilon\eta\rho} = \mathcal{E}^{C}_{\mu\kappa\lambda\nu,}{}^{\sigma\varepsilon\eta\rho} , \tag{4.1.8}$$

the structure constants $d^{(i)}$ are then given by

$$d^{(i)}_{\mu\kappa\lambda\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta} = \frac{1}{6}\frac{\delta}{\delta\mathcal{C}^{\mu\kappa\lambda\nu}}\frac{\delta}{\delta\mathcal{C}^{\sigma\varepsilon\eta\rho}}\frac{\delta}{\delta\mathcal{C}^{\alpha\beta\delta\gamma}}\,\mathcal{Q}^i . \tag{4.1.9}$$

We have

$$15\,\mathcal{C}_{\mu\kappa}{}^{[\lambda\nu}\mathcal{C}^{\mu\kappa}{}_{\eta\rho}\mathcal{C}^{\eta]\rho}{}_{\lambda\nu} = \mathcal{Q}^1 - 4\mathcal{Q}^2 . \tag{4.1.10}$$

Since expressions which are totally antisymmetric on five indices vanish in four dimensions, this yields

$$\mathcal{Q}^1 = 4\mathcal{Q}^2 \tag{4.1.11}$$

in four dimensions and hence also

$$d^{(1)}_{\mu\kappa\lambda\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta} = 4\,d^{(2)}_{\mu\kappa\lambda\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta} , \tag{4.1.12}$$

which is in agreement with the fact that there should be only one linearly independent anomaly free form in four dimensions.

Now four derivatives of the Weyl tensor three point function according to (4.1.2) are taken using FORM. The result of this calculation is expanded in the forms (3.5.8), giving expressions for the eight coefficients H, I, J, K, L, M, N, P. For the two independent parameters I, K we obtain

$$\begin{aligned} I_1 &= \frac{1}{4(d-2)^3(d-1)^2}(d^5 - 5d^4 - 14d^3 + 114d^2 - 164d + 64), \\ K_1 &= \frac{1}{8(d-1)(d-2)^3}(4d^3 - 35d^2 + 94d - 64) \end{aligned} \tag{4.1.13}$$

from $t^{(1)}$ and

$$
\begin{aligned}
I_2 &= \frac{1}{16(d-2)^3(d-1)^2}(d^6 - 19d^5 + 131d^4 - 393d^3 + 508d^2 - 284d + 64), \\
K_2 &= -\frac{1}{128(d-2)^3(d-1)}(d^6 - 21d^5 + 174d^4 - 706d^3 \\
&\qquad +1384d^2 - 1096d + 256) \qquad (4.1.14)
\end{aligned}
$$

from $t^{(2)}$.

Next we use (3.5.15) to determine the expansion of the anomaly free form $\langle T'_{\mu\nu}(x)T'_{\sigma\rho}(y)T'_{\alpha\beta}(z)\rangle$ in terms of the forms (3.4.15) in the collinear frame. We give here only the independent coefficients r, s, t. The remaining ones are in accord with (3.4.16) and (3.4.17). For the anomaly free form given by $t^{(1)}$ we have:

$$
\begin{aligned}
r_1 &= -\frac{d}{8(d-2)^3(d-1)^2}(d^7 - 8d^6 + 17d^5 - 40d^4 + 382d^3 - 1192d^2 + 1056d - 112)\,, \\
s_1 &= \frac{d}{8(d-2)^3(d-1)^2}(d^6 - 12d^5 + 53d^4 - 100d^3 + 70d^2 - 52d + 64)\,, \\
t_1 &= \frac{d}{4(d-2)^2(d-1)^2}(d^4 - 11d^3 + 42d^2 - 60d + 32)\,, \qquad (4.1.15)
\end{aligned}
$$

and for the form with $t^{(2)}$:

$$
\begin{aligned}
r_2 &= -\frac{d}{32(d-2)^3(d-1)^2}(d^7 - 35d^6 + 397d^5 - 1997d^4 + 4898d^3 \\
&\qquad -6024d^2 + 4104d - 1552), \\
s_2 &= -\frac{d}{32(d-2)^3(d-1)^2}(d^7 - 18d^6 + 126d^5 - 450d^4 \\
&\qquad +921d^3 - 1120d^2 + 652d - 64), \\
t_2 &= \frac{d}{32(d-2)^2(d-1)^2}(d^6 - 18d^5 + 121d^4 - 376d^3 + 552d^2 - 360d + 64)\,. \qquad (4.1.16)
\end{aligned}
$$

The two forms are independent in general. However, in three dimensions the Weyl tensor $C_{\mu\sigma\rho\nu}(x)$ is zero because there is no totally antisymmetric tensor with four indices when $d = 3$. Therefore we expect also the coefficients (4.1.15) and (4.1.16) to vanish in three dimensions. According to the discussion of section 3.4, there are only two independent forms in the energy momentum tensor three point function in three dimensions and

$\mathcal{A}^{TTT}_{ijklmn}$ depends only on the two combinations $r-4t$ and $s+2t$, rather than on r,s,t separately. These combinations do indeed vanish for (4.1.15) and (4.1.16) when $d=3$. Moreover, in three dimensions the coefficients (4.1.16) are related to the corresponding ones in (4.1.15) by a factor of -7 so that there are only two independent forms for the energy momentum tensor three point function in the limit $d\to 3$ as required by the general formalism.

In four dimensions, the coefficients from the first set are four times as large as those from the second set:

$$\begin{array}{ccccccccccccc} & \alpha & \beta & \gamma & \delta & \varepsilon & \rho & \tau & r & s & t & I & K \\ t^{(1)} & \frac{175}{3} & -\frac{175}{9} & \frac{35}{3} & 7 & -\frac{7}{9} & -7 & \frac{14}{3} & -\frac{17}{9} & -\frac{1}{3} & \frac{4}{9} & \frac{5}{18} & \frac{1}{24} \\ t^{(2)} & \frac{175}{12} & -\frac{175}{36} & \frac{35}{12} & \frac{7}{4} & -\frac{7}{36} & -\frac{7}{4} & \frac{7}{6} & -\frac{17}{36} & -\frac{1}{12} & \frac{1}{9} & \frac{5}{72} & \frac{1}{96} \end{array} \tag{4.1.17}$$

and the two forms are linearly dependent:

$$\langle T'_{\mu\nu}(x)T'_{\sigma\rho}(y)T'_{\alpha\beta}(z)\rangle^1 = 4\,\langle T'_{\mu\nu}(x)T'_{\sigma\rho}(y)T'_{\alpha\beta}(z)\rangle^2 \tag{4.1.18}$$

as expected from (4.1.12). This is consistent with the fact that there can at most be one independent anomaly-free form in four dimensions. For the linear combination

$$\chi_1\langle T'_{\mu\nu}(x)T'_{\sigma\rho}(y)T'_{\alpha\beta}(z)\rangle^1 + \chi_2\langle T'_{\mu\nu}(x)T'_{\sigma\rho}(y)T'_{\alpha\beta}(z)\rangle^2 \tag{4.1.19}$$

in four dimensions, (4.1.17) gives for $\mathcal{A},\mathcal{B},\mathcal{C}$ the values

$$\mathcal{A} = \tfrac{8}{9}\left(4\chi_1+\chi_2\right)\,, \quad \mathcal{B} = \tfrac{92}{9}\left(4\chi_1+\chi_2\right)\,, \quad \mathcal{C} = -\tfrac{2}{9}\left(4\chi_1+\chi_2\right)\,. \tag{4.1.20}$$

When inserted into the relations (3.4.33) or (3.5.17), the coefficients (4.1.17) give indeed

$$\beta_a = \beta_b = 0 \tag{4.1.21}$$

for the anomaly coefficients. The relations (3.4.33) were derived for free field theories only but we will show in the next section that they hold for a much larger class of theories.

The form (4.1.3) is not the most general form for the Weyl tensor three point function. Another set of structures with the required symmetries and scaling dimensions is

$$t^{(3)}_{\mu\kappa\lambda\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}(X) = \frac{1}{X^{d-2}}\mathcal{E}^C_{\mu\kappa\lambda\nu,\mu'\kappa'\lambda'\nu'}I_{\mu'\sigma'}(X)I_{\kappa'\varepsilon'}(X)$$
$$\times\ \mathcal{E}^C_{\sigma\varepsilon\eta\rho,\sigma'\varepsilon'\eta'\rho'}I_{\eta'\alpha'}(X)I_{\rho'\gamma'}(X)\mathcal{E}^C_{\alpha\gamma\delta\beta,\alpha'\gamma'\delta'\beta'}I_{\delta'\lambda'}(X)I_{\beta'\nu'}(X) \tag{4.1.22}$$

$$t^{(4)}_{\mu\kappa\lambda\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}(X) = \frac{1}{X^{d-2}}\mathcal{E}^C_{\mu\kappa\lambda\nu,\mu'\kappa'\lambda'\nu'}I_{\mu'\varepsilon'}(X)I_{\lambda'\rho'}(X)$$
$$\times\ \mathcal{E}^C_{\sigma\varepsilon\eta\rho,\sigma'\varepsilon'\eta'\rho'}I_{\sigma'\alpha'}(X)I_{\eta'\delta'}(X)\mathcal{E}^C_{\alpha\gamma\delta\beta,\alpha'\gamma'\delta'\beta'}I_{\gamma'\nu'}(X)I_{\beta'\kappa'}(X). \tag{4.1.23}$$

We can visualise these expressions by introducing a factor of $I_{\mu_1\mu_1'}(X)$ for each of the internal lines in the picture on page 69. In four dimensions, these forms should be linearly dependent on the form (4.1.3) discussed in this section since there is only one linearly independent form allowed when $d = 4$.

4.2 Three Point Function Involving Two Weyl Tensors

To calculate the coefficient β_b for general conformal field theories, we proceed further by constructing the three point function for one energy momentum tensor and two tensors with Weyl symmetry. We are then going to calculate its short distance limit and thus find β_b in terms of the parameters in $\langle T_{\mu\nu}(x)C_{\sigma\varepsilon\eta\rho}(y)C_{\alpha\gamma\delta\beta}(z)\rangle$. As opposed to the unregularised case (2.4.9), it will be possible to calculate the short distance limit of this three point function because there are no derivatives acting on the delta distributions here.

According to the general formalism (2.1.15), the three point function for one energy momentum tensor $T_{\mu\nu}$ and two tensors with Weyl symmetry $C_{\sigma\varepsilon\eta\rho}(y)$ and $C_{\alpha\gamma\delta\beta}(z)$ is given by

$$\langle T_{\mu\nu}(x)C_{\sigma\varepsilon\eta\rho}(y)C_{\alpha\gamma\delta\beta}(z)\rangle$$
$$= \frac{\mathcal{I}^C_{\sigma\varepsilon\eta\rho,\sigma'\varepsilon'\eta'\rho'}(y-x)\mathcal{I}^C_{\alpha\gamma\delta\beta,\alpha'\gamma'\delta'\beta'}(z-x)}{(y-x)^{2(d-2)}(z-x)^{2(d-2)}}\, t^{TCC}_{\mu\nu,\sigma'\varepsilon'\eta'\rho',\alpha'\gamma'\delta'\beta'}(X_{23}) \tag{4.2.1}$$
$$= \frac{\mathcal{I}^T_{\mu\nu,\mu'\nu'}(x-z)\mathcal{I}^C_{\sigma\varepsilon\eta\rho,\sigma'\varepsilon'\eta'\rho'}(y-z)}{(x-z)^{2d}(y-z)^{2(d-2)}}\, \hat{t}^{TCC}_{\mu'\nu',\sigma'\varepsilon'\eta'\rho',\alpha\gamma\delta\beta}(X_{12})$$

where X_{23} and X_{12} are given in (2.1.8) and in (2.1.10) and where

$$\hat{t}^{TCC}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}(X) = X^{-4}\mathcal{I}^{C}_{\sigma\varepsilon\eta\rho,\sigma'\varepsilon'\eta'\rho'}(X)t^{TCC}_{\mu\nu,\sigma'\varepsilon'\eta'\rho',\alpha\gamma\delta\beta}(-X) = \mathcal{O}(X^{-d}) \qquad (4.2.2)$$

since $t^{TCC}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}(X)$ is $\mathcal{O}(X^{-(d-4)})$.

From (4.2.1) we obtain an expression for the three point function $\langle T_{\mu\nu}(x){T'}_{\sigma\rho}(y){T'}_{\alpha\beta}(z)\rangle$ in which ${T'}_{\alpha\beta}(z) = \partial_\gamma\partial_\delta C_{\alpha\gamma\delta\beta}(z)$:

$$\begin{aligned} \langle T_{\mu\nu}(x){T'}_{\sigma\rho}(y){T'}_{\alpha\beta}(z)\rangle &= {\partial^y}_\varepsilon {\partial^y}_\eta {\partial^z}_\gamma {\partial^z}_\delta \langle T_{\mu\nu}(x)C_{\sigma\varepsilon\eta\rho}(y)C_{\alpha\gamma\delta\beta}(z)\rangle \\ &= \frac{\mathcal{I}^T_{\sigma\rho,\sigma'\rho'}(y-x)\mathcal{I}^T_{\alpha\beta,\alpha'\beta'}(z-x)}{(y-x)^{2d}(z-x)^{2d}} t^{TT'T'}_{\mu\nu\sigma'\rho'\alpha'\beta'}(X_{23}) \ , \end{aligned} \qquad (4.2.3)$$

where we have

$$t^{TT'T'}_{\mu\nu\sigma\rho\alpha\beta}(X) = \partial_\varepsilon\partial_\eta\partial_\gamma\partial_\delta t^{TCC}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}(X) \, . \qquad (4.2.4)$$

The most general expression for $t^{TCC}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}(X)$ which satisfies

$$t^{TCC}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}(X) = t^{TCC}_{\mu\nu,\alpha\gamma\delta\beta,\sigma\varepsilon\eta\rho}(-X) \ , \qquad (4.2.5)$$

is symmetric and traceless in $(\mu\nu)$ and has Weyl symmetry in $(\sigma\varepsilon\eta\rho)$ and in $(\alpha\gamma\delta\beta)$ can be constructed using $\mathcal{E}^C_{\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}$, the projection operator onto the space of Weyl tensors. There are ten possible forms with the required symmetry:

$$
\begin{aligned}
t^{TCC}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}(X) = & \\
& A \quad \mathcal{E}^C_{\sigma\varepsilon\eta\rho,(\mu|\sigma'\rho'\nu'}\mathcal{E}^C_{\alpha\gamma\delta\beta,\nu)\sigma'\rho'\nu'}\frac{1}{X^{(d-4)}} \; - \mathrm{trace}(\mu\nu) \\
+ & B \quad \mathcal{E}^C_{\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}\frac{X_\mu X_\nu}{X^{(d-2)}} \; - \mathrm{trace}(\mu\nu) \\
+ & C \quad \mathcal{E}^C_{\sigma\varepsilon\eta\rho,(\mu|\varphi\rho'\nu'}\mathcal{E}^C_{\alpha\gamma\delta\beta,\nu)\theta\rho'\nu'}\frac{X_\varphi X_\theta}{X^{(d-2)}} \; - \mathrm{trace}(\mu\nu) \\
+ & D \quad \mathcal{E}^C_{\sigma\varepsilon\eta\rho,(\mu|\rho'\varphi\nu'}\mathcal{E}^C_{\alpha\gamma\delta\beta,\nu)\rho'\theta\nu'}\frac{X_\varphi X_\theta}{X^{(d-2)}} \; - \mathrm{trace}(\mu\nu) \\
+ & E \quad \tfrac{1}{2}\Big(\mathcal{E}^C_{\sigma\varepsilon\eta\rho,\varphi\sigma'\rho'\nu'}\mathcal{E}^C_{\alpha\gamma\delta\beta,(\mu|\sigma'\rho'\nu'}\frac{X_{\nu)}X_\varphi}{X^{(d-2)}} \\
& \qquad\qquad + (\sigma\varepsilon\eta\rho \leftrightarrow \alpha\gamma\delta\beta)\Big) \; - \mathrm{trace}(\mu\nu) \\
+ & F \quad \tfrac{1}{2}\Big(\mathcal{E}^C_{\sigma\varepsilon\eta\rho,(\mu|\sigma'|\nu)\rho'}\mathcal{E}^C_{\alpha\gamma\delta\beta,\varphi\sigma'\theta\rho'}\frac{X_\varphi X_\theta}{X^{(d-2)}} \; + (\sigma\varepsilon\eta\rho \leftrightarrow \alpha\gamma\delta\beta)\Big) \\
+ & G \quad \mathcal{E}^C_{\sigma\varepsilon\eta\rho,(\mu|\varphi\rho'\theta}\mathcal{E}^C_{\alpha\gamma\delta\beta,\nu)\omega\rho'\chi}\frac{X_\varphi X_\theta X_\omega X_\chi}{X^d} \; - \mathrm{trace}(\mu\nu) \\
+ & Q \quad \mathcal{E}^C_{\sigma\varepsilon\eta\rho,\varphi\sigma'\rho'\nu'}\mathcal{E}^C_{\alpha\gamma\delta\beta,\theta\sigma'\rho'\nu'}\frac{X_\mu X_\nu X_\varphi X_\theta}{X^d} \; - \mathrm{trace}(\mu\nu) \qquad (4.2.6) \\
+ & R \quad \tfrac{1}{2}\Big(\mathcal{E}^C_{\sigma\varepsilon\eta\rho,(\mu|\rho'\varphi\sigma'}\mathcal{E}^C_{\alpha\gamma\delta\beta,\omega\rho'\theta\sigma'}\frac{X_{\nu)}X_\omega X_\varphi X_\theta}{X^d} \\
& \qquad\qquad + (\sigma\varepsilon\eta\rho \leftrightarrow \alpha\gamma\delta\beta)\Big) \; - \mathrm{trace}(\mu\nu) \\
+ & S \quad \mathcal{E}^C_{\sigma\varepsilon\eta\rho,\varphi\sigma'\theta\rho'}\mathcal{E}^C_{\alpha\gamma\delta\beta,\omega\sigma'\chi\rho'}\frac{X_\mu X_\nu X_\varphi X_\theta X_\omega X_\chi}{X^{(d+2)}} \; - \mathrm{trace}(\mu\nu)\,.
\end{aligned}
$$

These forms differ by the number of vectors X_μ they contain and by how many of these vectors are coupled to each of the $\mathcal{E}^C$ tensors. The vertical lines between the indices indicate that symmetrisation applies only to $(\mu\nu)$.

As a condition on the parameters in $\langle T_{\mu\nu}(x) C_{\sigma\varepsilon\eta\rho}(y) C_{\alpha\gamma\delta\beta}(z)\rangle$, we may require the conservation of $T_{\mu\nu}(x)$ for non-coincident points:

$$
\partial^x{}_\mu \langle T_{\mu\nu}(x) C_{\sigma\varepsilon\eta\rho}(y) C_{\alpha\gamma\delta\beta}(z)\rangle = 0 \;\;\Rightarrow\;\; \partial_\mu \hat{t}^{TCC}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}(X) = 0\,. \qquad (4.2.7)
$$

This yields two linear relations for the parameters in (4.2.6):

$$
\begin{aligned}
\mathcal{T}_1 \equiv \; & \tfrac{1}{2}(d-4)(d+4)A - 8B - \tfrac{1}{2}(d+2)C + \tfrac{1}{4}(d-4)D \\
& \qquad -E - \tfrac{3}{4}dF - Q \qquad\qquad = 0\,, \qquad (4.2.8)
\end{aligned}
$$

$$\mathcal{T}_2 \equiv \quad -(d-4)(d+4)A + 16B + dC + \tfrac{1}{4}d(d-4)D$$
$$-\tfrac{1}{4}d(d-10)F - \tfrac{1}{4}(d+4)G - R - S \qquad = 0\,. \qquad (4.2.9)$$

In order to determine the contribution (4.2.3) with (4.2.6) to the conformal energy momentum tensor three point function we begin by taking two derivatives of the three point function involving two Weyl tensors according to

$$\langle T_{\mu\nu}(x)T'_{\sigma\rho}(y)C_{\alpha\gamma\delta\beta}(z)\rangle = \partial^y{}_\varepsilon\partial^y{}_\eta\langle T_{\mu\nu}(x)C_{\sigma\varepsilon\eta\rho}(y)C_{\alpha\gamma\delta\beta}(z)\rangle\,. \qquad (4.2.10)$$

Requiring Bose symmetry for the energy momentum tensors $T_{\mu\nu}(x)$ and $T'{}_{\sigma\rho}(x)$ for non-coincident points,

$$\langle T_{\mu\nu}(x)T'_{\sigma\rho}(y)C_{\alpha\gamma\delta\beta}(z)\rangle \;=\; \langle T_{\sigma\rho}(y)T'{}_{\mu\nu}(x)C_{\alpha\gamma\delta\beta}(z)\rangle\,,\ \text{i.e.} \qquad (4.2.11)$$
$$\partial^y{}_\varepsilon\partial^y{}_\eta\langle T_{\mu\nu}(x)C_{\sigma\varepsilon\eta\rho}(y)C_{\alpha\gamma\delta\beta}(z)\rangle \;=\; \partial^x{}_\varepsilon\partial^x{}_\eta\langle T_{\sigma\rho}(y)C_{\mu\varepsilon\eta\nu}(x)C_{\alpha\gamma\delta\beta}(z)\rangle\,, \qquad (4.2.12)$$

which requires

$$\partial_\varepsilon\partial_\eta t^{TCC}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}(X) \;=\; \partial_\varepsilon\partial_\eta t^{TCC}_{\sigma\rho,\mu\varepsilon\eta\nu,\alpha\gamma\delta\beta}(X)\,, \qquad (4.2.13)$$

we obtain symmetry conditions for the ten parameters $A, B, C, D, E, F, G, Q, R, S$ in (4.2.6). The calculation of

$$t^{T'TC}_{\mu\nu,\sigma\rho,\alpha\gamma\delta\beta}(X) = t^{TT'C}_{\mu\nu,\sigma\rho,\alpha\gamma\delta\beta}(X) = \partial_\varepsilon\partial_\eta t^{TCC}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}(X) \qquad (4.2.14)$$

is performed using FORM. Comparing the components of (4.2.11) in the basis for $\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)C_{\alpha\gamma\delta\beta}(z)\rangle$ (3.5.8) discussed in section 3.5 yields:

$$\mathcal{S}_1 \equiv \quad (d-1)\,\Big[\tfrac{1}{3}(d-4)^2A + \tfrac{2}{3}(d-2)(d-4)B + \tfrac{1}{6}(d-4)^2E$$
$$-\tfrac{2}{3}(d-4)Q - \tfrac{1}{4}(d-4)R + S\Big]$$
$$+\tfrac{2}{3}(d-4)C - \tfrac{1}{6}(3d^2-13d+16)D - \tfrac{1}{2}d(d-3)F - G \quad = 0\,, \qquad (4.2.15)$$
$$\mathcal{S}_2 \equiv \quad (d-1)\,\Big[\tfrac{1}{6}(d-2)(d-4)^2A - \tfrac{4}{3}(d-2)(d-4)B$$
$$-\tfrac{1}{3}(d-4)(d-5)E + (d-4)Q + \tfrac{1}{2}(d-4)R - S\Big]$$
$$-\tfrac{1}{6}(d-4)(d^2-d-8)C - \tfrac{1}{6}(d^3-11d^2+26d-4)D$$
$$+\tfrac{1}{2}(d^2-5d+8)F + \tfrac{1}{4}(d^2-d-8)G \qquad = 0\,. \qquad (4.2.16)$$

The two symmetry conditions are not independent of the conservation conditions (4.2.15) and (4.2.16). In fact we have

$$-2(d-2)\mathcal{S}_1 + d\mathcal{S}_2 = (d-4)(d-1)\left[\tfrac{1}{3}(d-8)\mathcal{T}_1 - \mathcal{T}_2\right] . \qquad (4.2.17)$$

The conservation equations play an important role later when determining the singular non-integrable terms in the three point function $\langle T_{\mu\nu}(x) C_{\sigma\varepsilon\eta\rho}(y) C_{\alpha\gamma\delta\beta}(z)\rangle$ in section 4.3.

The completeness of the expansion (4.2.6) and the symmetry and conservation conditions can be checked by calculating the values for $A, B, C, D, E, F, G, Q, R, S$ for the contribution of the anomaly free form to (4.2.6). This contribution is obtained by taking two derivatives of the anomaly free form (4.1.1):

$$\langle T'_{\mu\nu}(x) C_{\sigma\varepsilon\eta\rho}(y) C_{\alpha\gamma\delta\beta}(z)\rangle = \partial^x{}_\kappa \partial^x{}_\lambda \langle C_{\mu\kappa\lambda\nu}(x) C_{\sigma\varepsilon\eta\rho}(y) C_{\alpha\gamma\delta\beta}(z)\rangle . \qquad (4.2.18)$$

For the two forms (4.1.3) we find:

$$\begin{array}{ccc}
 & t^{(1)} & t^{(2)} \\
A & -\frac{d(d^2-d-16)}{(d-2)(d-1)} & \frac{1}{2}\frac{(d^3-4d^2-7d-6)}{(d-2)(d-1)} \\
B & \frac{d^2}{(d-2)(d-1)} & \frac{1}{2}\frac{(2d^2-9d+6)}{(d-1)(d-2)} \\
C & -\frac{(d^4-3d^3-10d^2+20d+56)}{(d-2)(d-1)} & \frac{(d^3-11d^2+18d+24)}{(d-1)(d-2)} \\
D & -\frac{16(d-5)}{(d-2)(d-1)} & \frac{1}{2}\frac{(d^4-11d^3+42d^2-48d-48)}{(d-1)(d-2)} \\
E & \frac{2(d^3-d^2-28d+12)}{(d-2)(d-1)} & -\frac{2(2d^3-11d^2+7d-6)}{(d-2)(d-1)} \\
F & \frac{8(d^2-3d-6)}{(d-2)(d-1)} & \frac{1}{2}\frac{(d^4-9d^3+20d^2-12d+64)}{(d-2)(d-1)} \\
G & 0 & 0 \\
Q & \frac{48}{(d-2)} & \frac{6(d^2-5d+2)}{(d-2)} \\
R & -\frac{32(d-4)}{(d-2)} & -\frac{4(d-4)(d^2-5d+2)}{(d-2)} \\
S & \frac{32(d-4)}{(d-2)} & \frac{4(d-4)(d^2-5d+2)}{(d-2)} .
\end{array} \qquad (4.2.19)$$

This satisfies the symmetry conditions (4.2.15) and (4.2.16) as well as the conservation conditions (4.2.8) and (4.2.9). Furthermore we may check that the scale of the Weyl tensor two point function C_C as given by (4.3.55) vanishes for the two anomaly free forms.

We now determine the contribution of (4.2.3) to the general conformal energy momentum tensor three point function discussed in section 3.4. To this effect we calculate $t^{TT'T'}_{\mu\nu\sigma\rho\alpha\beta}(X)$ in (4.2.4) and relate the coefficients $A, B, C, D, E, F, G, Q, R, S$ to the coefficients $\mathcal{A}, \mathcal{B}, \mathcal{C}$ in the energy momentum tensor three point function defined in (3.4.7). By construction the result (4.2.4) automatically satisfies the conservation equation $\partial_\sigma t^{TT'T'}_{\mu\nu\sigma\rho\alpha\beta}(X) = 0$. This is sufficient to ensure that the symmetry condition

$$\langle T_{\mu\nu}(x) {T'}_{\sigma\rho}(y) {T'}_{\alpha\beta}(z)\rangle = \langle T_{\sigma\rho}(y) {T'}_{\mu\nu}(x) {T'}_{\alpha\beta}(z)\rangle \tag{4.2.20}$$

is satisfied and hence $\partial_\mu t^{TT'T'}_{\mu\nu\sigma\rho\alpha\beta}(X) = 0$ holds for non-coincident points. Therefore $t_{\mu\nu\sigma\rho\alpha\beta}(X)$ gives results for the coefficients $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ satisfying the conservation conditions (3.4.9) so it is sufficient to quote only $\mathcal{A}, \mathcal{B}, \mathcal{C}$. The results are lengthy and may be found in the appendix A.5. However it is important to note that they yield

$$C_T = 0 \tag{4.2.21}$$

for the scale of the energy momentum tensor two point function as given by (3.4.11). Thus the contribution (4.2.3) to the energy momentum tensor three point function contains only two independent forms.

In four dimensions the basis (4.2.6) is overcomplete since there are relations between the different forms from the vanishing of totally antisymmetric five index tensors. For the tensor $C_{\mu\sigma\rho\nu}$ with Weyl symmetry we have

$$\begin{aligned}
30\, C_{\alpha\kappa[\lambda\theta} X_\phi C_{\beta\kappa]\lambda\phi} X_\theta &= (2C_{\alpha\kappa\lambda\theta} C_{\beta\kappa\lambda\phi} + C_{\alpha\theta\kappa\lambda} C_{\beta\phi\kappa\lambda} - 2C_{\kappa\alpha\beta\lambda} C_{\kappa\theta\phi\lambda}) X_\theta X_\phi \\
&\quad - C_{\alpha\kappa\lambda\omega} C_{\beta\kappa\lambda\omega} X^2 + C_{\alpha\theta\kappa\omega} C_{\theta\kappa\lambda\omega} X_\theta X_\beta\,, \\
30\, C_{\varepsilon\eta[\kappa\lambda} X_\alpha C_{\varepsilon\eta]\kappa\lambda} &= C_{\varepsilon\eta\kappa\lambda} C_{\varepsilon\eta\kappa\lambda} X_\alpha - 4 C_{\alpha\eta\kappa\lambda} C_{\beta\eta\kappa\lambda} X_\beta\,.
\end{aligned} \tag{4.2.22}$$

For instance these relations imply that $\mathcal{E}^C_{\sigma\varepsilon\eta\rho,(\mu|\sigma'\rho'\nu'} \mathcal{E}^C_{\alpha\gamma\delta\beta,\nu)\sigma'\rho'\nu'} - \frac{1}{4}\delta_{\mu\nu} \mathcal{E}^C_{\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}$ vanishes in four dimensions such that t^{TCC} as given by (4.2.6) is independent of A, and depends only on the combinations

$$8B + 2E + F + 2Q\,,\ \ 2C + F\,,\ \ D + F\,,\ \ G,\ R,\ S\,. \tag{4.2.23}$$

In four dimensions the calculation of four derivatives according to (4.2.3) gives the following expressions for the coefficients r, s, t in (3.4.15) in terms of the coefficients $A, B, C, D, E, F, G, Q, R, S$:

$$\begin{aligned}
r &= \frac{1}{144}\left[6(8B+2E+F+2Q)+29(2C+F)+35(D+F)-\tfrac{23}{2}G+\tfrac{93}{2}R+\tfrac{111}{2}S\right] \\
s &= \frac{1}{48}\left[-3(8B+2E+F+2Q)-10(2C+F)-48(D+F)+G-9R+\tfrac{27}{2}S\right] \\
t &= \frac{1}{144}\left[6(8B+2E+F+2Q)+8(2C+F)+14(D+F)-G+15R+24S\right] .
\end{aligned} \tag{4.2.24}$$

This result may equivalently be written in terms of the parameters $\mathcal{A}, \mathcal{B}, \mathcal{C}$:

$$\begin{aligned}
\mathcal{A} &= \frac{1}{18}\left[6(8B+2E+F+2Q)+8(2C+F)+14(D+F)-G+15R+24S\right] \\
\mathcal{B} &= \frac{1}{18}\left[24(8B+2E+F+2Q)+2(2C+F)+26(D+F)+11G+15R+51S\right] \\
\mathcal{C} &= \frac{1}{18}\left[3(8B+2E+F+2Q)+7(2C+F)+10(D+F)-2G+12R+\tfrac{33}{2}S\right] .
\end{aligned} \tag{4.2.25}$$

Thus we have for the coefficient β_b in the free field case according to (3.4.33):

$$\beta_b = -\frac{\pi^4}{512\times 18}\left[(8B+2E+F+2Q)+2(2C+F)+3(D+F)-\tfrac{1}{2}G+\tfrac{7}{2}R+5S\right] . \tag{4.2.26}$$

β_a is zero for the form considered here. This is a consequence of the vanishing of C_T to which β_a is proportional. When inserting the values (4.2.19) for the anomaly free form into (4.2.26), β_b vanishes as required. In four dimensions (4.2.3) thus contains the anomaly free form and the form yielding the anomaly originating from the topological term $-\beta_b\, G$. In the next section we are going to determine the second of these two forms by calculating the singular non-integrable contribution in $\langle T_{\mu\nu}(x) C_{\sigma\varepsilon\eta\rho}(y) C_{\alpha\gamma\delta\beta}(z)\rangle$ as $d \to 4$.

In four dimensions the two symmetry conditions (4.2.15) and (4.2.16) become linearly dependent as can been seen from (4.2.17). They reduce to

$$\mathcal{S}_1\big|_{d=4} = G - 3S + 2(D+F) = 0 . \tag{4.2.27}$$

Furthermore the vanishing of totally antisymmetric five index tensors implies that the conservation condition $\mathcal{T}_1 = 0$ (4.2.8) is automatically satisfied. The only remaining condition which has to be imposed in four dimensions to ensure conservation of the energy momentum tensor is

$$4\mathcal{T}_1 + \mathcal{T}_2 = -2(8B + 2E + F + 2Q) - 4(2C + F) - R - S - 2G = 0\,. \qquad (4.2.28)$$

4.3 Topological Contribution to the Trace Anomaly

In this section we relate the coefficient β_b of the topological contribution to the trace anomaly to the parameters $\mathcal{A}, \mathcal{B}, \mathcal{C}$ in the energy momentum tensor three point function. For this purpose we construct a counterterm which removes the non-integrable singular terms present in $\langle T_{\mu\nu}(x) T'_{\sigma\rho}(y) T'_{\alpha\beta}(z)\rangle$ for $d \to 4$, and whose trace is the anomalous term required by the Ward identity. The tensorial structure of this counterterm is discussed in 4.3.1, whereas its coefficient is determined in 4.3.2 using the regularisation method developed in section 3.3.

4.3.1 Tensorial Structure of the Anomalous Contribution to the Three Point Function $\langle T_{\mu\nu}(x) T_{\sigma\rho}(y) T_{\alpha\beta}(z)\rangle$

From section 2.4 we recall that the trace anomaly on curved space arises from counterterms necessary to regularise the effective action which generates the energy momentum tensor. To be in accord with the Ward identity (2.4.9), the trace of the counterterms necessary for regularising the three point functions involving the energy momentum tensor on flat space must agree with the expression obtained from functionally differentiating the anomaly on curved space as given by (2.4.2).

Let us assume that $\beta_a = 0$. Furthermore we assume that there is an action W_G whose functional derivative with respect to the metric generates the anomalous contribution to the energy momentum tensor proportional to the Euler density G. According to (2.3.1) and (2.4.2), the topological part of the conformal trace anomaly is given by

$$g^{\mu\nu}\langle T_{\mu\nu}(x)\rangle_G = -g^{\mu\nu}(x)\frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}(x)}W_G = -\beta_b G(x)\ , \tag{4.3.1}$$

with $G(x)$ the Gauß-Bonnet invariant as in (2.4.2), such that on flat space

$$\langle T_{\mu\mu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle_G = -4\,\beta_b\,\mathcal{A}_{\sigma\rho,\alpha\beta}(x-y,x-z) \tag{4.3.2}$$

with $\mathcal{A}_{\sigma\rho,\alpha\beta}(x-y,x-z)$ defined in (2.4.11). All other contributions to the Ward identity (2.4.9) are zero for $\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle$ if we impose $C_T \propto \beta_a = 0$.

We now show that it is possible to construct a counterterm for the non-integrable singularities in $\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle$ which leads the G-anomaly in (4.3.2). In the framework of dimensional regularisation, this counterterm is given by a $1/\varepsilon$ pole, which must be a purely local term. The simplest tensorial expression with these properties may be obtained from functional derivatives of $\int \mathrm{d}^d x\sqrt{g}G$, the Gauß-Bonnet integral. For general dimensions the first variation

$$\begin{aligned} H_{\mu\nu}(x) &\equiv \frac{1}{\sqrt{g(x)}}\frac{\delta}{\delta g^{\mu\nu}(x)}\int \mathrm{d}^d x\ \sqrt{g}\,G \\ &= -\,15\,R^{\alpha\beta}{}_{[\alpha\beta}R^{\gamma\delta}{}_{\gamma\delta}g_{\mu]\nu} \end{aligned} \tag{4.3.3}$$

yields [6]

$$H_{\mu\nu} \;=\; 2R^{\alpha\beta\gamma}{}_{\mu}R_{\alpha\beta\gamma\nu} - 4R^{\alpha\beta}R_{\alpha\mu\beta\nu} - 4R^{\alpha}{}_{\mu}R_{\alpha\nu} + 2RR_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}G\ . \tag{4.3.4}$$

This vanishes in four dimensions where the Gauß-Bonnet integral is a topological invariant, as may be seen from the vanishing of totally antisymmetric five index tensors in four dimensions. $H_{\mu\nu}$ is a direct analogue of the Einstein tensor $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. It satisfies

$$g^{\mu\nu}H_{\mu\nu} = \tfrac{1}{2}\varepsilon G\,, \qquad \nabla^{\mu}H_{\mu\nu} = 0\,. \tag{4.3.5}$$

We may construct a fully symmetric counterterm compatible with the Bose symmetry of the energy momentum tensor three point function by considering functional derivatives of the expression for $H_{\mu\nu}$ given by (4.3.4), which yields

$$\begin{aligned} D^G_{\mu\nu\sigma\rho\alpha\beta}(x,y,z) &\equiv \frac{\delta^3}{\delta g^{\mu\nu}(x)\delta g^{\sigma\rho}(y)\delta g^{\alpha\beta}(z)} \int \mathrm{d}^d x \sqrt{g} G \Big|_{g=\delta} \\ &= \frac{\delta^2}{\delta g^{\sigma\rho}(y)\delta g^{\alpha\beta}(z)} H_{\mu\nu}(x)\Big|_{g=\delta} \\ &= -30 \left[\mathcal{E}^{(5)}_{\mu\sigma\alpha\kappa\varepsilon,\nu\rho\beta\lambda\eta} \partial_\kappa \partial_\lambda \delta^d(x-y) \partial_\varepsilon \partial_\eta \delta^d(x-z) + \sigma \leftrightarrow \rho\,,\ \alpha \leftrightarrow \beta \right] \end{aligned} \tag{4.3.6}$$

in general dimensions. $\mathcal{E}^{(5)}_{\mu\sigma\alpha\kappa\varepsilon,\nu\rho\beta\lambda\eta}$ denotes the projection operator onto totally antisymmetric five index tensors. From its definition $D^G_{\mu\nu\sigma\rho\alpha\beta}(x,y,z)$ is manifestly symmetric. It is also easy to see that $\partial^x{}_\mu D^G_{\mu\nu\sigma\rho\alpha\beta}(x,y,z) = 0$ such that the original derivative Ward identity is preserved.

An appropriately regularised expression for the energy momentum tensor three point function is therefore given by

$$\begin{aligned} \langle T_{\mu\nu}(x) T_{\sigma\rho}(y) T_{\alpha\beta}(z) \rangle_r &= \frac{\mathcal{I}^T_{\sigma\rho,\sigma'\rho'}(y-x)\mathcal{I}^T_{\alpha\beta,\alpha'\beta'}(z-x)}{(y-x)^{2d}(z-x)^{2d}} t^{TTT}_{\mu\nu\sigma'\rho'\alpha'\beta'}(X_{23}) \\ &\quad - 8\,\beta_b \frac{\mu^{-\varepsilon}}{\varepsilon} D^G_{\mu\nu\sigma\rho\alpha\beta}(x,y,z)\,. \end{aligned} \tag{4.3.7}$$

This counterterm generates the anomaly expected from the anomalous Ward identity (4.3.2) since

$$D^G_{\mu\mu\sigma\rho\alpha\beta}(x,y,z) = \tfrac{1}{2}\varepsilon \mathcal{A}^G_{\sigma\rho\alpha\beta}(x-y,x-z)\,. \tag{4.3.8}$$

This discussion displays the origin of the anomaly in a very clear way: The trace of the counterterm necessary to regularise the three point function gives a factor of ε which cancels the pole in ε arising from dimensional regularisation such that there remains a finite term when the limit $d \to 4$ is taken at the end of the calculation. It remains necessary to determine β_b in terms of the parameters in $t^{TTT}_{\mu\nu\sigma\rho\alpha\beta}(X)$.

There is an equivalent counterterm for $\langle T_{\mu\nu}(x) T'_{\sigma\rho}(y) T'_{\alpha\beta}(z) \rangle$ as defined in (4.2.3). To see this we note that $H_{\mu\nu}$ may be written as

$$H_{\mu\nu}(x) = H'_{\mu\nu}(x) + \varepsilon X_{\mu\nu}(x)\,, \tag{4.3.9}$$

$$\begin{aligned} H'_{\mu\nu} &= -\,15\, C^{\alpha\beta}{}_{[\alpha\beta} C^{\gamma\delta}{}_{\gamma\delta} g_{\mu]\nu} = 2\left(C^{\alpha\beta\gamma}{}_{\mu} C_{\alpha\beta\gamma\nu} - \tfrac{1}{4} g_{\mu\nu} C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta} \right)\,, \\ X_{\mu\nu} &= \tfrac{4}{d-2}\, R^{\gamma\delta} C_{\gamma\mu\delta\nu} + 2\tfrac{d-3}{(d-2)^2}(2R^{\gamma}{}_{\mu} R_{\gamma\nu} - g_{\mu\nu} R^{\gamma\delta} R_{\gamma\delta}) \\ &\qquad - \tfrac{1}{2}\tfrac{d-3}{(d-2)^2(d-1)}(4dRR_{\mu\nu} - (d+2) g_{\mu\nu} R^2)\,. \end{aligned} \tag{4.3.10}$$

We see that $\varepsilon X_{\mu\nu}(x)$ does not contribute to the residue of a simple pole in ε. Non-integrable singularities in $\langle T_{\mu\nu}(x) T'_{\sigma\rho}(y) T'_{\alpha\beta}(z)\rangle$ may thus be equivalently removed by a counterterm whose tensorial structure is given by

$$\frac{\delta}{\delta g^{\sigma\rho}(y)} \frac{\delta}{\delta g^{\alpha\beta}(z)} H'_{\mu\nu}(x) = 16 H^{(4)}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}\, \partial_\varepsilon \partial_\eta \delta^d(x-y) \partial_\gamma \partial_\delta \delta^d(x-z)\,, \tag{4.3.11}$$

with

$$H^{(4)}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta} \;\equiv\; \mathcal{E}^{C}_{\sigma\varepsilon\eta\rho,(\mu|\sigma'\rho'\nu'} \mathcal{E}^{C}_{\alpha\gamma\delta\beta,\nu)\sigma'\rho'\nu'} - \tfrac{1}{4}\delta_{\mu\nu} \mathcal{E}^{C}_{\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}\,. \tag{4.3.12}$$

Here we have discarded terms $\mathcal{O}(\varepsilon)$ and used that in an expansion around flat space

$$C_{\alpha\gamma\delta\beta} = 2\, \mathcal{E}^{C}_{\alpha\gamma\delta\beta,\mu\sigma\rho\nu} \partial_\sigma \partial_\rho \delta g_{\mu\nu}\,. \tag{4.3.13}$$

It is interesting to note that in $d = 4 - \varepsilon$ dimensions the trace of $H^{(4)}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}$ is

$$H^{(4)}_{\mu\mu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta} = \tfrac{1}{4}\varepsilon\, \mathcal{E}^{C}_{\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}\,. \tag{4.3.14}$$

Hence an equivalent counterterm for $\langle T_{\mu\nu}(x) T'_{\sigma\rho}(y) T'_{\alpha\beta}(z)\rangle$ is given by

$$-\,\beta_b\, \frac{\mu^{-\varepsilon}}{\varepsilon}\, 16\; H^{(4)}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}\, \partial_\varepsilon \partial_\eta \delta^d(x-y) \partial_\gamma \partial_\delta \delta^d(x-z) \tag{4.3.15}$$

although it does not satisfy the necessary symmetry conditions for $\langle T_{\mu\nu}(x) T_{\sigma\rho}(y) T_{\alpha\beta}(z)\rangle$. This is due to the fact that the tensor in (4.3.15) is required to be explicitly traceless in the indices $(\sigma\rho)$ and $(\alpha\beta)$ through the replacement $T_{\sigma\rho} \to T'_{\sigma\rho}$, which is not the case in the original Ward identity (4.3.2). Moreover the derivative with respect to x_μ of (4.3.15) is no longer zero as is the case for $\langle T_{\mu\nu}(x) T_{\sigma\rho}(y) T_{\alpha\beta}(z)\rangle$ where from the Ward identity (2.3.14) with $\beta_a \propto C_T = 0$ we have $\partial^x{}_\mu \langle T_{\mu\nu}(x) T_{\sigma\rho}(y) T_{\alpha\beta}(z)\rangle = 0$. However for the form (4.3.15) this is no longer valid since the conservation equation $\nabla^\mu H_{\mu\nu} = 0$ (4.3.5) follows from the Bianchi identity $R_{\alpha\beta[\gamma\delta;\varepsilon]} = 0$, which does not hold for the Weyl tensor in $H'_{\mu\nu}$.

4.3.2 Analysis of the Non-Integrable Singularities and Their Associated Anomalies

In this section we relate the anomaly coefficient β_b to the three parameters $\mathcal{A}, \mathcal{B}, \mathcal{C}$ in $\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle$. To this effect we determine the non-integrable singularities in the three point function $\langle T_{\mu\nu}(x)C_{\sigma\varepsilon\eta\rho}(y)C_{\alpha\gamma\delta\beta}(z)\rangle$ by following the same procedure as developed in section 3.3: First we analyse the subdivergences which arise for two of the three points coincident in order to eliminate them by a suitable choice of parameters. Then we contract the three point function in a suitable way and calculate the residue of the pole in $\varepsilon = 4 - d$ for all three points coincident. The detailed expressions arising in the calculation are however more complicated here than in section 3.3 due to the larger number of indices.

To check our procedure for finding the non-integrable singularities of tensorial expression by contracting them in a suitable way and subsequently applying results for scalars, we find the singular part of

$$\frac{\mathcal{I}^C_{\mu\sigma\rho\nu,\alpha\beta\gamma\delta}(x)}{x^d} \tag{4.3.16}$$

with $\mathcal{I}^C_{\mu\sigma\rho\nu,\alpha\beta\gamma\delta}(x)$ the inversion on the space of Weyl tensors gin by (3.5.3). Using standard methods such as (3.3.30) we find

$$\frac{\mathcal{I}^C_{\mu\sigma\rho\nu,\alpha\beta\gamma\delta}(x)}{x^{d+\omega}} \sim -\frac{S_d}{\omega}\delta^d(x)\frac{(d-4)(d-2)}{d(d+2)}\mathcal{E}^C_{\mu\sigma\rho\nu,\alpha\beta\gamma\delta} \tag{4.3.17}$$

using FORM. The same result is obtained using the contractions

$$\mathcal{I}^C_{\varepsilon\eta\kappa\lambda,\varepsilon\eta\kappa\lambda}(x) = \tfrac{1}{12}(d-4)(d-3)(d-2)(d+1) \qquad , \tag{4.3.18}$$

$$\mathcal{E}^C_{\varepsilon\eta\kappa\lambda,\varepsilon\eta\kappa\lambda}(x) = \tfrac{1}{12}(d-3)(d+2)(d+1)d \tag{4.3.19}$$

and by applying (3.3.26) to the resulting scalar expression x^{-d}. The tensorial structure of the right hand side of (4.3.17) is determined in this case by the fact that $\mathcal{E}^C$ is the unique invariant tensor with the required symmetries.

The analysis of the singularities in the three point function $\langle T_{\mu\nu}(x)C_{\sigma\varepsilon\eta\rho}(y)C_{\alpha\gamma\delta\beta}(z)\rangle$ for two points becoming coincident is performed by applying these techniques to the short distance limit, which for $x \to y$ with $s \equiv x - y$ is given by

$$\langle T_{\mu\nu}(x) C_{\sigma\varepsilon\eta\rho}(y) C_{\alpha\gamma\delta\beta}(z)\rangle \sim \hat{t}^{TCC}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha'\gamma'\delta'\beta'}(s) \, \frac{\mathcal{I}^{C}_{\alpha\gamma\delta\beta,\alpha'\gamma'\delta'\beta'}(z-y)}{(z-y)^{2(d-2)}} \,, \tag{4.3.20}$$

with $\hat{t}^{TCC}$ given by (4.2.2). We obtain for the divergent part of $\hat{t}^{TCC}(s)$ arising from the singularity at $s=0$ in (4.3.20), introducing $s^{-\omega}$ for regularisation,

$$s^{-\omega}\hat{t}^{TCC}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}(s) \sim \frac{S_d}{\omega}\, U \, H^{(d)}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}\delta^d(s) \,, \tag{4.3.21}$$

with

$$H^{(d)}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta} = \mathcal{E}^{C}_{\sigma\varepsilon\eta\rho,(\mu|\sigma'\rho'\nu'}\mathcal{E}^{C}_{\alpha\gamma\delta\beta,\nu)\sigma'\rho'\nu'} - \frac{1}{d}\delta_{\mu\nu}\mathcal{E}^{C}_{\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta} \,, \tag{4.3.22}$$

which is the unique invariant tensor with the symmetries of the three point function $\langle T_{\mu\nu}(x) C_{\sigma\varepsilon\eta\rho}(y) C_{\alpha\gamma\delta\beta}(z)\rangle$ in d dimensions. U is given by

$$\begin{aligned} U &= \frac{1}{d(d+2)(d+4)}\Big(-(d-4)(d+4)(d-2)A + 16(d-2)B + (d-4)(d+4)(C+D) \\ &\qquad +2(d-8)(E+Q) + 3dF - \tfrac{3}{2}(d+4)G - 6(R+S)\Big) \\ &= -2(d-8)\mathcal{T}_1 + 6\mathcal{T}_2 \,, \end{aligned} \tag{4.3.23}$$

where $\mathcal{T}_1, \mathcal{T}_2$ are defined in (4.2.8) and (4.2.9). If U is chosen to be zero, then there is no subdivergence when the two points x, y or x, z are coincident. This may be imposed as a constraint on the coefficients $A, B, C, D, E, F, G, Q, R, S$. The constraint $U = 0$ is linearly dependent on the two symmetry conditions (4.2.15) and (4.2.16), as may be seen from (4.2.17). In strictly four dimensions this condition is not necessary since $H^{(d)}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}$ is zero from the vanishing of totally antisymmetric five index tensors (cf. section 4.2). However it has to be imposed in order to allow for dimensional regularisation where d is kept as a variable until the subtraction of the ε pole. Note that since the three point function is $\mathcal{O}((y-z)^{d-4})$, there are no subdivergences when y, z become coincident.

Let us now turn to the calculation of the singular part of $\langle T_{\mu\nu}(x) C_{\sigma\varepsilon\eta\rho}(y) C_{\alpha\gamma\delta\beta}(z)\rangle$ which arises when all three points are coincident. Since $T_{\mu\nu}$ has dimension d and $C_{\alpha\gamma\delta\beta}$ has dimension $d-2$ such that

$$\langle T_{\mu\nu}(ax) C_{\sigma\varepsilon\eta\rho}(ay) C_{\alpha\gamma\delta\beta}(az)\rangle = \frac{1}{a^{3d-4}}\langle T_{\mu\nu}(x) C_{\sigma\varepsilon\eta\rho}(y) C_{\alpha\gamma\delta\beta}(z)\rangle \tag{4.3.24}$$

for the three point function, the singular behaviour as $d \to 4$ according to (3.3.25) with $q = 3d-4$ and $p = 2d$ must involve delta functions without derivatives. Then the tensorial structure requires that

$$\langle T_{\mu\nu}(x) C_{\sigma\varepsilon\eta\rho}(y) C_{\alpha\gamma\delta\beta}(z)\rangle \sim \frac{1}{\varepsilon} K H^{(d)}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta} \delta^d(x-y)\delta^d(x-z) \ . \tag{4.3.25}$$

K is a constant which remains to be determined and $H^{(d)}$ is given by (4.3.22) again.

A convenient way of contracting the three point function to avoid complicated tensorial expressions is

$$\delta_{\mu\sigma}\delta_{\nu\alpha}\delta_{\varepsilon\gamma}\delta_{\eta\delta}\delta_{\rho\beta}\langle T_{\mu\nu}(x) C_{\sigma\varepsilon\eta\rho}(y) C_{\alpha\gamma\delta\beta}(z)\rangle \ . \tag{4.3.26}$$

This contraction yields a sum of scalar three point functions of the form (3.3.27) in which the exponents λ_i are of the form $\lambda_i = d/2 + n_i$ where the n_i are positive or negative integers such that the λ_i satisfy

$$\sum_1^3 \lambda_i = \tfrac{1}{2}(3d-4) \ . \tag{4.3.27}$$

After the contraction, we denote the ten terms in (4.2.6) by $a_i(x,y,z), i = 1\ldots 10$ in the order they are listed, and the ten coefficients $A, B, C, D, E, F, G, Q, R, S$ accordingly by $A_i, i = 1\ldots 10$. The explicit form for the $a_i(x,y,z)$ may be found in the appendix A.6.

Since the formula (3.3.27) is singular if $\lambda_i = d/2 + n_i$ with $n_i = 0, 1, 2\ldots$, we regularise the subdivergences in $(x-y)^{-2\lambda_1}(x-z)^{-2\lambda_2}(y-z)^{-2\lambda_3}$ by letting

$$\lambda_i \to \lambda_i + \tfrac{1}{2}\omega_i \ , \tag{4.3.28}$$

with three small but finite quantities $\omega_1, \omega_2, \omega_3$. The sum of the three exponents is now

$$\sum_1^3 \lambda_i = \tfrac{1}{2}(3d - 4 + \omega_1 + \omega_2 + \omega_3) \ . \tag{4.3.29}$$

Later we let $\omega \to 0$. We now apply (3.3.27) to $\langle T_{\mu\nu}(x) C_{\mu\gamma\delta\beta}(y) C_{\nu\gamma\delta\beta}(z)\rangle$ which yields

$$\begin{aligned}(x-z)^{-\omega_1}(y-z)^{-\omega_2}(x-y)^{-\omega_3}&\langle T_{\mu\nu}(x) C_{\mu\gamma\delta\beta}(y) C_{\nu\gamma\delta\beta}(z)\rangle \\ &\sim \frac{2}{\varepsilon - \sum\omega_i}\, \mathcal{R}(\omega_1,\omega_2,\omega_3)\, \delta^d(x-y)\delta^d(x-z)\end{aligned} \tag{4.3.30}$$

with the residue

$$\mathcal{R}(\omega_1,\omega_2,\omega_3) = \frac{\pi^4}{\Gamma(d/2)} \left(\prod_i \frac{\Gamma(-\omega_i/2)}{\Gamma(1/2(d+\omega_i))} \right) \frac{\sum_{j=1}^{10} A_j b_j(\omega_1,\omega_2,\omega_3)}{H(\sum \omega_i)} \tag{4.3.31}$$

evaluated at the pole $d = 4 - \sum \omega_i$. The $b_i(\omega_1,\omega_2,\omega_3)$ are functions obtained by applying (3.3.27) to each of the scalar three point functions $a_i(x,y,z)$. The division by $H(\sum \omega_i)$ accounts for the contraction of $H^{(d)}$ defined in (4.3.22) which is given by

$$H^{(d)}_{\mu\nu,\mu\gamma\delta\beta,\nu\gamma\delta\beta} = \tfrac{1}{96}\tfrac{1}{(d-2)}(d-1)(d-3)(d-4)(d+4)(d+2)(d+1) \equiv H(\varepsilon)\ . \tag{4.3.32}$$

When $d \approx 4$ this reduces to

$$H(\varepsilon) \sim -\tfrac{15}{4}\,\varepsilon\ . \tag{4.3.33}$$

The factor of $(\sum \omega_i)$ in H in (4.3.31) cancels a corresponding factor present in the functions b_j. As opposed to the expected form for the local limit (4.3.25), there now seem to be poles in ω_i in the expression (4.3.31) arising from the Gamma functions. This is however only apparent. In the following it is shown how these poles cancel.

The expression (3.3.27) for the non-integrable singular contributions to the three point functions has a simple pole at $d = \sum \lambda_i$ which appears here, in the limit $d \to 4$, as a simple pole at $\varepsilon = \sum \omega_i$ in (4.3.30). The residue $\mathcal{R}(\omega_1,\omega_2,\omega_3)$ is defined unambiguously only if evaluated at this pole. In this case each of the $b_i(\omega_1,\omega_2,\omega_3)$ has an overall factor of ω_3 which cancels the pole arising from $\Gamma(-\omega_3/2)$ in (4.3.31) such that the limit $\omega_3 \to 0$ can be taken unambiguously. In this form, the $b_i(\omega_1,\omega_2,0)$ are listed in appendix A.7. Furthermore, if we impose (4.3.23) to vanish such that there are no two-point subdivergences, $\sum_j A_j b_j(\omega_1,\omega_2,0)$ is also proportional to $\omega_1\omega_2$ such that the remaining poles in (4.3.31) cancel as well and we are able to take the limit $\omega_i \to 0$. In this limit, which also implies $d \to 4$ for the residue, we obtain

$$K = \lim_{\omega_i \to 0} \frac{2\times 4}{15}\,\frac{\mathcal{R}(\omega_1,\omega_2,\omega_3)}{\sum \omega_i}\ , \tag{4.3.34}$$

for K in (4.3.25), where the constraint $U = 0$ for U in (4.3.23) has to be imposed. This calculation is carried out using FORM and gives

$$K = -\frac{\pi^4}{4\times 18}\Big[(8B + 2E + F + 2Q) + 2(2C+F) + 3(D+F) - \tfrac{1}{2}G + \tfrac{7}{2}R + 5S\Big]\ , \tag{4.3.35}$$

subject to using the constraint $U = 0$ to eliminate one coefficient, for example $S = \frac{1}{3}[16B - 4E - 4Q + 6F - 6G - 3R]$.

We are now able to write down a regularised expression for the conformally invariant three point function (4.2.1) which is well defined as a distribution even in the limit $d \to 4$ by subtracting the pole (4.3.25) with K as in (4.3.35):

$$\begin{aligned}\langle T_{\mu\nu}(x) C_{\sigma\varepsilon\eta\rho}(y) C_{\alpha\gamma\delta\beta}(z)\rangle_r \\ = \frac{\mathcal{I}^C_{\sigma\varepsilon\eta\rho,\sigma'\varepsilon'\eta'\rho'}(y-x)\mathcal{I}^C_{\alpha\gamma\delta\beta,\alpha'\gamma'\delta'\beta'}(z-x)}{(y-x)^{2(d-2)}(z-x)^{2(d-2)}} t^{TCC}_{\mu\nu,\sigma'\varepsilon'\eta'\rho',\alpha'\gamma'\delta'\beta'}(X_{23}) \\ -\mu^{-\varepsilon}\frac{K}{\varepsilon}\, H^{(4)}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}\, \delta^d(x-y)\delta^d(x-z)\,, \end{aligned} \tag{4.3.36}$$

where we have used the tensor $H^{(4)}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}$ defined in (4.3.12) instead of $H^{(d)}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}$ as in (4.3.25): To $\mathcal{O}(\varepsilon)$, both tensors are equivalent.

From the discussion of the tensorial structure of the singular contributions in the previous section follows that given the result (4.3.36) for the regularised three point function $\langle T_{\mu\nu}(x) C_{\sigma\varepsilon\eta\rho}(y) C_{\alpha\gamma\delta\beta}(z)\rangle_r$, the regularised energy momentum tensor three point function is given by

$$\begin{aligned}\langle T_{\mu\nu}(x) T'_{\sigma\rho}(y) T'_{\alpha\beta}(z)\rangle_r &= \frac{\mathcal{I}^T_{\sigma\rho,\sigma'\rho'}(y-x)\mathcal{I}^T_{\alpha\beta,\alpha'\beta'}(z-x)}{(y-x)^{2d}(z-x)^{2d}} t^{TT'T'}_{\mu\nu\sigma'\rho'\alpha'\beta'}(X_{23}) \\ &\quad - \mu^{-\varepsilon}\frac{K}{\varepsilon}\,\frac{1}{16}\, D^G_{\mu\nu\sigma\rho\alpha\beta}(x,y,z)\,. \end{aligned} \tag{4.3.37}$$

We may now use the relations (4.2.25) between the coefficients of $\langle T_{\mu\nu}(x) C_{\sigma\varepsilon\eta\rho}(y) C_{\alpha\gamma\delta\beta}(z)\rangle$ and $\langle T_{\mu\nu}(x) T_{\sigma\rho}(y) T_{\alpha\beta}(z)\rangle$ to express K as given by (4.3.35) in terms of the parameters $\mathcal{A}, \mathcal{B}, \mathcal{C}$ in the energy momentum tensor three point function defined in (3.4.7):

$$K = \frac{\pi^4}{4\times 90}(13\mathcal{A} - 2\mathcal{B} - 40\mathcal{C})\,. \tag{4.3.38}$$

Furthermore the comparison between (4.3.7) and (4.3.37) yields

$$\beta_b = \frac{K}{128}\,, \tag{4.3.39}$$

such that we have

$$\beta_b = \frac{\pi^4}{512\times 90}(13\mathcal{A} - 2\mathcal{B} - 40\mathcal{C})\,. \tag{4.3.40}$$

This relation holds for general conformally invariant theories, which may also be interacting. It agrees with the result (3.4.33) for free field theories which provides a consistency check.

4.3.3 Anomaly for the Tensor with Weyl Symmetry

So far the tensor $C_{\mu\kappa\lambda\nu}$ has only been introduced as an auxiliary field, such that $T'_{\mu\nu} = \partial_\kappa\partial_\lambda C_{\mu\kappa\lambda\nu}$, to enable us to regularise the energy momentum tensor three point function. In this section we discuss some issues which arise if we attribute a definite quantum field theoretical interpretation to $C_{\mu\kappa\lambda\nu}$.

We consider a tensor $C_{\mu\kappa\lambda\nu}$ of dimension $d-2$ and with Weyl symmetry as in (2.2.5), which is independent of the metric. For such a tensor we may introduce a source $\mathcal{C}_{\mu\kappa\lambda\nu}$ such that

$$-\frac{1}{\sqrt{g(x)}}\,\frac{\delta}{\delta\mathcal{C}^{\mu\kappa\lambda\nu}(x)}\,W[g,\mathcal{C}] = \langle C_{\mu\kappa\lambda\nu}(x)\rangle\,, \tag{4.3.41}$$

for an action $W[g,\mathcal{C}]$ depending both on the metric and on the source $\mathcal{C}_{\mu\kappa\lambda\nu}$. We have

$$\frac{\delta}{\delta\mathcal{C}^{\mu\kappa\lambda\nu}(x)}\mathcal{C}^{\sigma\varepsilon\eta\rho}(y) = \mathcal{E}^{C}_{\mu\kappa\lambda\nu,}{}^{\sigma\varepsilon\eta\rho}\delta^d(x-y)\,. \tag{4.3.42}$$

Using (4.3.41) we may construct a three point function

$$\begin{aligned}&\langle T_{\mu\nu}(x)C_{\sigma\varepsilon\eta\rho}(y)C_{\alpha\gamma\delta\beta}(z)\rangle\\ &\qquad = -2\,\frac{\delta}{\delta g^{\mu\nu}(x)}\frac{\delta}{\delta\mathcal{C}^{\sigma\varepsilon\eta\rho}(y)}\frac{\delta}{\delta\mathcal{C}^{\alpha\gamma\delta\beta}(z)}\,W[g,\mathcal{C}]\Big|_{g=\delta,\,\mathcal{C}=0}\,.\end{aligned} \tag{4.3.43}$$

In four dimensions we expect an anomaly

$$\delta_\sigma W[g,\mathcal{C}] = \frac{1}{2}\,\beta_{\mathcal{C}}\int \mathrm{d}^4x\sqrt{g}\,\sigma\,\mathcal{C}_{\mu\kappa\lambda\nu}\mathcal{C}^{\mu\kappa\lambda\nu} \tag{4.3.44}$$

for $\delta_\sigma g^{\mu\nu} = 2\sigma g^{\mu\nu}$, or equivalently

$$g^{\mu\nu}\langle T_{\mu\nu}(x)\rangle_{\mathcal{C}} = -\beta_{\mathcal{C}}\,\mathcal{C}_{\mu\kappa\lambda\nu}\mathcal{C}^{\mu\kappa\lambda\nu}\,. \tag{4.3.45}$$

By functionally differentiating this relation with respect to the source $\mathcal{C}$ we obtain

$$\langle T_{\mu\mu}(x)C_{\sigma\varepsilon\eta\rho}(y)C_{\alpha\gamma\delta\beta}(z)\rangle = -2\beta_{\mathcal{C}}\,\mathcal{E}^{C}_{\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}\delta^4(x-y)\delta^4(x-z)\,. \qquad (4.3.46)$$

By following the same procedure as in section 4.2, we may construct the possible forms in the three point function $\langle T_{\mu\nu}(x)C_{\sigma\varepsilon\eta\rho}(y)C_{\alpha\gamma\delta\beta}(z)\rangle$ and regularise it as calculated in section 4.3.2. Although the calculations are identical, it is important remember that here $C_{\mu\kappa\lambda\nu}$ is a field coupled to a source and independent of the metric, whereas in the previous sections $C_{\mu\kappa\lambda\nu}$ is defined only in the relation $T'_{\mu\nu} = \partial_\kappa\partial_\lambda C_{\mu\kappa\lambda\nu}$. Here we find

$$\begin{aligned}\langle T_{\mu\nu}(x)&C_{\sigma\varepsilon\eta\rho}(y)C_{\alpha\gamma\delta\beta}(z)\rangle_r \\ &= \frac{\mathcal{I}^{C}_{\sigma\varepsilon\eta\rho,\sigma'\varepsilon'\eta'\rho'}(y-x)\mathcal{I}^{C}_{\alpha\gamma\delta\beta,\alpha'\gamma'\delta'\beta'}(z-x)}{(y-x)^{2(d-2)}(z-x)^{2(d-2)}}t^{TCC}_{\mu\nu,\sigma'\varepsilon'\eta'\rho',\alpha'\gamma'\delta'\beta'}(X_{23}) \\ &\quad -\mu^{-\varepsilon}\frac{8\beta_{\mathcal{C}}}{\varepsilon}\,H^{(4)}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}\,\delta^d(x-y)\delta^d(x-z)\,, \qquad (4.3.47)\end{aligned}$$

with $H^{(4)}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}$ as in (4.3.12) and its trace given by (4.3.14). The trace of this regularised expression for the three point function agrees with (4.3.46). By comparison with with (4.3.36) we find

$$\begin{aligned}\beta_{\mathcal{C}} &= \frac{K}{8} \\ &= -\frac{\pi^4}{32\times 18}\left[(8B+2E+F+2Q)+2(2C+F)+3(D+F)-\tfrac{1}{2}G+\tfrac{7}{2}R+5S\right]\,, \end{aligned} \qquad (4.3.48)$$

with K given by (4.3.35), and where $S = \frac{1}{3}[16B-4E-4Q+6F-6G-3R]$ had to be imposed to remove subdivergences. Alternatively, $\beta_{\mathcal{C}}$ may be written as

$$\beta_{\mathcal{C}} = -\frac{\pi^4}{32\times 18}\left[-\tfrac{1}{2}(4\mathcal{T}_1+\mathcal{T}_2)-\tfrac{3}{2}\mathcal{S}_1+6(D+F)+3R\right] \qquad (4.3.49)$$

in terms of the the symmetry and conservation conditions (4.2.27) and (4.2.28). Imposing these conditions thus reduces $\beta_{\mathcal{C}}$ to

$$\beta_{\mathcal{C}} = -\frac{\pi^4}{32\times 6}\,[2(D+F)+R]\,. \qquad (4.3.50)$$

This derivation is valid for general conformal field theories.

With the definition of this section we may relate the coefficients in the three point function (4.2.1) given in (4.2.6), subject to the conservation conditions (4.2.8) and (4.2.9), to the scale of the two point function

$$\langle C_{\sigma\varepsilon\eta\rho}(x) C_{\alpha\gamma\delta\beta}(0)\rangle \;=\; C_C \frac{\mathcal{I}^C{}_{\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}(x)}{x^{2(d-2)}} \,, \tag{4.3.51}$$

by virtue of the Ward identity (2.3.18). From the short distance limit for the three point function $\langle T_{\mu\nu}(x) C_{\sigma\varepsilon\eta\rho}(y) C_{\alpha\gamma\delta\beta}(z)\rangle$ given by (4.3.20) and with $\hat{t}^{TCC}$ as defined in (4.2.2), we find for the symmetric and antisymmetric parts of the Ward identity

$$\int \mathrm{d}\Omega_{\hat{x}}\, \hat{x}_\mu \hat{x}_\nu \hat{t}^{TCC}_{\mu\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}(\hat{x}) = -(d-2) C_{C,s} \mathcal{E}^C_{\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta} \tag{4.3.52}$$

$$\int \mathrm{d}\Omega_{\hat{x}}\, \hat{x}_\nu \hat{x}_{[\mu} \hat{t}^{TCC}_{\omega]\nu,\sigma\varepsilon\eta\rho,\alpha\gamma\delta\beta}(\hat{x}) = -2 C_{C,a} \mathcal{E}^C_{\sigma\varepsilon\eta\rho,\phi\theta\chi[\mu} \mathcal{E}^C_{\omega]\phi\theta\chi,\alpha\gamma\delta\beta} \,, \tag{4.3.53}$$

where

$$\begin{aligned} C_{C,s} \;&=\; -S_d \frac{1}{d^2(d-2)(d+2)} \Big[-(d-1)(d-4)\,(2A+E-B+Q) \\ &\qquad + (d-4)C + \tfrac{3}{2}(d-1)(R+S) \\ &\qquad + \tfrac{1}{2}((5d-8)D + 3dF - 3G) \Big] \end{aligned} \tag{4.3.54}$$

$$C_{C,a} \;=\; -S_d \frac{d-1}{8d(d-2)(d+2)} \left[-2(d-4)(2A+E) + 6(D+F) + 3R \right] . \tag{4.3.55}$$

We therefore have

$$C_{C,s} - C_{C,a} = -\tfrac{3}{2} \frac{S_d}{d^2(d+2)(d-2)} \mathcal{S}_1 \tag{4.3.56}$$

with $\mathcal{S}_1$ given in (4.2.15). Hence when $\mathcal{S}_1 = 0$, $C_{C,s} = C_{C,a} = C_C$ are equal if the Ward identity is satisfied.

In four dimensions, subject to $\mathcal{S}_1 = 0$ as given by (4.2.27), the scale C_C is given by

$$C_C = -\tfrac{3}{64} \pi^2 \left[2(D+F) + R \right] . \tag{4.3.57}$$

Comparing the expression (4.3.50) for $\beta_{\mathcal{C}}$ with (4.3.57) for C_C, we find

$$\beta_{\mathcal{C}} = \frac{\pi^2}{9} C_C \,, \tag{4.3.58}$$

if the condition $\mathcal{S}_1 = 0$ is imposed as an extra condition. This is an interesting result since unitarity requires any two point function to be positive. Therefore C_C must be positive and thus (4.3.58) implies that β_C is positive as well. It is conceivable that these results may lead to a similar relation for β_b.

5 Effective Actions

In this chapter we specify further the action functional $W[g,\mathcal{A}]$ introduced in section 2.3 to derive Ward identities, where correlation functions involving the energy momentum tensor or the vector current V_μ are obtained by functionally differentiating $W[g,\mathcal{A}]$ with respect to the metric or to the gauge field $\mathcal{A}_\mu$ as in (2.3.1). In section 2.3 the action $W[g,\mathcal{A}]$ is a scalar invariant under diffeomorphisms and local Weyl rescalings $\delta_\sigma g^{\mu\nu} = 2\sigma g^{\mu\nu}$ as well as under $\delta_\sigma \mathcal{A}_\mu = 0$, such that the resulting quantum field theory on flat space is conformally invariant, apart from local anomalies which give rise to a non-zero trace of the energy momentum tensor.

In particular in view of describing the back-reaction of matter fields on the geometry within quantum field theories on a curved space background it would be useful to find a non-local covariant action which upon functional differentiation gives rise to the two and three point functions discussed in the previous sections, and which under local Weyl rescalings generates the terms in the conformal anomaly.

5.1 Conformal Action in Two Dimensions

In two dimensions an action generating the conformal anomaly was constructed by Polyakov some time ago [12]. In two dimensions, the trace anomaly may be integrated uniquely to give

$$W[g] = -\frac{c}{96\pi}\iint \mathrm{d}^2x\,\mathrm{d}^2y\,\sqrt{g(x)}R(x)G_\Delta(x,y)\sqrt{g(y)}R(y)\,,$$
$$\text{where}\quad \Delta_x G_\Delta(x,y) = \delta^2(x-y)\,,\quad \Delta = -\sqrt{g}\nabla^2\,. \tag{5.1.1}$$

By considering a Weyl rescaling of this non-local action, which is defined by $\delta_\sigma g^{\mu\nu} = 2\sigma g^{\mu\nu}$, we recover the two dimensional conformal anomaly

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle_g = \frac{c}{24\pi}R\,, \tag{5.1.2}$$

since under Weyl rescalings, Δ is invariant, $\delta_\sigma\Delta = 0$, and $\delta_\sigma\sqrt{g}R = -2\Delta\sigma$.

The second order operator $\Delta \equiv \Delta_{2,2}$ is the two dimensional version of the operator

$$\Delta_{d,2} = \sqrt{g}\left(-\nabla^2 + \frac{d-2}{4(d-1)}R\right) \tag{5.1.3}$$

acting on scalars in general dimension d, which transforms covariantly under conformal variations,

$$\delta_\sigma \Delta_{d,2} = -\tfrac{1}{2}\,(d-2)\,(\sigma\Delta_{d,2} + \Delta_{d,2}\sigma)\ . \tag{5.1.4}$$

There is an extensive discussion of d dimensional conformally covariant differential operators in the literature [27, 28].

Although $W[g]$ itself is zero on flat space where the curvature vanishes, second or higher functional derivatives of W are non-zero even on flat space. We may thus construct flat space two and three point functions by considering appropriate functional derivatives of W and taking $g^{\mu\nu} = \delta^{\mu\nu}$ at the end of the calculation. For the energy momentum tensor two point function we take two functional derivatives with respect to the metric and using $\delta_g\sqrt{g}R = -\sqrt{g}(\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^2)\delta g^{\mu\nu}$ and $G_0(x-y) \equiv G_\Delta(x,y)|_{g=\delta} = -\ln\mu^2(x-y)^2/4\pi$ we obtain

$$\begin{aligned} \langle T_{\mu\nu}(x)T_{\sigma\rho}(y)\rangle &= 4\frac{\delta^2}{\delta g^{\mu\nu}(x)\delta g^{\sigma\rho}(y)}\,W[g]\Big|_{g=\delta} \\ &= -\frac{c}{48\pi^2}S^x_{\mu\nu}S^y_{\sigma\rho}\ln(x-y)^2\,,\quad S_{\mu\nu} = \partial_\mu\partial_\nu - \delta_{\mu\nu}\partial^2\,. \end{aligned} \tag{5.1.5}$$

For complex coordinates $z = x_1 + ix_2$, $T(z) = -2\pi T_{zz}(x)$ this gives rise to the standard two dimensional conformal field theory result

$$\langle T(z_1)T(z_2)\rangle = \frac{c}{2}\frac{1}{(z_1-z_2)^4}\,. \tag{5.1.6}$$

This calculation shows that the conformal energy momentum tensor two point function on flat space is completely determined by the conformal anomaly on curved space.

Similarly we obtain the non-local part of the energy momentum tensor three point function. For the variation of G_Δ with the metric we have

$$\begin{aligned} \delta_g G_\Delta(x,y) &= -\int \mathrm{d}^2w\, G_\Delta(x-w)\delta_g\Delta^w G_\Delta(w-y) \\ \text{with}\ \ \delta_g\Delta &= -\partial_\mu\sqrt{g}\left(\delta g^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}g_{\sigma\rho}\delta g^{\sigma\rho}\right)\partial_\nu\ . \end{aligned} \tag{5.1.7}$$

Using this we find

$$\begin{aligned}\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(w)\rangle &= \frac{c}{6\pi}\Big(S^x_{\mu\nu}G_0(x-w)\overleftarrow{\partial}_{(\alpha}\, S^y_{\sigma\rho}G_0(y-w)\overleftarrow{\partial}_{\beta)} \\ &\quad -\tfrac{1}{2}\delta_{\alpha\beta}S^x_{\mu\nu}G_0(x-w)\overleftarrow{\partial}_{\gamma}\, S^y_{\sigma\rho}G_0(y-w)\overleftarrow{\partial}_{\gamma} \\ &\qquad +\text{permutations}\Big). \end{aligned} \tag{5.1.8}$$

Again this gives the standard two dimensional conformal field theory result in complex coordinates:

$$\begin{aligned}&\langle T(z_1)T(z_2)T(z_3)\rangle \\ &\quad = -\frac{c}{3}\left[\frac{1}{(z_1-z_3)^3(z_2-z_3)^3}+\frac{1}{(z_2-z_1)^3(z_3-z_1)^3}+\frac{1}{(z_1-z_2)^3(z_3-z_2)^3}\right] \\ &\quad = \frac{c}{(z_1-z_3)^2(z_2-z_3)^2(z_1-z_2)^2}. \end{aligned} \tag{5.1.9}$$

Like the two point function, the three point function on flat space is therefore fully determined by the conformal anomaly. The correlation functions (5.1.6) and (5.1.9) are in accord with standard conformal field theory results.

5.2 Discussion of the Riegert Action

In four dimensions a non-local action equivalent to the two dimensional case (5.1.1) in the sense that it generates the trace anomaly when functionally differentiated with respect to σ has been found by Riegert [14]. At the basis of this action is a fourth order conformally covariant differential operator acting on scalars. Riegert constructed this operator for four dimensions only. A d-dimensional version was found almost simultaneously by Paneitz [26]. Subsequently it was generalised to differential forms by Branson and others [27, 28]. Denoting the exterior derivative on forms by d and its adjoint by δ, such that $(\mathrm{d}\mathcal{O})_\mu = \nabla_\mu\mathcal{O}$ on functions and $(\delta V) = -\nabla^\mu V_\mu$ on vectors, the Laplacian on functions is given by $\delta\mathrm{d}$. Following Branson we define

$$J \equiv \frac{1}{2(d-1)}R \tag{5.2.1}$$

$$K_{\mu\nu} \equiv \frac{1}{(d-2)}\left(R_{\mu\nu}-Jg_{\mu\nu}\right) \tag{5.2.2}$$

with $R_{\mu\nu}$ and R the Ricci tensor and scalar, which transform under local Weyl rescalings as

$$\delta_\sigma J = 2\sigma J + \nabla^2\sigma\,, \quad \delta_\sigma K_\mu{}^\nu = 2\sigma K_\mu{}^\nu + \nabla_\mu\nabla^\nu\sigma\,. \tag{5.2.3}$$

$K_{\mu\nu}$ is the same tensor as used for the definition of the Weyl tensor in (2.4.4). With these definitions the second order conformal operator defined in (5.1.3) may be written as

$$\Delta_{d,2} = \sqrt{g}(\delta\mathrm{d} + \tfrac{1}{2}(d-2)J)\,. \tag{5.2.4}$$

The fourth order operator

$$\Delta_{d,4} = \sqrt{g}\left(\delta\mathrm{d}\delta\mathrm{d} \,+\, \delta L\mathrm{d} \,+ \tfrac{1}{2}(d-4)\,M\right) \tag{5.2.5}$$

acting on scalars with

$$(LV)_\mu \;\;\equiv\;\; (d-2)JV_\mu - 4K_\mu{}^\nu V_\nu \tag{5.2.6}$$

acting on vectors V_μ and

$$M \;\;\equiv\;\; \delta\mathrm{d}\,J + \tfrac{1}{2}dJ^2 - 2K^{\mu\nu}K_{\mu\nu} \tag{5.2.7}$$

is also conformally covariant in the sense that

$$\delta_\sigma\Delta_{d,4} = -\tfrac{1}{2}(d-4)\left(\sigma\Delta_{d,4} + \Delta_{d,4}\sigma\right)\,. \tag{5.2.8}$$

In four dimensions this reduces to the operator introduced by Riegert

$$\Delta^{\mathrm{R}} \equiv \Delta_{4,4} = \sqrt{g}\nabla^2\nabla^2 + \partial_\mu H^{\mu\nu}\partial_\nu\,, \quad H^{\mu\nu} = 2\sqrt{g}(R^{\mu\nu} - \tfrac{1}{3}g^{\mu\nu}R)\,, \tag{5.2.9}$$

which is conformally invariant, $\delta_\sigma\Delta^{\mathrm{R}} = 0$. Its Green function is defined by

$$\Delta^{\mathrm{R}}G^{\mathrm{R}}(x,y) = \delta^4(x-y)\,. \tag{5.2.10}$$

Furthermore the quantity

$$\mathcal{G} \equiv \sqrt{g}(G - \tfrac{2}{3}\nabla^2 R)\,, \tag{5.2.11}$$

with G the Gauß-Bonnet term defined in (2.4.3), plays an important role since its conformal variation is

$$\delta_\sigma\mathcal{G} = -4\Delta^{\mathrm{R}}\sigma\,, \tag{5.2.12}$$

and hence

$$\delta_\sigma\Big(\int \mathrm{d}^4y\ G^{\mathrm{R}}(x,y)\mathcal{G}(y)\Big) = 4\sigma(x)\,. \tag{5.2.13}$$

With these ingredients Riegert constructed the non-local action

$$\begin{aligned} & W_{\mathrm{Riegert}}[g,\mathcal{A}] \\ = & -\int \mathrm{d}^4x\,\mathrm{d}^4y\ \Big[\frac{\kappa}{4}\sqrt{g}F_{\mu\nu}F^{\mu\nu} + \beta_a\sqrt{g}C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + \frac{\beta_b}{2}\mathcal{G}\Big](x)\,G^{\mathrm{R}}(x,y)\,\frac{1}{4}\mathcal{G}(y) \\ & + \Big(h - \frac{2}{3}\beta_b\Big)\int \mathrm{d}^4x\,\sqrt{g}R^2(x)\,, \end{aligned} \tag{5.2.14}$$

from which the trace anomaly (2.4.2) may be derived using $\delta_\sigma(\sqrt{g}R^2) = 12\sqrt{g}R\nabla^2\sigma$ as well as $\delta_\sigma(\sqrt{g}F_{\mu\nu}F^{\mu\nu}) = \delta_\sigma(\sqrt{g}C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}) = 0$.

Just as in two dimensions we may obtain correlation functions involving the energy momentum tensor by functionally differentiating the action and then restricting to flat space. There are no contributions to the two point functions from this action from any of the terms. Thus this action violates the expected proportionality relations (3.3.48) between C_V and κ and (2.4.22) between C_T and and β_a. The term in the action (5.2.14) which gives rise to the topological term $-\beta_b\,G$ in the trace anomaly is

$$\begin{aligned} W_{\mathrm{Riegert},G}[g] &= -\frac{\beta_b}{8}\int \mathrm{d}^4x\mathrm{d}^4y\ \mathcal{G}(x)G^{\mathrm{R}}(x,y)\mathcal{G}(y) \\ &\quad + \frac{\beta_b}{18}\int \mathrm{d}^4x\,\sqrt{g}R^2\,, \end{aligned} \tag{5.2.15}$$

as may seen by integrating by parts and using the results above. (5.2.15) does not give rise to a two point function on flat space since the only non-zero second derivative on flat space is of the form

$$\begin{aligned} & \frac{\delta}{\delta g^{\mu\nu}(x)}\frac{\delta}{\delta g^{\sigma\rho}(y)}W^{(2)}{}_{\mathrm{Riegert},G}[g]\Big|_{g=\delta} \\ = & -\frac{\beta_b}{9}\int \mathrm{d}^4x'\mathrm{d}^4y'\ \nabla^2\,\frac{\delta}{\delta g^{\mu\nu}(x)}R(x')G^{\mathrm{R}}(x',y')\nabla^2\frac{\delta}{\delta g^{\sigma\rho}(y')}R(y')\Big|_{\text{flat space}} \\ & + \frac{\beta_b}{9}\int \mathrm{d}^4x'\,\sqrt{g}\frac{\delta}{\delta g^{\mu\nu}(x)}R(x')\frac{\delta}{\delta g^{\sigma\rho}(y)}R(x')\Big|_{\text{flat space}}\,. \end{aligned} \tag{5.2.16}$$

This vanishes since $\Delta^{\rm R}|_{\rm flat\ space} = (\partial^2)^2$ and hence

$$(\partial^x)^2(\partial^y)^2 G^R(x,y)\Big|_{g=\delta} = \delta^4(x-y)\,, \tag{5.2.17}$$

so that integrating by parts the two Laplacians in the first term of (5.2.16) gives the negative of the second term.

There is however a non-local contribution to the three point function from (5.2.15) which we may calculate using

$$\begin{aligned}&\delta_g\left(\iint \mathrm{d}^4x\mathrm{d}^4y\,\sqrt{g(x)}\sqrt{g(y)}\,\nabla^2X(x)G^{\rm R}(x,y)\nabla^2X(y) - \int \mathrm{d}^4x\,\sqrt{g}X^2\right)\Big|_{\rm flat\ space}\\ &\quad = \iiint \mathrm{d}^4x\mathrm{d}^4y\mathrm{d}^4z\;\delta g^{\alpha\beta}(z)\mathcal{D}^z{}_{\alpha\beta,\varepsilon\eta}\partial_\varepsilon G_0(z-x)X(x)\,\partial_\eta G_0(z-y)X(y)\,,\end{aligned} \tag{5.2.18}$$

where X is an arbitrary scalar whose dependence on the metric is disregarded and G_0 is the flat space restriction of $-\nabla^2 G^{\rm R}$,

$$G_0(x) = \frac{1}{4\pi^2x^2}\,. \tag{5.2.19}$$

$\mathcal{D}_{\sigma\rho,\varepsilon\eta}$ is a differential operator acting on any second rank tensor $X_{\varepsilon\eta}$, which is defined by

$$\begin{aligned}\int \mathrm{d}^4x\,X_{\varepsilon\eta}\delta_g H^{\varepsilon\eta}\Big|_{\rm flat\ space} &= -\int \mathrm{d}^4x\,\delta g^{\sigma\rho}\mathcal{D}_{\sigma\rho,\varepsilon\eta}X_{\varepsilon\eta}\,,\\ \mathcal{D}_{\sigma\rho,\varepsilon\eta}X_{\varepsilon\eta} = -\partial^2X_{\sigma\rho} - \delta_{\sigma\rho}\partial_\varepsilon\partial_\eta X_{\varepsilon\eta} &+ \partial_\sigma\partial_\varepsilon X_{\varepsilon\rho} + \partial_\rho\partial_\eta X_{\sigma\eta} + \tfrac{2}{3}(\delta_{\sigma\rho}\partial^2 - \partial_\sigma\partial_\rho)X_{\varepsilon\varepsilon}\,.\end{aligned} \tag{5.2.20}$$

Taking $X(x) = R(x)$ and differentiating (5.2.18) further with respect to $g^{\sigma\rho}(y)$ and to $g^{\mu\nu}(x)$ as well as substituting in

$$X_{\varepsilon\eta} = \frac{1}{(4\pi^2)^2}\partial^z{}_\varepsilon\frac{1}{(x-z)^2}\partial^z{}_\eta\frac{1}{(z-y)^2} \tag{5.2.21}$$

after restricting to flat space we obtain

$$\begin{aligned}\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle_{\rm Riegert} \;=\; -\tfrac{8}{9}\beta_b\Big\{&\mathcal{D}^z{}_{\alpha\beta,\gamma\delta}(S^x{}_{\mu\nu}G_0(x-z)\overleftarrow{\partial}^z{}_{(\gamma}\partial^z{}_{\delta)}G_0(z-y)\overleftarrow{S}^y{}_{\sigma\rho})\\ &+\text{cyclic permutations}\Big\}\end{aligned} \tag{5.2.22}$$

for the three point function arising from the action (5.2.15) for non-coincident points.

This result is not conformally invariant, as can be seen by considering the short distance limit $s = x - y \to 0$:

$$\langle T_{\mu\nu}(x) T_{\sigma\rho}(y) T_{\alpha\beta}(z)\rangle_{\text{Riegert}} \sim \tfrac{8}{27}\beta_b\, I_{\mu\nu\sigma\rho\gamma\delta}(s)\, \partial_\gamma\partial_\delta G_0(y-z)\overleftarrow{S}^z{}_{\alpha\beta}\,,$$
$$I_{\mu\nu\sigma\rho\gamma\delta}(s) = (4\mathcal{E}^T{}_{\sigma\rho,\lambda(\gamma}\partial_{\delta)}\partial_\lambda S_{\mu\nu} + 4\mathcal{E}^T{}_{\mu\nu,\lambda(\gamma}\partial_{\delta)}\partial_\lambda S_{\sigma\rho} - s_{(\gamma}\partial_{\delta)} S_{\mu\nu} S_{\sigma\rho}) G_0(s)\,. \quad (5.2.23)$$

Since $I_{\mu\nu\sigma\rho\gamma\delta}(s) = \mathcal{O}(s^{-6})$ this result is incompatible with what would be expected from a conformally covariant three point function, where according to the expressions obtained in section 3.4 the leading singularity should be $\mathcal{O}(s^{-4})$, reflecting the contribution of the energy momentum tensor itself in the operator product expansion of two energy momentum tensors.

The failure of the Riegert action to lead to results on flat space in agreement with conformal invariance is a consequence of its large distance behaviour. Diffeomorphism and Weyl invariance, up to the anomalies, impose the following constraint on the gravitational effective action:

$$\int \mathrm{d}^4x\, (\mathcal{L}_v g^{\mu\nu} + 2\sigma g^{\mu\nu}) \frac{\delta}{\delta g^{\mu\nu}}\, W[g] = \int \mathrm{d}^4x \sqrt{g}\, \sigma(\beta_a F + \beta_b G)\,, \quad (5.2.24)$$

with the Lie derivative

$$\mathcal{L}_v g^{\mu\nu} = v^\lambda \partial_\lambda g^{\mu\nu} - \partial_\lambda v^\mu g^{\lambda\nu} - \partial_\lambda v^\nu g^{\mu\lambda} = -\nabla^\mu v^\nu - \nabla^\nu v^\mu\,. \quad (5.2.25)$$

On flat space $(\mathcal{L}_v g^{\mu\nu} + 2\sigma g^{\mu\nu})|_{\text{flat space}} = 0$ is equivalent to the condition for conformal invariance (2.1.3) on v. By taking functional derivatives of (5.2.24) and restricting to flat space, we may derive conformal Ward identities including the anomalies such as (2.4.15) for the two point function. From (5.2.24) we may also derive

$$\nabla^\mu \langle T_{\mu\nu}\rangle = 0 \quad (5.2.26)$$

using the symmetry of the metric and the fact that v satisfies the condition (2.1.3) for conformal invariance but is undetermined otherwise. Vice versa it is possible to derive (5.2.24) from $\nabla^\mu\langle T_{\mu\nu}\rangle = 0$ and $g^{\mu\nu}\langle T_{\mu\nu}\rangle = -\beta_a\, F - \beta_b\, G$. However both these derivations necessitate the absence of surface terms when performing the integration by parts involved which is not the case for the Riegert action when $v(x) = \mathcal{O}(x^2)$. From (5.2.15) and (5.2.22) we see that the leading term, i.e. the term with the lowest power in x in the denominator, in the energy momentum tensor expectation value on asymptotically flat spaces is $\langle T_{\mu\nu}(x)\rangle_{g,\,\mathrm{Riegert}} = \mathcal{O}(|x|^{-5})$ for $|x| \to \infty$ since it involves $S_{\mu\nu}\partial^2 G^{\mathrm{R}}(x,y)\overleftarrow{\partial}_{\alpha}$. With this asymptotic behaviour it is necessary to restrict v to be $\mathcal{O}(|x|)$ to avoid surface terms in the integration by parts. Hence there is no longer any identity involving special conformal transformations for which $v(x) = \mathcal{O}(x^2)$ and thus conformal invariance is spoiled. Note that if κ, β_a are non-zero in the Riegert action (5.2.14), we expect $\langle T_{\mu\nu}(x)\rangle_{g,\,\mathrm{Riegert}} = \mathcal{O}(|x|^{-4})$, such that even scale invariance Ward identities are expected to be violated.

Calculating the exact contributions from the surface terms to the conformal identities for the Riegert action would involve the calculation of four point functions which we shall not undertake here. However it is possible to demonstrate the importance of surface terms by calculating their contribution to the conformal identities in the easier situation of two dimensions. In two dimensions with complex coordinates x_1, x_2, x_3 and the abbreviation $\delta(1,2) \equiv \delta^2(x_1 - x_2)$, the trace and derivative Ward identities (2.3.13) and (2.3.14) become

$$4\,\langle T_{\bar{z}z}(x_1)T_{zz}(x_2)T_{zz}(x_3)\rangle = 2\left(\delta(1,2) + \delta(1,3)\right)\langle T_{zz}(x_2)T_{zz}(x_3)\rangle\,, \tag{5.2.27}$$

$$\begin{aligned} &2\left[\partial_{\bar{z}_1}\langle T_{zz}(x_1)T_{zz}(x_2)T_{zz}(x_3)\rangle + \partial_{z_1}\langle T_{\bar{z}z}(x_1)T_{zz}(x_2)T_{zz}(x_3)\rangle\right] \\ &\qquad = \partial_{z_1}\delta(1,2)\langle T_{zz}(x_1)T_{zz}(x_3)\rangle + \partial_{z_1}\delta(1,3)\langle T_{zz}(x_2)T_{zz}(x_1)\rangle \\ &\qquad\qquad + 2\,\partial_{z_1}\left[\delta(1,2)\langle T_{zz}(x_2)T_{zz}(x_3)\rangle + \delta(1,3)\langle T_{zz}(x_2)T_{zz}(x_1)\rangle\right]\,, \end{aligned} \tag{5.2.28}$$

where we have used

$$\partial^\mu T_{\mu z} = \partial^z T_{zz} + \partial^{\bar{z}} T_{\bar{z}z} = 2\left(\partial_{\bar{z}} T_{zz} + \partial_z T_{\bar{z}z}\right)\,,\quad g^{\mu\nu}T_{\mu\nu} = 4T_{\bar{z}z}\,. \tag{5.2.29}$$

(5.2.29) implies $\partial_{\bar{z}} T_{zz}(x) = 0$ for conformal theories. Thus $T_{zz}(x)$ is analytic and we may define $T(z) = -2\pi T_{zz}(x)$ as before. Differentiating (5.2.27) with respect to $\partial^{\bar{z}_1} = 2\partial_{z_1}$ and subtracting the two Ward identities from each other we obtain in the new coordinates

$$\begin{aligned} \partial_{\bar{z}_1}\langle T(z_1)T(z_2)T(z_3)\rangle = -2\pi\Big[&\partial_{z_1}\left(\delta(1,2) + \delta(1,3)\right)\langle T(z_2)T(z_3)\rangle \\ &- \tfrac{1}{2}\delta(1,2)\partial_{z_1}\langle T(z_1)T(z_3)\rangle - \tfrac{1}{2}\delta(1,3)\partial_{z_1}\langle T(z_1)T(z_2)\rangle\Big]\,. \end{aligned} \tag{5.2.30}$$

There is a similar Ward identity for the component involving $T(\bar{z}) = -2\pi T_{\bar{z}\bar{z}}(x)$. Using $\partial_{\bar{z}} z^{-2} = -\pi \partial_z \delta^2(x)$ [23] we may verify that the conformal correlation functions (5.1.6) and (5.1.9) indeed satisfy the identity (5.2.30). When multiplying both sides of this identity with $v(z)$ and integrating over two dimensional space we find that by naively integrating by parts there is no contribution from the left hand side of (5.2.30), and for the right hand side we have

$$\begin{aligned} -2\pi \int \mathrm{d}^2 z_1 \;\; v(z_1) \;\; & \Big[\partial_{z_1} \left(\delta(1,2) + \delta(1,3) \right) \langle T(z_2) T(z_3) \rangle \\ & -\tfrac{1}{2} \delta(1,2) \partial_{z_1} \langle T(z_1) T(z_3) \rangle - \tfrac{1}{2} \delta(1,3) \partial_{z_1} \langle T(z_1) T(z_2) \rangle \Big] \\ = \quad & \pi \Big[\langle L_v T(z_2) T(z_3) \rangle + \langle T(z_2) L_v T(z_3) \rangle \Big] \,, \end{aligned} \tag{5.2.31}$$

where L_v is the generator of conformal transformations defined in (2.1.12) which when acting on $T(z)$ is given by

$$L_v T(z) = v(z) T'(z) + 2 v'(z) T(z) \,. \tag{5.2.32}$$

Thus we may expect

$$\langle L_v T(z_2) T(z_3) \rangle + \langle T(z_2) L_v T(z_3) \rangle = 0 \,. \tag{5.2.33}$$

By inserting the explicit expression for the two point function (5.1.6) into this Ward identity we may check that it is satisfied for $v(z) = \{1, z, z^2\}$. However for $v(z) = z^n$, $n \geq 3$ equation (5.2.33) is not true. For the case $v(z) = z^3$ for example we find

$$\langle L_v T(z_2) T(z_3) \rangle + \langle T(z_2) L_v T(z_3) \rangle = 2\pi \frac{c/2}{(z_2 - z_3)^2} \tag{5.2.34}$$

for the integral of the right hand side of (5.2.30) using the two point function (5.1.6). This discrepancy is due to the fact that, in this case, there is a surface term in the integration by parts of the left hand side of (5.2.30.) To calculate the surface contribution we may use the coordinates $z \equiv r \, \exp{(i\varphi)}$ and the explicit expression (5.1.9) for the three point function,

$$\begin{aligned}
\int \mathrm{d}^2z\ z^3 \partial_{\bar z} \langle T(z)T(z_2)T(z_3)\rangle &= \int \mathrm{d}^2 z\ z^3 \partial_{\bar z} \frac{c}{(z-z_3)^2(z_2-z_3)^2(z-z_2)^2} \\
&= \lim_{r\to\infty} \int_0^{2\pi} \mathrm{d}\varphi\, r\, r^3\, e^{3i\varphi} \frac{1}{2e^{-i\varphi}} \frac{1}{r^4 e^{4i\varphi}} \frac{c}{(z_2-z_3)^2} \\
&= 2\pi \frac{c/2}{(z_2-z_3)^2}, \qquad (5.2.35)
\end{aligned}$$

using

$$\frac{\partial}{\partial \bar z} = \frac{1}{2e^{-i\varphi}} \Big(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \varphi} \Big) . \qquad (5.2.36)$$

The results (5.2.34) and (5.2.35) for the integration of both sides of the Ward identity (5.2.30) with $v(z) = z^3$ coincide, which shows that the surface term (5.2.35) gives exactly the anomalous term in the conformal Ward identity (5.2.34). For the general case $v(z) = z^n$, $n \geq 3$ we choose $z_3 = 0$. Expanding $(z - z_2)^{-2}$ for $|z_2| < |z|$,

$$\frac{1}{(z-z_2)^2} = \sum_{k=0}^{\infty} k \left(\frac{z_2}{z}\right)^{k-1} , \qquad (5.2.37)$$

we find for the surface term

$$\int \mathrm{d}^2 z\ z^n \partial_{\bar z} \langle T(z)T(z_2)T(0)\rangle = \pi c(n-2) z_2^{n-5} . \qquad (5.2.38)$$

This agrees with

$$\begin{aligned}
\langle L_v T(z_2)T(0)\rangle\Big|_{z_3=0} &= \pi \left({z_2}^n \frac{\partial}{\partial z_2} + 2n {z_2}^{n-1} \right) \frac{c/2}{{z_2}^4} \\
&= \pi c(n-2) z_2^{n-5} , \qquad (5.2.39)
\end{aligned}$$

since $L_v T(0) = 0$ in this case.

For $v(z) = \{1, z, z^2\}$ however the conformal Ward identity (5.2.33) is anomaly free as we expect since $v(z) = \{1, z, z^2\}$ corresponds to the three generators $\mathrm{L}_{-1}, \mathrm{L}_0, \mathrm{L}_1$ in the mode expansion of $T_{\mu\nu}$ in which we have

$$\mathrm{L}_n = \oint \frac{\mathrm{d}z}{2\pi i}\, z^{n+1} T(z) . \qquad (5.2.40)$$

$\mathrm{L}_{-1}, \mathrm{L}_0, \mathrm{L}_1$ satisfy the Lie algebra for the group $\mathrm{SL}(2, \mathbb{C})$

$$[\mathrm{L}_n, \mathrm{L}_n] = (n-m)\mathrm{L}_{n+m} , \qquad (5.2.41)$$

which is the subalgebra of the Virasoro algebra with no anomaly term. On the other hand, since $T(z)$ is not a primary field we do expect an anomaly if $v(z) = z^n$ with $n \geq 3$. Up to factors, $\mathrm{SL}(2, \mathbb{C})$ is isomorphic to the conformal group $O(3,1)$ in two dimensions.

We conclude from this discussion that surface terms play a crucial role when determining the conformal transformation properties of a given correlation function. Considering the two dimensional action (5.1.1) which leads to the conformal correlation functions (5.1.6) and (5.1.9), we see that there are no surface terms spoiling the conformal invariance in this case since the two-dimensional action (5.1.1) gives $\langle T_{\mu\nu}(x)\rangle_g = \mathcal{O}(|x|^{-4})$ for asymptotically flat space which allows for conformal invariance identities to be derived for $v(z) = \mathcal{O}(z^2)$. This is sufficient for invariance under the two-dimensional conformal group $O(3,1)$ which is in accord with the conformal invariance of the correlation functions (5.1.6) and (5.1.9). However in four dimensions, as mentioned earlier, we have $\langle T_{\mu\nu}(x)\rangle_{g,\,\mathrm{Riegert}} = \mathcal{O}(|x|^{-5})$ from the Riegert action (5.2.15) so that special conformal transformations for which $v(z) = \mathcal{O}(z^2)$ lead to surface terms in the Ward identity which spoil the conformal symmetry of the correlation functions.

5.3 Conformal Actions in Four Dimensions

In order to cure the problems arising from the Riegert action we need to construct non-local actions for which $\langle T_{\mu\nu}(x)\rangle_g \sim |x|^{-6}$. To second order in the curvature such actions have been discussed by Barvinsky et al. [25]. They found

$$\begin{aligned} W[g,A]^{(2)} &= \tfrac{1}{2}\beta_a \int \mathrm{d}^4x\sqrt{g}\, C^{\mu\sigma\rho\nu}\Big(\ln(-\nabla^2/\hat{\mu}^2) - 1\Big)C_{\mu\sigma\rho\nu} \\ &\quad -\tfrac{1}{2}\kappa \int \mathrm{d}^4x\sqrt{g}\, F^{\mu\nu}\Big(\ln(-\nabla^2/\hat{\mu}^2) - 1\Big)F_{\mu\nu}\,, \end{aligned} \tag{5.3.1}$$

with $\hat{\mu} = 2e^{-\gamma}\mu$ as an action which is compatible with the trace anomaly. When differentiated twice with respect to the background gauge field or the metric, this expression generates exactly the expressions (2.2.13) and (2.2.14) for the two point functions with C_V and C_T according to (3.3.48) and (2.4.22). In this action the logarithm ensures a suitable asymptotical behaviour. Barvinsky et al. have also constructed a basis for the contributions to third order in the curvature, but it is not clear at present which linear combination of these terms must be taken in order to generate both the conformal anomaly and the three point functions discussed earlier.

In the following we construct non-local effective actions leading to conformal three point functions by making use of the features discussed so far. Since in four dimensions all the contributions to the trace anomaly involve the square of some operator, we may evaluate each of the two factors at different points. The propagator then necessary between the two operators is constructed using a conformally covariant differential operator. As a first example we consider the scalar contribution to the trace anomaly which is given by

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle = \tfrac{1}{2}p\mathcal{J}^2 \tag{5.3.2}$$

with p the anomaly coefficient and $\mathcal{J}$ a scalar source which is coupled to a dimension two operator $\mathcal{O}$. This anomaly gives rise to an anomalous term in the trace of three point function involving the energy momentum tensor and two scalar operators $\mathcal{O}$ even on flat space,

$$\begin{aligned}\langle \mathcal{O}(x)\mathcal{O}(y)T_{\alpha\alpha}(z)\rangle &= \frac{\delta^2}{\delta\mathcal{J}(x)\delta\mathcal{J}(y)}\langle T^{\alpha}{}_{\alpha}\rangle\Big|_{\text{flat space}} \\ &= p\,\delta^4(x-z)\delta^4(y-z)\,. \end{aligned}\tag{5.3.3}$$

To obtain the action W generating this anomaly we make use of the operator Δ_d acting on scalars defined in (5.1.3) and (5.2.4). In four dimensions (5.1.3) gives if now $\Delta_{4,2} \equiv \Delta$

$$\Delta = \sqrt{g}\left(-\nabla^2 + \tfrac{1}{6}R\right)\,, \quad \Delta_x G_\Delta(x,y) = \delta^4(x-y)\,. \tag{5.3.4}$$

On flat space $G_\Delta(x,y)|_{g=\delta} = G_0(x-y)$ as in (5.2.19). Under variations with respect to σ we have from the general case (5.1.4)

$$\delta_\sigma\Delta = -\left(\sigma\Delta + \Delta\sigma\right)\,, \tag{5.3.5}$$

from which follows

$$\delta_\sigma G_\Delta(x,y) = (\sigma(x)+\sigma(y))G_\Delta(x,y)\,. \tag{5.3.6}$$

The contribution to the effective action involving $\mathcal{J}$ may now be taken as

$$W[g,\mathcal{J}] = 4\pi^2 p \iint \mathrm{d}^4x\mathrm{d}^4y\, \sqrt{g(x)}\mathcal{J}(x)\mathcal{R}(G_\Delta(x,y)^2)\sqrt{g(y)}\mathcal{J}(y)\,, \tag{5.3.7}$$

with the regularised product

$$\mathcal{R}(G_\Delta(x,y)^2) = \left(\mu^{2\omega}G_\Delta(x,y)^{2-\omega} - \frac{1}{16\pi^2\omega}\delta^4(x,y)\right)\Big|_{\omega\to 0} \tag{5.3.8}$$

defined similarly as in (2.4.17), with with $\delta^4(x,y) = \delta^4(x-y)/\sqrt{g(x)}$. The regularisation is necessary since on flat space, $G_0(x-y)^2 = (16\pi^4)^{-1}(x-y)^{-4}$ is singular when $x \to y$ as in (3.3.26). Varying (5.3.8) with respect to σ we obtain

$$\begin{aligned} &\delta_\sigma \mathcal{R}(G_\Delta(x,y)^2) - 2(\sigma(x)+\sigma(y))\mathcal{R}(G_\Delta(x,y)^2) \\ &= -\omega\,(\sigma(x)+\sigma(y))\,G_\Delta(x,y)^2|_{\omega\to 0} = -\frac{1}{8\pi^2}\,\sigma(x)\delta^4(x,y)\,. \end{aligned} \tag{5.3.9}$$

Using (5.3.6), (5.3.9) as well as $\delta_\sigma\mathcal{J} = 2\sigma\mathcal{J}$ we may now see that the variation of (5.3.7) yields indeed the trace anomaly (5.3.2). By functionally differentiating the action (5.3.7) twice with respect to the source $\mathcal{J}$ and restricting to flat space we find the two point function

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{C_\mathcal{O}}{(x-y)^4}\,, \quad C_\mathcal{O} = \frac{p}{2\pi^2} \tag{5.3.10}$$

which is conformally invariant. Moreover we may calculate the three point function involving the energy momentum tensor and the two scalar operators by varying (5.3.7) with respect to the metric. Using

$$\begin{aligned} \delta_g G_\Delta(x,y) &= -\int \mathrm{d}^4z\, G_\Delta(x,z)\delta_g\Delta_z G_\Delta(z,y)\,, \\ \delta_g\Delta_z|_{g=\delta} &= -\partial_\mu\left(-\tfrac{1}{2}g_{\alpha\beta}\delta g^{\alpha\beta}g^{\mu\nu} + \delta g^{\mu\nu}\right)\partial_\nu + \tfrac{1}{6}\left(-\partial_\mu\partial_\nu\delta g^{\mu\nu} + \partial^2\delta g^{\mu\nu}g_{\mu\nu}\right) \end{aligned} \tag{5.3.11}$$

we may obtain

$$\begin{aligned} \frac{\delta}{\delta g^{\alpha\beta}(z)}G_\Delta(x,y)\Big|_{\text{flat space}} &= \frac{1}{12\pi^4}\,\frac{(x-y)^2}{(x-z)^4(y-z)^4}\left(\frac{X_{12\,\alpha}X_{12\,\beta}}{X_{12}^2} - \tfrac{1}{4}\delta_{\alpha\beta}\right) \\ &\quad + \frac{1}{32\pi^2}\,\frac{1}{(x-y)^2}\delta_{\alpha\beta}(\delta^4(x-z)+\delta^4(y-z))\,, \end{aligned} \tag{5.3.12}$$

and hence

$$
\begin{aligned}
\langle \mathcal{O}(x)\mathcal{O}(y)T_{\alpha\beta}(z)\rangle &= -2\frac{\delta^3}{\delta\mathcal{J}(x)\delta\mathcal{J}(y)\delta g^{\alpha\beta}(z)}W[g,\mathcal{J}]\Big|_{\text{flat space}} \\
&= -\frac{4}{3\pi^2}\,\frac{C_{\mathcal{O}}}{(x-z)^4(y-z)^4}\left(\frac{X_{12\,\alpha}X_{12\,\beta}}{X_{12}^2}-\tfrac{1}{4}\delta_{\alpha\beta}\right) \\
&\quad +\tfrac{1}{4}\,p\,\delta_{\alpha\beta}\delta^4(x-z)\delta^4(y-z)\,. \qquad (5.3.13)
\end{aligned}
$$

This agrees with what is expected from the general formalism for conformal three point functions presented in section 2.1. Furthermore when taking the trace we obtain the purely local term

$$
\langle \mathcal{O}(x)\mathcal{O}(y)T_{\alpha\alpha}(z)\rangle = p\,\delta^4(x-z)\delta^4(y-z) \qquad (5.3.14)
$$

which agrees with what is expected from (5.3.3).

In order to extend these results to the tensors $F_{\mu\nu}$ and $C_{\mu\sigma\rho\nu}$, we need differential operators acting on tensors and associated Green functions with conformal transformation properties analogous to the scalar operator and Green function defined in (5.3.4). The simplest such differential operator is a second order operator acting on two forms. For general dimensions and general k-forms $\omega_{\mu_1\cdots\mu_k} = \omega_{[\mu_1\cdots\mu_k]}$, a second order conformal operator has been found by Branson [27]. Using the notation (5.2.7) and $\gamma \equiv (d-2k)/2$, it is given by

$$
\begin{aligned}
\Delta^{(k)}\,\omega_{\mu_1\cdots\mu_k} &= \sqrt{g}\Big(\,(\gamma+1)\delta\mathrm{d}\,\omega + (\gamma-1)\mathrm{d}\delta\,\omega\; + (\gamma+1)(\gamma-1)J\,\omega\Big)_{\mu_1\cdots\mu_k} \\
&\quad -2\sqrt{g}\,(\gamma+1)(\gamma-1)\,k\,K_{[\mu_1}{}^{\nu}\omega_{|\nu|\mu_2\cdots\mu_k]}\,, \qquad (5.3.15)
\end{aligned}
$$

where J and K are defined in (5.2.1) and (5.2.2). The operators $\delta\mathrm{d}$ and $\mathrm{d}\delta$ acting on ω are given by

$$
\begin{aligned}
(\delta\mathrm{d}\omega)_{\mu_1\cdots\mu_k} &= -(k+1)\frac{1}{\sqrt{g}}g_{\mu_1\nu_1}\cdots g_{\mu_k\nu_k}\partial_\lambda\Big(\sqrt{g}g^{\lambda\tau}g^{\nu_1\rho_1}\cdots g^{\nu_k\rho_k}\partial_{[\tau}\omega_{\rho_1\cdots\rho_k]}\Big) \\
&= -(k+1)\nabla^\lambda\nabla_{[\lambda}\omega_{\mu_1\cdots\mu_k]} \qquad (5.3.16) \\
(\mathrm{d}\delta\omega)_{\mu_1\cdots\mu_k} &= -k\partial_{[\mu_1}\left(\frac{1}{\sqrt{g}}g_{\mu_2|\nu_2}\cdots g_{\mu_k]\nu_k}\partial_\lambda(\sqrt{g}g^{\lambda\tau}g^{\nu_2\rho_2}\cdots g^{\nu_k\rho_k}\omega_{\tau\rho_2\cdots\rho_k})\right) \\
&= -k\nabla_{[\mu_1}\nabla^\lambda\omega_{|\lambda|\mu_2\cdots\mu_k]}\,. \qquad (5.3.17)
\end{aligned}
$$

The conformal variation of the operator $\Delta^{(k)}$ given by (5.3.15) may be found by using $\delta_\sigma g^{\mu\nu} = 2\sigma g^{\mu\nu}$, $\delta_\sigma \sqrt{g} = -d\sqrt{g}$ in d dimensions. In (5.3.16, 5.3.17) we then have

$$\begin{aligned}\delta_\sigma(\delta \mathrm{d}\omega)_{\mu_1\cdots\mu_k} &= 2\gamma\sigma(\delta\mathrm{d}\omega)_{\mu_1\cdots\mu_k} \\ &\quad +2(\gamma-1)(k+1)\nabla^\lambda(\sigma\nabla_{[\lambda}\omega_{\mu_1\cdots\mu_k]}) \\ \delta_\sigma(\mathrm{d}\delta\omega)_{\mu_1\cdots\mu_k} &= -2\gamma(\mathrm{d}\delta(\sigma\omega))_{\mu_1\cdots\mu_k} \\ &\quad -2(\gamma+1)k\nabla_{[\mu_1}(\sigma\nabla^\lambda\omega_{|\lambda|\mu_2\cdots\mu_k]}) \ .\end{aligned} \tag{5.3.18}$$

Using

$$(k+1)\nabla^\lambda(\sigma\nabla_{[\lambda}\omega_{\mu_1\cdots\mu_k]}) = \nabla^\lambda(\sigma\nabla_\lambda\omega_{\mu_1\cdots\mu_k}) - k\nabla^\lambda(\sigma\nabla_{[\mu_1}\omega_{|\lambda|\mu_2\cdots\mu_k]}) \tag{5.3.19}$$

we find

$$\begin{aligned}&\delta_\sigma\Big((\gamma+1)(\delta\mathrm{d}\omega)_{\mu_1\cdots\mu_k} + (\gamma-1)(\mathrm{d}\delta\omega)_{\mu_1\cdots\mu_k}\Big) \\ &= 2\gamma(\gamma+1)\sigma(\delta\mathrm{d}\omega)_{\mu_1\cdots\mu_k} + 2\gamma(\gamma-1)(\mathrm{d}\delta(\sigma\omega))_{\mu_1\cdots\mu_k} \\ &\quad -2(\gamma-1)(\gamma+1)k\Big[\nabla^\lambda(\sigma\nabla_{[\mu_1}\omega_{|\lambda|\mu_2\cdots\mu_k]}) + \nabla_{[\mu_1}(\sigma\nabla^\lambda\omega_{|\lambda|\mu_2\cdots\mu_k]})\Big] \\ &\quad +2(\gamma-1)(\gamma+1)\nabla^\lambda(\sigma\nabla_\lambda\omega_{\mu_1\cdots\mu_k})\,.\end{aligned} \tag{5.3.20}$$

Eliminating single derivatives acting on σ by virtue of the identity

$$\begin{aligned}\nabla^\lambda\sigma\nabla_\mu + \nabla_\mu\sigma\nabla^\lambda &= \tfrac{1}{2}(\nabla^\lambda\nabla_\mu + \nabla_\mu\nabla^\lambda)\sigma \\ &\quad -\tfrac{1}{2}\sigma(\nabla^\lambda\nabla_\mu + \nabla_\mu\nabla^\lambda) - (\nabla_\mu\nabla^\lambda\sigma) \ ,\end{aligned} \tag{5.3.21}$$

we obtain for the operator

$$(\mathcal{D}^{(k)}\omega)_{\mu_1\cdots\mu_k} \equiv (\gamma+1)(\delta\mathrm{d}\omega)_{\mu_1\cdots\mu_k} + (\gamma-1)(\mathrm{d}\delta\omega)_{\mu_1\cdots\mu_k} \tag{5.3.22}$$

from (5.3.20)

$$\begin{aligned}\delta_\sigma(\mathcal{D}^{(k)}\omega)_{\mu_1\cdots\mu_k} &= (\gamma+1)\sigma(\mathcal{D}^{(k)}\omega)_{\mu_1\cdots\mu_k} - (\gamma-1)(\mathcal{D}^{(k)}\sigma\omega)_{\mu_1\cdots\mu_k} \\ &\quad +2(\gamma-1)(\gamma+1)k(\nabla_{[\mu_1}\nabla^\lambda\sigma)\omega_{|\lambda|\mu_2\cdots\mu_k]} \\ &\quad - (\gamma-1)(\gamma+1)\nabla^2\sigma\omega_{\mu_1\cdots\mu_k} \ .\end{aligned} \tag{5.3.23}$$

The last two terms involving second derivatives acting on σ can be cancelled by terms involving J and $K_\mu{}^\nu$ in $\Delta^{(k)}$ giving

$$\Delta^{(k)} = \sqrt{g}\mathcal{D}^{(k)} + (\gamma+1)(\gamma-1)\sqrt{g}\,(J\ -2\,k\,K) \tag{5.3.24}$$

for $\Delta^{(k)}$ defined in (5.3.15) with

$$(K\omega)_{\mu_1\cdots\mu_k} \equiv\ K_{[\mu_1}{}^\nu\omega_{|\nu|\mu_2\cdots\mu_k]}\,. \tag{5.3.25}$$

The variation of J and $K_\mu{}^\nu$ is given in (5.2.3). Thus the conformal variation of the operator $\Delta^{(k)}$ is

$$\delta_\sigma\Delta^{(k)} = (\gamma-d+1)\sigma\Delta^{(k)} - (\gamma-1)\Delta^{(k)}\sigma\,. \tag{5.3.26}$$

On functions, when $k = 0$, (5.3.15) reduces to $(d+2)/2\cdot\Delta_{d,2}$, the operator defined in (5.1.3).

In appendix A.8 we calculate the flat space Green function for this operator for general d and k. The result is

$$G^{(k)}_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(x) = \frac{\Gamma(\frac{1}{2}d)}{\pi^{d/2}(d-2k-2)(d-2k+2)}\,\mathcal{I}^A_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(x)\frac{1}{x^{d-2}}\,, \tag{5.3.27}$$

with $\mathcal{I}^A_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(x)$ the inversion on k-forms. Note that this Green function does not exist if $\gamma = \pm1$ or $k = \pm(d-2)/2$, when $\Delta^{(k)} \sim 2\sqrt{g}\delta\mathrm{d}$ and $\Delta^{(k)} \sim -2\sqrt{g}\mathrm{d}\delta$ have solutions satisfying $\Delta^{(k)}\psi = 0$ and hence are not invertible. The Green function (5.3.27) is of a form expected from the general results for conformal two point functions discussed in section 2.1. It will be used later on for the construction of conformal actions.

The conformal variation of the operator $\Delta^{(k)}$ defined in (5.3.15) as given by (5.3.26) remains unchanged if we add a term of the form

$$\omega'{}_{\mu_1\cdots\mu_k} \equiv C_{[\mu_1\mu_2}{}^{\sigma\rho}\omega_{|\sigma\rho|\mu_3\cdots\mu_k]} \tag{5.3.28}$$

whose conformal variation is

$$\delta_\sigma\omega'{}_{\mu_1\cdots\mu_k} = 2\sigma\omega'{}_{\mu_1\cdots\mu_k}\,. \tag{5.3.29}$$

In the following we restrict ourselves to the case $d = 4$ and $k = 2$ giving $\gamma = 0$ which is relevant for the construction of conformal actions in four dimensions. In this case we may define

$$\begin{aligned}\Delta^F F_{\mu\nu} &= \Delta^{(2)} F_{\mu\nu} + t\sqrt{g} C_{\mu\nu}{}^{\sigma\rho} F_{\sigma\rho} \\ &= \sqrt{g}\Big([(\delta \mathrm{d} - \mathrm{d}\delta)F]_{\mu\nu} + R_\mu{}^\lambda F_{\lambda\nu} - R_\nu{}^\lambda F_{\lambda\mu} - \tfrac{1}{2} R F_{\mu\nu} + t\, C_{\mu\nu}{}^{\sigma\rho} F_{\sigma\rho}\Big)\,.\end{aligned} \tag{5.3.30}$$

t is an arbitrary parameter. Under local rescalings of the metric Δ^F satisfies from (5.3.26)

$$\delta_\sigma \Delta^F = -3\sigma\Delta^F + \Delta^F \sigma\,, \tag{5.3.31}$$

which is in agreement with the general case (5.3.26). This will enable us to construct an action similarly to the scalar case. Δ^F may be expressed alternatively by

$$\begin{aligned}\Delta^F F_{\mu\nu} = \sqrt{g}\Big(&- \nabla^2 F_{\mu\nu} + 2\nabla_\mu \nabla^\lambda F_{\lambda\nu} - 2\nabla_\nu \nabla^\lambda F_{\lambda\mu} \\ &+ R_\mu{}^\lambda F_{\lambda\nu} - R_\nu{}^\lambda F_{\lambda\mu} - \tfrac{1}{6} R F_{\mu\nu} + (t-1) C_{\mu\nu}{}^{\sigma\rho} F_{\sigma\rho}\Big)\,,\end{aligned} \tag{5.3.32}$$

and the corresponding Green function is defined by

$$(\Delta^F{}_x G^F)_{\mu\nu}{}^{\sigma\rho}(x,y) = \delta_\mu^{[\sigma} \delta_\nu^{\rho]}\, \delta^4(x-y)\,. \tag{5.3.33}$$

Using (5.3.31) we may see that

$$\delta_\sigma G^F_{\mu\nu\sigma\rho}(x,y) = -(\sigma(x) + \sigma(y)) G^F_{\mu\nu\sigma\rho}(x,y)\,, \tag{5.3.34}$$

in agreement with the general case (5.3.27). On flat space the Green function may be found explicitly by considering Fourier transforms. On flat space the operator Δ^F is given by

$$\Delta^F F_{\mu\nu}|_{\text{flat space}} = -\partial^2 F_{\mu\nu} + 2\partial_\mu \partial^\lambda F_{\lambda\nu} - 2\partial_\nu \partial^\lambda F_{\lambda\mu}\,. \tag{5.3.35}$$

For the Fourier transform $(\widetilde{\Delta^F F})_{\mu\nu}(k)$ we find

$$(\widetilde{\Delta^F F})_{\mu\nu}(k) = P_{\mu\nu,\sigma\rho}(k) \tilde{F}_{\sigma\rho}(k)\,, \tag{5.3.36}$$

where

$$P_{\mu\nu,\sigma\rho}(k) = \mathcal{E}^F_{\mu\nu,\sigma\rho}k^2 - 4\mathcal{E}^F_{\mu\nu,\varepsilon\lambda}\mathcal{E}^F_{\eta\lambda,\sigma\rho}k_\varepsilon k_\eta \,. \tag{5.3.37}$$

The inverse is

$$P^{-1}_{\mu\nu,\sigma\rho}(k) = \mathcal{E}^F_{\mu\nu,\sigma\rho}\frac{1}{k^2} - 4\mathcal{E}^F_{\mu\nu,\varepsilon\lambda}\mathcal{E}^F_{\eta\lambda,\sigma\rho}\frac{k_\varepsilon k_\eta}{k^4} \,. \tag{5.3.38}$$

Fourier transforming back to position space, using $k^2 \to -\partial^2$, $k^{-2} \to (4\pi^2x^2)^{-1}$ and $k^{-4} \to (16\pi^2)^{-1}\ln x^2$, we find the Green function on flat space

$$G^F_{\ \mu\nu\sigma\rho}(x,y)|_{\text{flat space}} = -\frac{1}{4\pi^2 s^2}\,\mathcal{I}^F_{\ \mu\nu,\sigma\rho}(s)\,, \quad s = x-y\,. \tag{5.3.39}$$

Using the analogous result to (5.3.11), we may calculate the variation of this Green function with respect to the metric. The variation of the differential operator Δ^F is obtained by varying the explicit expressions for $(\mathrm{d}\delta F)_{\mu\nu}$ and $(\delta\mathrm{d}F)_{\mu\nu}$ in (5.3.30). For the variation of the Green function we find

$$\begin{aligned}\frac{\delta}{\delta g^{\alpha\beta}(z)}G^F_{\ \mu\nu\sigma\rho}(x,y)\Big|_{g=\delta} \;=\;& \frac{\mathcal{I}^F_{\ \mu\nu,\mu'\nu'}(x-z)\mathcal{I}^F_{\ \sigma'\rho',\sigma\rho}(z-y)}{(4\pi^2)^2(x-z)^2(z-y)^2}\,h_{\mu'\nu',\sigma'\rho',\alpha\beta}(X_{12}) \\ & - 2t\,\mathcal{E}^C_{\ \alpha\gamma\delta\beta,\kappa\lambda\varepsilon\eta}\partial^z_{\ \gamma}\partial^z_{\ \delta}\Big(\frac{\mathcal{I}^F_{\ \mu\nu,\kappa\lambda}(x-z)\mathcal{I}^F_{\ \varepsilon\eta,\sigma\rho}(z-y)}{(4\pi^2)^2(x-z)^2(z-y)^2}\Big)\,,\end{aligned} \tag{5.3.40}$$

where

$$\begin{aligned} h_{\mu\nu,\sigma\rho,\alpha\beta}(X) \;=\;& 16\,\mathcal{E}^F_{\ \mu\nu,\lambda\varepsilon}\mathcal{E}^F_{\ \sigma\rho,\lambda\eta}\mathcal{E}^T_{\ \varepsilon\eta,\alpha\beta}X^2 \\ & -8\,\mathcal{E}^F_{\ \mu\nu,\lambda\varepsilon}\mathcal{E}^F_{\ \sigma\rho,\lambda\eta}(\mathcal{E}^T_{\ \varepsilon\kappa,\alpha\beta}X_\eta + \mathcal{E}^T_{\ \eta\kappa,\alpha\beta}X_\varepsilon)X_\kappa \\ & +4\,\mathcal{E}^F_{\ \mu\nu,\sigma\rho}(X_\alpha X_\beta - \tfrac{1}{4}\delta_{\alpha\beta}X^2)\,.\end{aligned} \tag{5.3.41}$$

With these results we may now construct a contribution to the effective action involving the gauge field tensor $F_{\mu\nu}$:

$$W[g,\mathcal{A}]^F = -\tfrac{1}{16}U\int\!\!\int \mathrm{d}^4x\mathrm{d}^4y\,\sqrt{g(x)}F^{\mu\nu}(x)G^F_{\mu\nu\sigma\rho}(x,y)G_\Delta(x,y)\sqrt{g(y)}F^{\sigma\rho}(y)\,. \tag{5.3.42}$$

This action does not require any regularisation since

$$G^F_{\mu\nu\sigma\rho}(x,y)G_\Delta(x,y)\big|_{g=\delta} = -\frac{1}{(4\pi^2)^2}\frac{\mathcal{I}^F_{\mu\nu,\sigma\rho}(x-y)}{(x-y)^4} \tag{5.3.43}$$

is regular as a distribution in four dimensions (cf. (3.2.17)). The absence of any singularity is due to the fact that when contracting $\mathcal{I}^F_{\mu\nu,\sigma\rho}(s)/s^4$ in order to apply (3.3.26) as in the previous sections, we have in general dimension

$$\mathcal{I}^F_{\mu\nu,\mu\nu} = \tfrac{1}{2}(d-4)(d-1)\,, \tag{5.3.44}$$

which vanishes in four dimensions. Consequently we may check that $\delta_\sigma W[g,\mathcal{A}]^F = 0$. We thus expect to obtain the anomaly free contribution to the associated conformal three point function upon functional differentiation of the effective action (5.3.42). There is indeed no non-local contribution to the vector operator two point function since the derivative of (5.3.43) with respect to x_ν and y_ρ vanishes for non-coincident points as a consequence of (3.2.17). For the three point function involving the energy momentum tensor we have

$$\begin{aligned}\langle V_\mu(x)V_\sigma(y)T_{\alpha\beta}(z)\rangle^F &= -2\frac{\delta^3}{\delta\mathcal{A}^\mu(x)\delta\mathcal{A}^\sigma(y)\delta g^{\alpha\beta}(z)}W[g,\mathcal{A}]^F\Big|_{\text{flat space}} \\ &= \partial^x{}_\nu\partial^y{}_\rho\langle F_{\mu\nu}(x)F_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle\,. \end{aligned} \tag{5.3.45}$$

Using the results of this section, as well as the relation

$$\mathcal{I}^F{}_{\mu\nu,\sigma\rho}(x-y) = \mathcal{I}^F{}_{\mu\nu,\varepsilon\eta}(x-z)\mathcal{I}^F{}_{\varepsilon\eta,\sigma\rho}(z-y) - 4\mathcal{I}^F{}_{\mu\nu,\lambda\varepsilon}(x-z)\mathcal{I}^F{}_{\lambda\eta,\sigma\rho}(z-y)\frac{X_{12\,\varepsilon}X_{12\,\eta}}{X_{12}^2}\,, \tag{5.3.46}$$

$\langle F_{\mu\nu}(x)F_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle^F$ is expressible exactly in the form (3.3.2) with (3.3.3) in four dimensions, with the coefficients

$$A = \hat{U}(16+\tfrac{8}{3}t)\,,\ B = -\hat{U}\tfrac{16}{3}t\,,\ C = -\hat{U}(8+\tfrac{8}{3}t)\,,\ D = \hat{U}(\tfrac{8}{3}+\tfrac{16}{3}t)\,,\ E = \hat{U}\tfrac{16}{3}\,, \tag{5.3.47}$$

where $\hat{U} \equiv U/(4\pi^2)^3$. These results satisfy $K=0$ for K as in (3.3.8) and also $R=0$ with R as in (3.3.39), such that there are no anomalous contributions to the Ward identities. The effective action (5.3.45) thus yields indeed the anomaly free form in the conformal three point function involving the energy momentum tensor and two vector operators.

Similarly we may also construct an anomaly free contribution to the purely gravitational effective action:

$$W[g]^F = \tfrac{1}{64}V\iint \mathrm{d}^4x\mathrm{d}^4y\,\sqrt{g(x)}C^{\mu\kappa\lambda\nu}(x)G^F_{\mu\kappa\sigma\varepsilon}(x,y)G^F_{\lambda\nu\eta\rho}(x,y)\sqrt{g(y)}C^{\sigma\varepsilon\eta\rho}(y)\,. \tag{5.3.48}$$

The singularity as $x \to y$ again does not require regularisation and, as for (5.3.42), $W[g]^F$ does not contribute to the energy momentum tensor two point function on flat space but there is a corresponding expression for the three point function given by

$$\begin{aligned} \langle T_{\mu\nu}(x) T_{\sigma\rho}(y) T_{\alpha\beta}(z)\rangle^F &= -8 \frac{\delta^3}{\delta g^{\mu\nu}(x)\delta g^{\sigma\rho}(y)\delta g^{\alpha\beta}(z)} W(g)^F \Big|_{\text{flat space}} \\ &= \partial^x{}_\kappa \partial^x{}_\lambda \partial^y{}_\varepsilon \partial^y{}_\eta \langle C_{\mu\kappa\lambda\nu}(x) C_{\sigma\varepsilon\eta\rho}(y) T_{\alpha\beta}(z)\rangle^F \\ &\qquad + \text{cyclic permutations}\,. \end{aligned} \tag{5.3.49}$$

Again this is of the form may be represented just as in (4.2.1) with (4.2.6) for $d = 4$ where, by applying (5.3.40) and (5.3.46) again, the coefficients are

$$\begin{aligned} &A = -\tfrac{1}{4}D = \hat{V}(32 + \tfrac{16}{3}t)\,, \quad B = -\tfrac{1}{4}Q = \hat{V}(8 + \tfrac{32}{3}t)\,, \\ &E = -\tfrac{1}{4}R = \hat{V}(8 + \tfrac{8}{3}t)\,, \quad C = -\tfrac{1}{4}G = -\hat{V}\tfrac{32}{3}t\,, \\ &F = S = 0\,, \end{aligned} \tag{5.3.50}$$

for $\hat{V} = V/(4\pi^2)^3$. With these values, $U = 0$ for U in (4.3.23) and $K = 0$ for K in (4.3.35) such that the three point function is anomaly free and satisfies the Ward identities for the non-anomalous case. The relations (4.2.25) give

$$\mathcal{A} = -4 \times \tfrac{32}{3}\hat{V}\,, \ \ \mathcal{B} = -46 \times \tfrac{32}{3}\hat{V}\,, \ \ \mathcal{C} = \tfrac{32}{3}\hat{V}\,. \tag{5.3.51}$$

The ratios are just as in (4.1.20) as expected since both trace anomalies are absent for the gravitational effective action (5.3.48).

The construction of contributions to the effective action which give rise to the gauge field and gravitational anomalies is subject of current investigations. The first step involves the construction of a conformally covariant second order differential operator acting on the Weyl tensor $C_{\mu\sigma\rho\nu}$ which in general may not be treated as a differential form since it is not totally antisymmetric.

To construct such an operator we follow a procedure inspired by a paper by O'Raifeartaigh, Sachs and Wiesendanger [29]. We consider the d-dimensional action

$$S_0[g, \mathcal{C}] = -\tfrac{1}{2}\int \mathrm{d}^d x \, \sqrt{g}\,\mathcal{C}^{\mu\sigma\rho\nu}\left[a\, \nabla^2 \mathcal{C}_{\mu\sigma\rho\nu} + b \nabla_\mu \nabla^\alpha \mathcal{C}_{\alpha\sigma\rho\nu}\right], \tag{5.3.52}$$

which after integrating by parts is equal to

$$S_0[g,\mathcal{C}] \;=\; \tfrac{1}{2}\int \mathrm{d}^d x\,\sqrt{g}\;\left[a\,\nabla^\alpha \mathcal{C}^{\mu\sigma\rho\nu}\nabla_\alpha \mathcal{C}_{\mu\sigma\rho\nu} \;+\; b\,\nabla_\mu \mathcal{C}^{\mu\sigma\rho\nu}\nabla^\alpha \mathcal{C}_{\alpha\sigma\rho\nu}\right]. \tag{5.3.53}$$

Here $\mathcal{C}_{\mu\sigma\rho\nu}$ is a general tensor with Weyl symmetry which is not necessarily the genuine Weyl tensor constructed from the metric in (2.4.4), i.e. $\mathcal{C}_{\mu\sigma\rho\nu}$ has the symmetries of (2.2.5) but is here regarded as independent of the curvature. The two terms in (5.3.53) with coefficients a and b are the only possible independent scalars involving $\mathcal{C}_{\mu\sigma\rho\nu}$ and two covariant derivatives. In the following we determine the values for the two coefficients a and b necessary for a conformally covariant second order differential operator acting on $\mathcal{C}_{\mu\sigma\rho\nu}$ by requiring $\delta_\sigma S[g,\mathcal{C}] = 0$, where $S^C[g,\mathcal{C}]$ may contain curvature dependent terms in addition to $S_0[g,\mathcal{C}]$.

For the variation of $\mathcal{C}_{\mu\sigma\rho\nu}$ with respect to σ we assume

$$\delta_\sigma \mathcal{C}_{\mu\sigma\rho\nu} \;=\; -\,k\,\sigma\,\mathcal{C}_{\mu\sigma\rho\nu} \tag{5.3.54}$$

with k an arbitrary number. Since the action S_0 is required to be an invariant scalar, cancellation of terms without derivatives on σ immediately implies

$$k = -\tfrac{1}{2}(d-10)\,. \tag{5.3.55}$$

In general we have

$$\begin{aligned}
\delta_\sigma\left(\nabla^\alpha \mathcal{C}^{\mu\sigma\rho\nu}\nabla_\alpha \mathcal{C}_{\mu\sigma\rho\nu}\right) \;&=\; (10-2k)\,\sigma\,\nabla^\alpha \mathcal{C}^{\mu\sigma\rho\nu}\nabla_\alpha \mathcal{C}_{\mu\sigma\rho\nu} + 2(4-k)\nabla^\alpha \mathcal{C}^{\mu\sigma\rho\nu}(\partial_\alpha\sigma)\mathcal{C}_{\mu\sigma\rho\nu} \\
&\quad + 8\,\nabla^\alpha \mathcal{C}^{\mu\sigma\rho\nu}(\partial_\mu\sigma)\mathcal{C}_{\alpha\sigma\rho\nu} \;-\; 8\,\nabla_\mu \mathcal{C}^{\mu\sigma\rho\nu}(\partial^\alpha\sigma)\mathcal{C}_{\alpha\sigma\rho\nu} \qquad (5.3.56)\\
\delta_\sigma \nabla^\alpha \mathcal{C}_{\alpha\sigma\rho\nu} \;&=\; (2-k)\,\sigma\,\nabla^\alpha \mathcal{C}_{\alpha\sigma\rho\nu} \;+\; (5-k-d)\,(\partial^\alpha\sigma)\mathcal{C}_{\alpha\sigma\rho\nu}\,. \qquad (5.3.57)
\end{aligned}$$

Using these results for (5.3.53) we obtain

$$\begin{aligned}
\delta_\sigma S_0[g,\mathcal{C}] \;&=\; \tfrac{1}{2}\int \mathrm{d}^d x\sqrt{g}\,\Big[a\Big((d-2)(\partial^\alpha\sigma)\mathcal{C}^{\mu\sigma\rho\nu}\nabla_\alpha \mathcal{C}_{\mu\sigma\rho\nu} + 16\nabla^\alpha \mathcal{C}^{\mu\sigma\rho\nu}(\partial_\mu\sigma)\mathcal{C}_{\alpha\sigma\rho\nu} \\
&\quad + 8\,(\nabla_\mu\partial^\alpha\sigma)\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\alpha\sigma\rho\nu}\Big) \;-\; b\,d\,\nabla_\mu \mathcal{C}^{\mu\sigma\rho\nu}(\partial^\alpha\sigma)\mathcal{C}_{\alpha\sigma\rho\nu}\Big]\,.
\end{aligned} \tag{5.3.58}$$

Hence if we choose

$$16\,a = -d\,b\,, \tag{5.3.59}$$

$\delta_\sigma S_0$ may be written as

$$\begin{aligned}
\delta_\sigma S_0[g,\mathcal{C}] &= \tfrac{1}{2}\,a \int \mathrm{d}^d x \sqrt{g}\,\Big[\tfrac{1}{2}(d-2)(\partial_\alpha\sigma)\nabla^\alpha(\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\mu\sigma\rho\nu}) \\
&\qquad +16(\partial_\mu\sigma)\nabla^\alpha(\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\alpha\sigma\rho\nu}) + 8(\nabla_\mu\partial^\alpha\sigma)\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\alpha\sigma\rho\nu}\Big]\,, \\
&= -\tfrac{1}{2}\,a \int \mathrm{d}^d x \sqrt{g}\,\Big[\tfrac{1}{2}(d-2)(\nabla^2\sigma)\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\mu\sigma\rho\nu} + 8(\nabla^\alpha\partial_\mu\sigma)\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\alpha\sigma\rho\nu}\Big]\,.
\end{aligned} \tag{5.3.60}$$

These terms may be cancelled by the conformal variation of the the action

$$S_1[g,\mathcal{C}] \equiv \tfrac{1}{2}\,a \int \mathrm{d}^d x\,\sqrt{g}\,\Big[\tfrac{1}{2}(d-2)J\,\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\mu\sigma\rho\nu} \;+\; 8\,K_\mu{}^\alpha\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\alpha\sigma\rho\nu}\Big] \tag{5.3.61}$$

involving the curvature dependent terms J and $K_{\mu\nu}$ defined in (5.2.1) and (5.2.2), such that for $a = d/16$ we obtain the conformally invariant action

$$\begin{aligned}
S^C[g,\mathcal{C}] &= S_0[g,\mathcal{C}] + S_1[g,\mathcal{C}] \\
&= \tfrac{1}{2} \int \mathrm{d}^d x\,\mathcal{C}^{\mu\sigma\rho\nu}(\Delta^C\mathcal{C})_{\mu\sigma\rho\nu}\;,
\end{aligned} \tag{5.3.62}$$

with the conformally covariant differential operator Δ^C given by

$$\begin{aligned}
(\Delta^C\mathcal{C})_{\mu\sigma\rho\nu} &= -\tfrac{1}{16}d\,\sqrt{g}\Big(\nabla^2 - \tfrac{1}{2}(d-2)J\Big)\mathcal{C}_{\mu\sigma\rho\nu} \\
&\quad + \sqrt{g}\,\mathcal{E}^C_{\mu\sigma\rho\nu,}{}^{\mu'\sigma'\rho'\nu'}\Big(\nabla_{\mu'}\nabla^\alpha + \tfrac{1}{2}d\,K_{\mu'}{}^\alpha\Big)\mathcal{C}_{\alpha\sigma'\rho'\nu'}\,.
\end{aligned} \tag{5.3.63}$$

As a consequence of $\delta_\sigma S^C[g,\mathcal{C}] = 0$ as well as (5.3.54) and (5.3.55) it must satisfy

$$\delta_\sigma\Delta^C = -\tfrac{1}{2}(d+6)\sigma\Delta^C - \tfrac{1}{2}(d-10)\Delta^C\sigma\,. \tag{5.3.64}$$

For generality we could also add $S_2[g,\mathcal{C}]$ containing the invariants

$$S_2[g,\mathcal{C}] = \tfrac{1}{2} \int \mathrm{d}^d x \sqrt{g}\,\Big(c\,\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\nu\beta\sigma\alpha}C^\alpha{}_\rho{}^\beta{}_\mu + e\,\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\alpha\beta\rho\nu}C^{\alpha\beta}{}_{\mu\sigma}\Big)\,, \tag{5.3.65}$$

where $C_{\mu\kappa\lambda\nu}$ is the Weyl tensor depending on the metric defined in (2.4.4). c and e are arbitrary parameters.

The operator Δ^C vanishes identically when $d=4$, as may be seen using the identities

$$
\begin{aligned}
15\,\mathcal{C}^{\mu\sigma}{}_{[\rho\nu}\nabla_\alpha\mathcal{C}_{\mu\sigma]}{}^{\rho\nu} &= -2\,\mathcal{C}_\alpha{}^{\sigma\rho\nu}\nabla_\mu\mathcal{C}^\mu{}_{\sigma\rho\nu} - 2\,\mathcal{C}^{\mu\sigma\rho\nu}\nabla_\mu\mathcal{C}_{\alpha\sigma\rho\nu} + \mathcal{C}^{\mu\sigma\rho\nu}\nabla_\alpha\mathcal{C}_{\mu\sigma\rho\nu} = 0\,, \quad (5.3.66)\\
15\,\mathcal{C}^{\mu\sigma}{}_{[\rho\nu}K_\alpha{}^\alpha\mathcal{C}_{\mu\sigma]}{}^{\rho\nu} &= -4\,\mathcal{C}^{\mu\sigma\rho\nu}K_\mu{}^\alpha\mathcal{C}_{\alpha\sigma\rho\nu} + \mathcal{C}^{\mu\sigma\rho\nu}J\mathcal{C}_{\mu\sigma\rho\nu} = 0\,, \quad (5.3.67)
\end{aligned}
$$

which follow from the vanishing of totally antisymmetric five index tensors in four dimensions together with $K_\alpha{}^\alpha = J$. In order to be able to apply these identities we write the action S^C given in (5.3.62) as

$$
\begin{aligned}
S^C[g,\mathcal{C}] \;=\; \tfrac{1}{2}\int \mathrm{d}^d x\sqrt{g}\,\Big[&\tfrac{1}{16}d\,\nabla^\alpha\mathcal{C}^{\mu\sigma\rho\nu}\nabla_\alpha\mathcal{C}_{\mu\sigma\rho\nu} - \tfrac{1}{2}\nabla_\alpha\mathcal{C}^{\alpha\sigma\rho\nu}\nabla^\mu\mathcal{C}_{\mu\sigma\rho\nu}\\
&-\tfrac{1}{2}\nabla^\alpha\mathcal{C}^{\mu\sigma\rho\nu}\nabla_\mu\mathcal{C}_{\alpha\sigma\rho\nu} + \tfrac{1}{2}\mathcal{C}^{\mu\sigma\rho\nu}[\nabla_\mu,\nabla^\alpha]\mathcal{C}_{\alpha\sigma\rho\nu}\\
&+\tfrac{1}{16}d\,\tfrac{1}{2}(d-2)J\,\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\mu\sigma\rho\nu} + \tfrac{1}{2}d\,\mathcal{C}^{\mu\sigma\rho\nu}K_\mu{}^\alpha\mathcal{C}_{\alpha\sigma\rho\nu}\Big]\,. \qquad (5.3.68)
\end{aligned}
$$

The commutator may be evaluated by using the Ricci identity and the definitions of J and $K_\mu{}^\alpha$. Using the definition of the Weyl tensor $C_{\mu\kappa\lambda\nu}$ given by (2.4.4), we find

$$
\begin{aligned}
\mathcal{C}^{\mu\sigma\rho\nu}[\nabla_\mu,\nabla^\alpha]\mathcal{C}_{\alpha\sigma\rho\nu} \;=\; &-(d-2)\,\mathcal{C}^{\mu\sigma\rho\nu}K_\mu{}^\alpha\mathcal{C}_{\alpha\sigma\rho\nu} - J\,\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\mu\sigma\rho\nu}\\
&+2\,\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\nu\lambda\sigma\alpha}C^\alpha{}_\rho{}^\lambda{}_\mu + \mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\alpha\lambda\rho\nu}C^{\alpha\lambda}{}_{\mu\sigma}\,. \qquad (5.3.69)
\end{aligned}
$$

In four dimensions we thus have

$$
\begin{aligned}
S^C[g,\mathcal{C}] = \int \mathrm{d}^4 x\sqrt{g}\,\Big[&\tfrac{1}{4}\,\nabla^\alpha\mathcal{C}^{\mu\sigma\rho\nu}\nabla_\alpha\mathcal{C}_{\mu\sigma\rho\nu} - \tfrac{1}{2}\nabla_\mu\mathcal{C}^{\mu\sigma\rho\nu}\nabla^\alpha\mathcal{C}_{\alpha\sigma\rho\nu} - \tfrac{1}{2}\nabla^\alpha\mathcal{C}^{\mu\sigma\rho\nu}\nabla_\mu\mathcal{C}_{\alpha\sigma\rho\nu}\\
&+\mathcal{C}^{\mu\sigma\rho\nu}K_\mu{}^\alpha\mathcal{C}_{\alpha\sigma\rho\nu} - \tfrac{1}{4}\,\mathcal{C}^{\mu\sigma\rho\nu}J\,\mathcal{C}_{\mu\sigma\rho\nu}\\
&+\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\nu\lambda\sigma\alpha}C^\alpha{}_\rho{}^\lambda{}_\mu + \mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\alpha\lambda\rho\nu}C^{\alpha\lambda}{}_{\mu\sigma}\Big]\,. \qquad (5.3.70)
\end{aligned}
$$

The terms involving derivatives or J or $K_\mu{}^\alpha$ vanish for $d=4$ by virtue of (5.3.66) and (5.3.67). If we define, by adding $\tilde{S}^C = S^C + S_2$ with the definitions (5.3.62) and (5.3.65),

$$
\begin{aligned}
(\tilde{\Delta}^C\mathcal{C})_{\mu\sigma\rho\nu} \;=\; &(\Delta^C\mathcal{C})_{\mu\sigma\rho\nu}\\
&+\sqrt{g}\,\mathcal{E}^C_{\mu\sigma\rho\nu,}{}^{\mu'\sigma'\rho'\nu'}\Big(c\,C^\alpha{}_{\rho'}{}^\beta{}_{\mu'}\mathcal{C}_{\nu'\beta\sigma'\alpha} + e\,C^{\alpha\beta}{}_{\mu'\sigma'}\mathcal{C}_{\rho'\nu'\alpha\beta}\Big)\,, \qquad (5.3.71)
\end{aligned}
$$

then $(\tilde{\Delta}^C\mathcal{C})_{\mu\sigma\rho\nu} = 0$ for $d=4$ if

$$
c + 4\,e = -3\,. \qquad (5.3.72)
$$

In general dimensions, since the tensor $\mathcal{C}_{\mu\sigma\rho\nu}$ with Weyl symmetry considered here is non-zero on flat space in general, we may calculate the Green function defined by

$$(\Delta^C{}_x G^C)_{\mu\sigma\rho\nu,}{}^{\alpha\gamma\delta\beta}(x,y) = \mathcal{E}^C_{\mu\sigma\rho\nu,}{}^{\alpha\gamma\delta\beta}\delta^d(x-y) \tag{5.3.73}$$

on flat space in a similar fashion as previously for the operators acting on forms. This calculation, which involves inverting the Fourier transform of Δ^C, is performed in appendix A.9 and yields

$$G^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(x,0)\Big|_{\text{flat space}} = \frac{8\,\Gamma(d/2)}{\pi^{d/2}(d-4)(d-6)}\,\frac{1}{x^{d-2}}\,\mathcal{I}^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(x)\,. \tag{5.3.74}$$

(5.3.74) is clearly in accord with the general expression expected by conformal invariance given by (2.1.14) and thus provides a consistency check on the derivation of the conformal operator Δ^C. We see that Δ^C is invertible neither for $d=4$, which is in agreement with $\widetilde{\Delta}^C = 0$ in four dimensions, nor for $d=6$.

The operator Δ^C may be of relevance for constructing a gravitational effective action generating the F-anomaly as well as the form in the energy momentum tensor three point function with coefficient β_a.

6 Conclusions and Perspectives

Conformal invariance is a strong symmetry constraint which has important consequences for the correlation functions in d dimensional quantum field theories. Here we have investigated these consequences in particular for two and three point functions involving conserved currents and the energy momentum tensor. In general dimensions these three point functions are not unique, but are given in terms of a small number of independent forms.

We have used the freedom in the definition of V_μ and $T_{\mu\nu}$ which is at the origin of the number of independent forms in the three point functions to develop a regularisation method involving the extraction of derivatives. This method allows us to establish relations between the different independent forms and the axial and conformal anomalies. The results obtained are listed in the following.

- By introducing projection operators onto the tensor spaces with the relevant rank and symmetries we are able to write the three point functions involving vector operators and the energy momentum tensor in a compact form which displays the number of independent components explicitly. The three point functions considered in detail are
  - $\langle V^p{}_\mu(x) V^q{}_\nu(y) V^r{}_\omega(z)\rangle$ involving three conserved vector operators which has two independent forms in general dimensions,
  - $\langle V_\mu(x) V_\nu(y) A_\omega(z)\rangle$ involving two conserved vector operators and one axial vector current for which there is only one form in accord with the required symmetries,
  - $\langle T_{\mu\nu}(x) V_\sigma(y) V_\rho(z)\rangle$ involving the energy momentum tensor and two conserved vector currents. There are two independent forms in this case.
  - $\langle T_{\mu\nu}(x) T_{\sigma\rho}(y) T_{\alpha\beta}(z)\rangle$ involving three energy momentum tensors for which there are three independent forms.

The number of independent components may be understood in terms of an arbitrariness in the definition of the vector operator V_μ and the energy momentum tensor $T_{\mu\nu}$ up to terms which satisfy their properties automatically, ${V'}_\mu = \partial_\nu F_{\nu\mu}$ and ${T'}_{\mu\nu} = \partial_\alpha \partial_\beta C_{\mu\alpha\beta\nu}$, where $F_{\nu\mu}$ is antisymmetric and $C_{\mu\alpha\beta\nu}$ has the symmetries of the Weyl tensor. To be in accord with conformal invariance and conservation, V_μ must be of scale dimension $d-1$ and $T_{\mu\nu}$ of scale dimension d while both $F_{\mu\nu}$ and $C_{\mu\alpha\beta\nu}$ are required to be of scale dimension $d-2$.

- We have developed a regularisation method which allows to classify the independent forms in the conformal three point functions according to their relation to the anomalous terms in the conformal Ward identities.

  The derivatives introduced into the three point functions by replacing V_μ and $T_{\mu\nu}$ by ${V'}_\mu$ and ${T'}_{\mu\nu}$ are used to regularise the corresponding forms in the three point functions. By extracting the derivatives from the three point functions the power of the spatial variables in the denominator and thus the degree of the singularities is reduced. This method is applied to the axial anomaly as well as to the different contributions to the conformal anomaly.

- The axial anomaly was reproduced by calculating the Ward identity for the form in the three point function involving two vector and one axial vector operators in which the two vector operators trivially satisfy their conservation equations. In this case extracting the derivatives introduced to impose vector operator conservation is sufficient to ensure that the singularities in the three point function are well-defined integrable distributions with an unambiguous Fourier transform. Therefore it is possible to differentiate the regularised expression for the three point function in an unambiguous way with respect to the variable of the axial current to obtain the anomalous Ward identity. Our calculation emphasises the independence of the axial anomaly from any renormalisation scheme chosen once vector conservation has been imposed.

- For the case of the conformal anomaly, which is present in three point functions involving the energy momentum tensor, we have developed our regularisation method further. In this case, extracting the derivatives reduces the non-integrable singularities to terms whose spatial dependence involves just delta functions without derivatives. To ensure that subdivergences for two of the three points coincident are absent, we impose appropriate relations for the parameters in the three point functions. We then find counterterms for removing the non-integrable singular terms by adapting techniques from dimensional regularisation.

- Of the two forms in the three point function for the energy momentum tensor and two conserved vector currents, one may be associated in four dimensions with the term in the conformal anomaly involving the background gauge field, $-\frac{1}{4}\kappa F_{\mu\nu}F^{\mu\nu}$, while the other is anomaly free. For the anomalous form we have related the parameters in the conformal three point function to the coefficient of the energy momentum tensor trace anomaly term involving the background gauge field. To obtain this relation we apply our regularisation method to construct the counterterm necessary for removing the non-integrable singular terms. This counterterm involves an ε pole whose residue is determined by the tensorial structure and the symmetry of the three point function. The regularised three point function in which the counterterm has been subtracted gives rise to an anomalous term in the Ward identity. This term is of the same structure as the anomalous contribution to the Ward identity originating from the background gauge field. By comparing the coefficients we find the anomaly coefficient in terms of the parameters in the three point function.

  The mechanism which generates the anomaly in the three point function relies on a cancellation $\mathcal{O}(\varepsilon/\varepsilon)$: The counterterm which has to be subtracted within dimensional regularisation to ensure a well-defined integrable expression for the three point function involves an $1/\varepsilon$-pole. The tensorial structure of this counterterm, which is dictated by the symmetries of the three point function, is such that its trace contains a factor of ε which cancels the pole. Thus there remains a finite term when the limit $d \to 4$ is taken at the end of the calculation.

We may check the relation (3.3.44) between the anomaly coefficient κ and the parameters in the three point function obtained by using our regularisation method since the gauge field anomaly is scale dependent. Hence the anomaly coefficient κ, the scale of the vector current two point function C_V and the parameters in the three point function are all related to each other by Ward identities. This provides a consistency check for our regularisation method.

Furthermore we have constructed a contribution to the anomaly free form. This form is given by the three point function where all three operators have been replaced by their trivially conserved part.

- In the energy momentum tensor three point function there are two anomalous terms present in four dimensions, one of which arises from the scale dependent part of the anomaly, $-\beta_a F$, while the second originates from the Gauß-Bonnet topological invariant $-\beta_b G$. But since there are three independent forms in the energy momentum tensor three point function we conclude that there must be an anomaly-free form as well. It is given by the three point function for which all the energy momentum tensors are trivially conserved, and is obtained by differentiating the three point function for three Weyl tensors. We construct two contributions to this anomaly-free form explicitly for general dimensions. In four dimensions these become linearly dependent. This is in accord with the fact that the energy momentum tensor three point function has three independent forms in general, and that there are two anomalous forms in four dimensions. In three dimensions, the two forms obtained for the Weyl tensor three point function vanish as required from the properties of the Weyl tensor.

  Moreover we have shown using the Ward identity linking the three point function and the conformal variation of the two point function that the anomaly coefficient β_a of the scale dependent part of the anomaly is proportional to C_T, the scale of the two point function. C_T is expressed in terms of the parameters $\mathcal{A}, \mathcal{B}, \mathcal{C}$ in the three point function by inserting the operator product expansion for $T_{\mu\nu}(x)T_{\sigma\rho}(y)$ into the

Ward identity on flat space. Thus also β_a is proportional to a linear combination of $\mathcal{A}, \mathcal{B}, \mathcal{C}$.

- The main result of this work is a similar relation for the coefficient β_b. Since this coefficient belongs to the topological invariant term in the anomaly, it is not possible to find this relation just by using the Ward identities since there is no contribution to the two point function from $-\beta_b G$. Here we obtain this relation by applying our regularisation method. To this effect we construct the form in the three point function for which two of the three energy momentum tensors have been differentially regularised, and subsequently we calculate the counterterm necessary for removing the non-integrable singular terms. The trace of the three point function regularised by subtracting the counterterm is of the same structure as the anomalous term with coefficient β_b in the Ward identity. By comparing the coefficients we find the required relation. The mechanism by which the counterterm generates the anomalous trace in the three point function relies on a $\mathcal{O}(\varepsilon/\varepsilon)$ cancellation in the same way as described for the gauge field anomaly above. This mechanism, which here we find for the three point functions on flat space, is similar to the cancellation mechanism discussed by Deser and Schwimmer [21] for the trace of the energy momentum tensor expectation value on curved space.

  Using our results the anomaly coefficients κ, β_a and β_b may now be calculated for arbitrary conformal field theories which can be interacting since the parameters in the three point function are in principle calculable for any conformal field theory.

- Using our results we may write the energy momentum tensor three point function in four dimensions as

$$\begin{aligned}\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle \quad = \quad & \beta_a \,\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle_{\beta_a F} \\ & + \beta_b \,\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle_{\beta_b G} \\ & + C_{AF}\,\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle_{AF} \qquad (6.1)\end{aligned}$$

for general four-dimensional conformal field theories. β_a and β_b are given by

$$\begin{aligned}
\beta_a &= \frac{\pi^4}{64 \times 120}\left(9\mathcal{A} - \mathcal{B} - 10\mathcal{C}\right) \ , \\
\beta_b &= \frac{\pi^4}{512 \times 90}(13\mathcal{A} - 2\mathcal{B} - 40\mathcal{C}) \, .
\end{aligned} \tag{6.2}$$

as calculated in sections 3.4 and 4.3 in terms of the three parameters in the energy momentum tensor three point function defined in section 3.4. Using (6.2) we may construct linear combinations of the three independent forms in the energy momentum tensor three point function which belong to the two anomalous forms in (6.1). The anomaly free form with coefficient C_{AF} is discussed in section 4.1. Two of the contributions to this form which are linearly dependent in four dimensions have been constructed explicitly. It would be interesting to find a way to project out the forms in (6.1) unambiguously since C_{AF} has not been uniquely determined yet.

- Furthermore we have constructed and discussed conformal effective actions depending on the metric and on classical background fields which yield the conformal anomaly as well as conformal two and three point functions on flat space. From our discussion of the Riegert action it is clear that additional constraints on the fall off of the energy momentum tensor expectation value at large distances need to be imposed if results obtained from this action, when reducing to flat space, are to correspond to standard field theory results. This is important since the Riegert action is widely used in the literature, for instance for constructing quantum states in agreement with spatial diffeomorphism invariance and for relating the Casimir energy for a constrained geometry to the trace anomaly [30].

  We propose an approach for constructing four-dimensional effective actions in agreement with conformal invariance using second order conformal differential operators acting on k-forms. We have found an action giving rise to the scalar contribution to the trace anomaly which also yields the conformal three point function involving the energy momentum tensor and two scalar operators on flat space. Furthermore we have found effective actions generating anomaly free contributions to the three point functions involving the energy momentum tensor and two conserved vector currents or three energy momentum tensors. Unlike the action constructed by Riegert, these new actions lead to an expectation value for the energy momentum tensor which falls off rapidly enough at large distances to generate consistent results for the conformal three point functions on flat space.

Furthermore we have constructed a conformally covariant second order differential operator acting on tensors with Weyl symmetry, and also its associated Green function on flat space. The fact that this operator is not invertible in $d = 4$ or $d = 6$ dimensions may be related to the anomaly.

By pursuing this approach it should also be feasible to construct actions which give rise to the gauge field and gravitational anomalies and which generate the corresponding anomalous forms in the three point functions. This will be an interesting result for the study of quantum field theories in a constrained geometry as well as for investigating the back reaction of matter fields on the geometry in black holes in four dimensions.

- The coefficient of the topological part of the trace anomaly has been suggested as a candidate for a four dimensional C-function which could be used for a proof of Zamolodchikov's C-theorem in four dimensions [9]. This function reduces to β_b in the conformal limit. Like the central charge c in two dimensions, β_b is the coefficient of the topological contribution to the trace anomaly in four dimensions. For a consistent physical interpretation in the framework of the C-theorem it is necessary that $\beta_b \geq 0$.

  A very encouraging result as far as the positivity of β_b is concerned was found in section 4.3.3, where β_c, the coefficient of the anomaly involving the tensor with Weyl symmetry $\mathcal{C}_{\mu\kappa\lambda\nu}$ defined in (4.3.41), is related to the scale C_C of the two point function involving two tensors conjugate to $\mathcal{C}_{\mu\kappa\lambda\nu}$. The relevant relation is

$$\beta_c = \frac{\pi^2}{9} C_C \, . \tag{6.3}$$

  C_C is positive since two point functions must be positive by unitarity and reflection positivity. The tensor $\mathcal{C}_{\mu\kappa\lambda\nu}$ is independent of the metric in general. It is conceivable that it is possible to find a similar relation for the genuine Weyl tensor as defined in (2.4.4), which depends on the metric. This would lead to a similar relation for β_b.

The importance of relation (6.3) is enhanced by the fact that other approaches towards proving the conjectured positivity of β_b have not been successful so far. A particular approach of this kind involves the energy momentum tensor three point function. For Euclidean three point functions the requirement of positive energy density translates into the condition [7]

$$\langle \Psi T_{11}(0) \Theta \Psi \rangle \ \leq 0 \tag{6.4}$$

in the collinear frame , for Ψ some function of the fields restricted to $x_1 \geq 0$. Θ is the antilinear conjugation operator combined with reflection in the plane $x_1 = 0$. It satisfies $\Theta^2 = 1$, and for the components of the energy momentum tensor we have $\Theta T_{i1}(y) = -T_{i1}(\bar{y})$, $\Theta T_{ij}(y) = T_{ij}(\bar{y})$ where $\bar{y}$ denotes the reflection of y through the plane $x_1 = 0$. If we insert these components of $T_{\mu\nu}$ for Ψ in condition (6.4), we obtain $r > 0, t > 0$ but $s < 0$ for the collinear frame parameters. Given the collinear frame expression (3.4.32) for β_b this does not imply that $\beta_b \geq 0$. Even the imposition of stronger conditions like the weak energy conditions of general relativity does not seem to yield the desired result. Hence in order to find positivity relations for three point functions, it is worthwhile to investigate whether it is feasible to adapt correlation inequalities for three point functions known from statistical mechanics [31] to continuous quantum field theories.

Moreover in view of a possible proof of the Zamolodchikov C-theorem in four dimensions it will be necessary to study the implications of breaking conformal invariance by some relevant operators and to determine the structure of the two and three point functions along a RG flow. Although away from the fixed points where only Poincaré invariance is maintained there are significantly more terms than in two dimensions, it may be possible to construct such terms using algebraic computing programs like FORM, making use of the experience gained by using this program for the calculations presented here.

- Our results also open up the interesting possibility of calculating the anomaly coefficients β_a and β_b for interacting conformal field theories at non-trivial renormalisation group fixed points. This should be of interest for understanding the new supersymmetric conformal field theories [2]. Some of the implications of conformal invariance in these theories have already been studied [32], but there are no general rules for constructing three point functions or calculating anomaly coefficients yet. Recent results indicate that the C-theorem holds also in this context [33]. Therefore it will be interesting to extend our results to supersymmetric theories.

A Appendix

A.1 Projection Operator onto the Space of Tensors with Weyl Symmetry

The projection operator $\mathcal{E}^C$ has the explicit form

$$\begin{aligned}
\mathcal{E}^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta} \;=\;& \tfrac{1}{12}\left(\delta_{\mu\alpha}\delta_{\nu\beta}\delta_{\sigma\gamma}\delta_{\rho\delta}+\delta_{\mu\delta}\delta_{\sigma\beta}\delta_{\rho\alpha}\delta_{\nu\gamma}-\mu\leftrightarrow\sigma,\nu\leftrightarrow\rho\right)\\
&+\tfrac{1}{24}\left(\delta_{\mu\alpha}\delta_{\nu\gamma}\delta_{\rho\delta}\delta_{\sigma\beta}-\mu\leftrightarrow\sigma,\nu\leftrightarrow\rho,\alpha\leftrightarrow\gamma,\beta\leftrightarrow\delta\right)\\
&-\frac{1}{d-2}\tfrac{1}{8}\left(\delta_{\mu\rho}\delta_{\alpha\delta}\delta_{\sigma\gamma}\delta_{\nu\beta}+\delta_{\mu\rho}\delta_{\alpha\delta}\delta_{\sigma\beta}\delta_{\nu\gamma}-\mu\leftrightarrow\sigma,\nu\leftrightarrow\rho,\alpha\leftrightarrow\gamma,\beta\leftrightarrow\delta\right)\\
&+\frac{1}{(d-2)(d-1)}\tfrac{1}{2}\left(\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\nu}\delta_{\sigma\rho}\right)\left(\delta_{\alpha\delta}\delta_{\beta\gamma}-\delta_{\alpha\beta}\delta_{\gamma\delta}\right)\ . \qquad \text{(A.1.1)}
\end{aligned}$$

A.2 Derivation of the Singular Contribution to Scalar Three Point Functions

Here we give a derivation of equation (3.3.27). To derive the singular term in the scalar three point function

$$\frac{1}{(x-z)^{2\lambda_1}(y-z)^{2\lambda_2}(y-x)^{2\lambda_3}}\,, \qquad \text{(A.2.1)}$$

we use its Fourier transform which we calculate using integral representations of the Gamma function. We define $x_1 = x - y$, $x_2 = x - z$ and consider

$$I(x_1,x_2) = \frac{1}{{x_1}^{2\lambda_1}{x_2}^{2\lambda_2}(x_1-x_2)^{2\lambda_3}}\,. \qquad \text{(A.2.2)}$$

Using the identity

$$\Gamma(\mu)z^{-\mu} = \int_0^\infty \mathrm{d}\alpha\,\alpha^{-(\mu+1)}e^{-z/\alpha}\,, \qquad \text{(A.2.3)}$$

for each of the factors in the three point function, we find

$$I(x_1,x_2) = \prod_{i=1}^{3}\frac{1}{\Gamma(\lambda_i)}\int_0^\infty \mathrm{d}\alpha_i\,\alpha_i^{-(\lambda_i+1)}e^{-F(x_1,x_2)} \qquad \text{(A.2.4)}$$

with

$$F(x_1,x_2) = \frac{1}{\alpha_2}x_1^2 + \frac{1}{\alpha_1}x_2^2 + \frac{1}{\alpha_3}(x_1-x_2)^2$$
$$= \frac{\alpha_1+\alpha_3}{\alpha_1\alpha_3}{x_2'}^2 + \frac{\alpha_1+\alpha_2+\alpha_3}{\alpha_2(\alpha_1+\alpha_3)}x_1^2\,, \tag{A.2.5}$$

where we have defined

$$x_2' = x_2 - \frac{\alpha_1}{\alpha_1+\alpha_3}x_1\,. \tag{A.2.6}$$

Next we calculate the Fourier transform

$$\tilde{I}(p_1,p_2) = \int \mathrm{d}^d x_1 \mathrm{d}^d x_2\, e^{ip_1\cdot x_1 + ip_2\cdot x_2}\, I(x_1,x_2) \tag{A.2.7}$$

using

$$\int \mathrm{d}^d x\, e^{ip\cdot x - sx^2} = \left(\frac{\pi}{s}\right)^{d/2} e^{-p^2/4s} \tag{A.2.8}$$

for both x_1 and x_2 and by writing

$$p_1\cdot x_1 + p_2\cdot x_2 = \left(p_1 + \frac{\alpha_1}{\alpha_1+\alpha_3}p_2\right)\cdot x_1 + p_2\cdot x_2'\,, \tag{A.2.9}$$

by which we obtain

$$\tilde{I}(p_1,p_2) = \pi^d \prod_{i=1}^{3} \frac{1}{\Gamma(\lambda_i)} \int_0^\infty \mathrm{d}\alpha_i\, \alpha_i^{d/2-\lambda_i-1} \left(\sum_i \alpha_i\right)^{-d/2} e^{-\frac{1}{4}E}\,, \tag{A.2.10}$$

where

$$E = \frac{\alpha_2\alpha_3 p_1^2 + \alpha_1\alpha_3 p_2^2 + \alpha_1\alpha_2 p_3^2}{\alpha_1+\alpha_2+\alpha_3}\,. \tag{A.2.11}$$

We now perform a change of variables $\alpha_i = \beta\gamma_i$ where we impose $\sum_i \gamma_i = 1$, so that $\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3 = \beta^2\mathrm{d}\beta\mathrm{d}\gamma_1\mathrm{d}\gamma_2$. Using (A.2.3) again we carry out the β integration

$$\int_0^\infty \mathrm{d}\beta\, \beta^{d-\lambda-1} e^{-\beta X} = X^{\lambda-d}\Gamma(d-\lambda)\,. \tag{A.2.12}$$

Then

$$\tilde{I}(p_1,p_2) = \pi^d \Gamma(d - \sum_i \lambda_i) \int_0^1 \mathrm{d}\gamma_1 \int_0^{1-\gamma_1} \mathrm{d}\gamma_2 \prod_{i=1}^{3} \frac{\gamma_i^{d/2-\lambda_i-1}}{\Gamma(\lambda_i)} \left(\tfrac{1}{4}E'\right)^{-(d-\sum\lambda_i)}\,, \tag{A.2.13}$$

where

$$E' = \gamma_2\gamma_3 {p_1}^2 + \gamma_1\gamma_3 {p_2}^2 + \gamma_1\gamma_2 {p_3}^2\,, \qquad \gamma_3 = 1 - \gamma_1 - \gamma_2\,. \tag{A.2.14}$$

When determining the singular behaviour, we are interested in the situation where the overall $\Gamma(d - \sum_i \lambda_i)$ factor has a pole, i.e. where $\sum \lambda_i = d$. This is the case in both sections 3.3 and 4.2 where $\sum \lambda_i = \frac{1}{2}(3d-4)$ and $d \to 4$. Here there is a pole

$$\tilde{I}(p_1, p_2) \;\sim\; \frac{1}{d - \sum \lambda_i} R_I \tag{A.2.15}$$

with the residue

$$R_I = \pi^d \int_0^1 \mathrm{d}\gamma_1 \int_0^{1-\gamma_1} \mathrm{d}\gamma_2 \prod_{i=1}^{3} \frac{\gamma_i^{d/2-\lambda_i-1}}{\Gamma(\lambda_i)}\,, \tag{A.2.16}$$

since the exponent of the p-dependent term in (A.2.13) vanishes at the pole. In (A.2.16) we have substituted $\gamma_3 = 1 - \gamma_1 - \gamma_2$ using the condition $\sum \gamma_i = 1$. We may evaluate the integral in the residue by applying

$$\int_0^1 \mathrm{d}x \, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{A.2.17}$$

twice which yields

$$\int_0^1 \mathrm{d}\gamma_1 \int_0^{1-\gamma_1} \mathrm{d}\gamma_2 \prod_{i=1}^{3} \gamma_i^{s_i-1} = \frac{\Gamma(s_1)\Gamma(s_2)\Gamma(s_3)}{\Gamma(s_1+s_2+s_3)}\,. \tag{A.2.18}$$

Then the residue is

$$R_I \;=\; \frac{\pi^d}{\Gamma(\frac{1}{2}d)} \prod_{i=1}^{3} \frac{\Gamma(\frac{1}{2}d - \lambda_i)}{\Gamma(\lambda_i)}\,. \tag{A.2.19}$$

Fourier transforming back to position space we then have

$$\begin{aligned} I(x_1, x_2) \;&=\; \frac{1}{(2\pi)^{2d}} \int \mathrm{d}^d p_1 \mathrm{d}^d p_2 \, e^{-i(p_1 \cdot x_1 + p_2 \cdot x_2)} \tilde{I}(p_1, p_2) \\ &\sim\; \frac{1}{d - \sum \lambda_i} \frac{\pi^d}{\Gamma(\frac{1}{2}d)} \prod_{i=1}^{3} \frac{\Gamma(\frac{1}{2}d - \lambda_i)}{\Gamma(\lambda_i)} \, \delta^d(x_1)\delta^d(x_2)\,. \end{aligned} \tag{A.2.20}$$

This is equation (3.3.27), recalling that $x_1 = x - y$, $x_2 = x - z$.

A.3 Forms in $\langle TFF \rangle$ Necessary for Calculating its Non-Integrable Singular Terms

Here we give the expression for the three point function involving the energy momentum tensor and two field strength tensors of section 3.3, after contraction according to (3.3.35).

$$
\begin{aligned}
&(z-x)^d(y-x)^d(y-z)^{d-4}\langle T_{\mu\nu}(x)F_{\mu\alpha}(y)F_{\nu\alpha}(z)\rangle \\
&= \quad A \quad \Bigg[-\frac{1}{8}(d-2)(d^2-8)\frac{1}{d}\left(\frac{(y-z)^4}{(y-x)^2(z-x)^2}+\frac{(z-x)^2}{(y-x)^2}+\frac{(y-x)^2}{(z-x)^2}\right) \\
&\qquad\qquad +\frac{1}{4}(d-2)(d^2-8)\frac{1}{d}\left(\frac{(y-z)^2}{(z-x)^2}+\frac{(y-z)^2}{(y-x)^2}\right) \\
&\qquad\qquad -\frac{1}{8}(d-2)(d^3-5d^2+2d+16)\frac{1}{d}\Bigg] \\
&+ \quad B \quad \Bigg[-\frac{1}{16}(d-2)(d^2-d-4)\frac{1}{d}\frac{(y-z)^4}{(y-x)^2(z-x)^2} \\
&\qquad\qquad +\frac{1}{16}(d-2)(2d^2-3d-8)\frac{1}{d}\left(\frac{(y-z)^2}{(z-x)^2}+\frac{(y-z)^2}{(y-x)^2}\right) \\
&\qquad\qquad -\frac{1}{4}(d-2)\frac{1}{d} \\
&\qquad\qquad -\frac{1}{16}(d-2)(d-4)(d+1)\frac{1}{d}\left(\frac{(z-x)^2}{(y-x)^2}+\frac{(y-x)^2}{(z-x)^2}\right) \\
&\qquad\qquad +\frac{1}{16}(d-2)\left(\frac{(z-x)^2}{(y-z)^2}+\frac{(y-x)^2}{(y-z)^2}\right) \\
&\qquad\qquad -\frac{1}{16}(d-2)\left(\frac{(z-x)^4}{(y-z)^2(y-x)^2}+\frac{(y-x)^4}{(z-x)^2(y-z)^2}\right)\Bigg] \\
&+ \quad C \quad \Bigg[-\frac{1}{8}(d-2)(d-4)\frac{1}{d}\frac{(y-z)^4}{(z-x)^2(y-x)^2} \\
&\qquad\qquad +\frac{1}{16}(d-2)(d^2+2d-16)\frac{1}{d}\left(\frac{(y-z)^2}{(z-x)^2}+\frac{(y-z)^2}{(y-x)^2}\right) \\
&\qquad\qquad -\frac{1}{2}(d-2)\frac{1}{d}-\frac{1}{8}(d-2)(d^2-d-4)\frac{1}{d}\left(\frac{(z-x)^2}{(y-x)^2}+\frac{(y-x)^2}{(z-x)^2}\right) \\
&\qquad\qquad -\frac{1}{16}(d-2)^2\left(\frac{(z-x)^2}{(y-z)^2}+\frac{(y-x)^2}{(y-z)^2}\right) \\
&\qquad\qquad +\frac{1}{16}(d-2)^2\left(\frac{(z-x)^4}{(y-z)^2(y-x)^2}+\frac{(y-x)^4}{(z-x)^2(y-z)^2}\right)\Bigg] \\
&+ \quad D \quad \Bigg[-\frac{1}{16}(d-2)(d-4)\frac{1}{d}\frac{(y-z)^4}{(y-x)^2(z-x)^2} \\
&\qquad\qquad +\frac{1}{8}(d-2)(d-4)\frac{1}{d}\left(\frac{(y-z)^2}{(y-x)^2}+\frac{(y-z)^2}{(z-x)^2}+1\right) \\
&\qquad\qquad -\frac{1}{16}(d-2)(d-4)\frac{1}{d}\left(\frac{(z-x)^2}{(y-x)^2}+\frac{(y-x)^2}{(z-x)^2}\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
+ \quad E \quad & \Bigg[-\frac{1}{64}(d-2)(d-4)\frac{1}{d}\frac{(y-z)^4}{(y-x)^2(z-x)^2} \\
& +\frac{1}{32}(d-2)(d-4)\frac{1}{d}\left(\frac{(y-z)^2}{(y-x)^2}+\frac{(y-z)^2}{(z-x)^2}-2\right) \\
& -\frac{1}{32}(d-2)^2\frac{1}{d}\left(\frac{(y-x)^2}{(z-x)^2}+\frac{(z-x)^2}{(y-x)^2}\right) \\
& -\frac{1}{32}(d-2)\left(\frac{(z-x)^2}{(y-z)^2}+\frac{(y-x)^2}{(y-z)^2}\right) \\
& +\frac{1}{32}(d-2)\left(\frac{(z-x)^4}{(y-z)^2(y-x)^2}+\frac{(y-x)^4}{(z-x)^2(y-z)^2}\right) \\
& +\frac{1}{16}(d-2)\left(\frac{(z-x)^4}{(y-z)^4}+\frac{(y-x)^4}{(y-z)^4}\right) \\
& -\frac{1}{64}(d-2)\left(\frac{(z-x)^6}{(y-z)^4(y-x)^2}+\frac{(y-x)^6}{(z-x)^2(y-z)^4}\right) \\
& -\frac{3}{32}(d-2)\frac{(y-x)^2(z-x)^2}{(y-z)^4}\Bigg] \ . \qquad \text{(A.3.1)}
\end{aligned}
$$

A.4 Coefficients , $\mathcal{B}, \mathcal{C}$ for Free Field Theories in General Dimensions

In general dimensions, there is no conformal vector field theory and the three coefficients in the energy momentum tensor three point function are given by:

$$
\begin{aligned}
\mathcal{A} &= \frac{1}{S_d^3}\frac{d^3}{(d-1)^3}n_\varphi \\
\mathcal{B} &= -\frac{1}{S_d^3}\left(\frac{d^3(d-2)}{(d-1)^3}n_\varphi + 2d^2\tilde{n}_\psi\right) \\
\mathcal{C} &= -\frac{1}{S_d^3}\left(\frac{d^2(d-2)}{4(d-1)^3}n_\varphi + d^2\tilde{n}_\psi\right) \ . \qquad \text{(A.4.1)}
\end{aligned}
$$

Here $\tilde{n}_\psi = \frac{1}{4}\mathrm{tr}_D\,(1)n_\psi$, where tr_D is the Dirac trace. From (3.4.11) we obtain the standard result

$$
C_T = \frac{1}{S_d^2}\left(\frac{d}{d-1}n_\varphi + 2d\tilde{n}_\psi\right) \ . \qquad \text{(A.4.2)}
$$

A.5 Relations Between the Coefficients of $\langle TTT\rangle$ and $\langle TCC\rangle$ in d Dimensions

Here we give the results for the coefficients defined by (3.4.7) for the three point function given by (4.2.3) and (4.2.6) in d dimensions. For $d = 4$ these reduce to the results given in (4.2.25).

$$
\begin{aligned}
\mathcal{A} \;=\;& -\frac{d(d-4)^2}{2(d-2)}A + 2\,\frac{d(d-3)}{d-1}B \\
&+\frac{d}{(d-1)^2(d-2)}\Big\{\tfrac{1}{2}(d^3-6d^2+9d+4)C+\tfrac{1}{2}(d^3-7d^2+15d-5)D \\
&+\tfrac{1}{2}(d-3)(d^2-d+2)F\Big\} \\
&+\tfrac{1}{2}\frac{d}{(d-1)(d-2)}\Big\{(d^2-7d+14)E-(d-5)Q-\tfrac{1}{4}(d-9)R+2S\Big\} \\
&-\frac{d}{4(d-1)^2(d-2)^2}(3d^3-16d^2+15d+6)G\,, \qquad\qquad (\text{A.5.1})
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B} \;=\;& -\frac{d(d-4)^2(d-1)}{4(d-2)}A + 2\,\frac{d^2(d-3)}{d-1}B+\frac{d}{4(d-1)(d-2)}(3d^3-18d^2+27d+4)E \\
&+\frac{d}{(d-1)^2(d-2)}\Big\{-\tfrac{1}{4}(d^4-7d^3+11d^2+11d-32)C \\
&+\tfrac{1}{4}(d^4-7d^3+17d^2-17d+14)D+\tfrac{1}{2}(d-3)(d^3-3d^2+2d+2)F\Big\} \\
&-\frac{d}{2(d-1)(d-2)^2}\Big\{(d^3-10d^2+21d-4)Q \\
&+\tfrac{1}{4}d(d-5)(3d-7)R-(3d^2-8d+1)S\Big\} \\
&+\frac{d}{8(d-1)^2(d-2)^2}(d^4-7d^3+11d^2+27d-48)G\,, \qquad\qquad (\text{A.5.2})
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C} \;=\;& -\frac{d(d-4)^2(d-1)}{8(d-2)}A + \frac{d(d-2)(d-3)}{2(d-1)}B+\frac{d}{8(d-1)(d-2)}(d^3-10d^2+33d-32)E \\
&+\frac{d}{4(d-1)^2(d-2)}\Big\{+\tfrac{1}{2}(d^4-5d^3-d^2+37d-40)C \\
&+\tfrac{1}{2}(d^4-8d^3+21d^2-14d-4)D+\tfrac{1}{2}(d-3)(d^3-3d^2+8d-8)F\Big\} \\
&-\frac{d}{4(d-1)(d-2)^2}\Big\{(d^3-8d^2+19d-16)Q \\
&+\tfrac{1}{4}(d^3-14d^2+41d-36)R-(2d^2-7d+7)S\Big\} \\
&-\frac{d}{16(d-1)^2(d-2)^2}(3d^4-15d^3+d^2+63d-60)G\,. \qquad\qquad (\text{A.5.3})
\end{aligned}
$$

A.6 Forms in $\langle TCC\rangle$ Necessary for Calculating its Non-Integrable Singular Terms

In the following, we list the forms in $\langle T_{\mu\nu}(x)C_{\sigma\varepsilon\eta\rho}(y)C_{\alpha\gamma\delta\beta}(z)\rangle$ corresponding to the ten terms in (4.2.6) after substituting (2.1.10) for X_{23} and contracting according to (4.3.26):

$$\langle T_{\mu\nu}(x)C_{\mu\gamma\delta\beta}(y)C_{\nu\gamma\delta\beta}(z)\rangle = \frac{(d-3)(d-4)}{d(d-2)}\sum_i A_i \frac{a_i(x,y,z)}{(y-x)^d(z-x)^d(z-y)^{(d-4)}} \qquad \text{(A.6.1)}$$

We label the $a_i^{A_i}(x,y,z)$ by their index i and their coefficient A_i:

$$\begin{aligned}
a_1^A(x,y,z) \;=\;& \frac{d}{32}(d^2+4d-8)\left[\frac{(y-x)^4}{(z-x)^4}+\frac{(z-x)^4}{(y-x)^4}\right.\\
&-4\frac{(z-y)^2(y-x)^2}{(z-x)^4}-4\frac{(z-y)^2(z-x)^2}{(y-x)^4}\\
&\left.-\frac{(z-y)^6}{(y-x)^2(z-x)^4}-\frac{(z-y)^6}{(z-x)^2(y-x)^4}+\frac{(z-y)^8}{(y-x)^4(z-x)^4}\right]\\
+\;& \frac{d}{16}(3d^2+12d-24)\left[\frac{(z-y)^4}{(z-x)^4}+\frac{(z-y)^4}{(y-x)^4}\right]\\
+\;& \frac{1}{96}(d-2)(d^4-2d^3-9d^2+28d-32)\\
+\;& \frac{1}{96}(3d^4-5d^3-36d^2+72d-32)\left[\frac{(y-x)^2}{(z-x)^2}+\frac{(z-x)^2}{(y-x)^2}\right]\\
+\;& \frac{1}{48}(-3d^4-d^3+12d^2-24d+32)\left[\frac{(z-y)^2}{(z-x)^2}+\frac{(z-y)^2}{(y-x)^2}\right]\\
+\;& \frac{1}{96}(3d^4+19d^3+80d^2-120d-32)\frac{(z-y)^4}{(y-x)^2(z-x)^2} \qquad \text{(A.6.2)}\\
a_2^B(x,y,z) \;=\;& -\frac{1}{12}(d-2)^3\\
+\;& \frac{d}{8}(d-2)\left[\frac{(y-x)^4}{(z-x)^4}+\frac{(z-x)^4}{(y-x)^4}+6\frac{(z-y)^4}{(z-x)^4}+6\frac{(z-y)^4}{(y-x)^4}\right.\\
&-4\frac{(z-y)^2(y-x)^2}{(z-x)^4}-4\frac{(z-y)^2(z-x)^2}{(y-x)^4}\\
&\left.-4\frac{(z-y)^6}{(y-x)^2(z-x)^4}-4\frac{(z-y)^6}{(z-x)^2(y-x)^4}+\frac{(z-y)^8}{(y-x)^4(z-x)^4}\right]\\
+\;& \frac{1}{24}(d-2)(d^2-7d+4)\left[\frac{(y-x)^2}{(z-x)^2}+\frac{(z-x)^2}{(y-x)^2}\right]
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{12}(d-2)(d^2-d+4)\left[\frac{(z-y)^2}{(z-x)^2}+\frac{(z-y)^2}{(y-x)^2}\right] \\
&+ \frac{1}{24}(d-2)(d^2+17d+4)\frac{(z-y)^4}{(y-x)^2(z-x)^2} \qquad \text{(A.6.3)}
\end{aligned}
$$

$$
\begin{aligned}
a_3^C(x,y,z) = &\frac{d^2}{64}\left[\frac{(y-x)^6}{(z-y)^2+(z-x)^4}+\frac{(z-x)^6}{(z-y)^2\,(y-x)^4}\right] \\
&+\frac{d}{192}(d-2)^2\left[\frac{(y-x)^4}{(z-y)^2\,(z-x)^2}+\frac{(z-x)^4}{(z-y)^2\,(y-x)^2}\right] \\
&+\frac{d}{64}(d^2-2d-4)\left[\frac{(y-x)^4}{(z-x)^4}+\frac{(z-x)^4}{(y-x)^4}\right] \\
&-\frac{d}{192}\left[\frac{(y-x)^2}{(z-y)^2}+\frac{(z-x)^2}{(z-y)^2}\right] \\
&+\frac{(d-4)}{192}(d^3+d^2-4d+8)\left[\frac{(y-x)^2}{(z-x)^2}+\frac{(z-x)^2}{(y-x)^2}\right] \\
&-\frac{d}{32}(2d^2+d-8)\left[\frac{(z-y)^2\,(y-x)^2}{(z-x)^4}+\frac{(z-y)^2\,(z-x)^2}{(y-x)^4}\right] \\
&+\frac{1}{96}(2d^3+d^2-34d+24) \\
&-\frac{1}{192}(2d^4+9d^3-7d^2+12d-64)\left[\frac{(z-y)^2}{(z-x)^2}+\frac{(z-y)^2}{(y-x)^2}\right] \\
&+\frac{d}{32}(3d^2+4d-12)\left[\frac{(z-y)^4}{(z-x)^4}+\frac{(z-y)^4}{(y-x)^4}\right] \\
&+\frac{1}{192}(d^4+23d^3+26d^2-64d-32)\frac{(z-y)^4}{(y-x)^2\,(z-x)^2} \\
&-\frac{d}{64}(4d^2+7d-16)\left[\frac{(z-y)^6}{(y-x)^2\,(z-x)^4}+\frac{(z-y)^6}{(y-x)^4\,(z-x)^2}\right] \\
&+\frac{d}{64}\left(d^2+2\,d-4\right)\frac{(z-y)^8}{(y-x)^4\,(z-x)^4} \qquad \text{(A.6.4)}
\end{aligned}
$$

$$
\begin{aligned}
a_4^D(x,y,z) = &\frac{d}{128}(d^2+4d-8)\left[\frac{(y-x)^4}{(z-x)^4}+\frac{(y-x)^4}{(z-x)^4}+\frac{(z-y)^8}{(y-x)^4(z-x)^4}\right. \\
&\quad -4\frac{(z-y)^6}{(y-x)^2(z-x)^4}-4\frac{(z-y)^6}{(z-x)^2(y-x)^4} \\
&\quad \left.-4\frac{(z-x)^2(z-y)^2}{(y-x)^4}-4\frac{(z-y)^2(y-x)^2}{(z-x)^4}\right] \\
&+\frac{d}{384}(d-8)(d+1)\left[\frac{(y-x)^2}{(z-x)^2}+\frac{(z-x)^2}{(y-x)^2}\right.
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{(y-x)^4}{(z-y)^2(z-x)^2}-\frac{(z-x)^4}{(z-y)^2(y-x)^2}\Bigg] \\
& +\frac{d}{64}\left[\frac{(z-y)^4}{(z-x)^4}+\frac{(z-y)^4}{(z-x)^4}\right] \\
& +\frac{1}{384}(2d^4-3d^3-25d^2+48d-64)\left[\frac{(y-x)^2}{(z-x)^2}+\frac{(z-x)^2}{(y-x)^2}\right] \\
& +\frac{(d+4)\,(4\,d^2-17\,d+12)}{192} \\
& -\frac{1}{384}(4d^4+3d^3+19d^2+24d-128)\left[\frac{(z-y)^2}{(y-x)^2}+\frac{(z-y)^2}{(z-x)^2}\right] \\
& +\frac{1}{384}(2d^4+19d^3+85d^2-128d-64)\frac{(z-y)^4}{(y-x)^2\,(z-x)^2}
\end{aligned}
\tag{A.6.5}
$$

$$
\begin{aligned}
a_5^E(x,y,z) \;=\; & -\frac{d^3}{128}\left[(y-x)^6(z-y)^2\,(z-x)^4+\frac{(z-x)^6}{(z-y)^2\,(y-x)^4}\right] \\
& -\frac{d}{384}\left(d^3-6\,d^2-6\,d-8\right)\left[\frac{(y-x)^4}{(z-y)^2\,(z-x)^2}+\frac{(z-x)^4}{(z-y)^2\,(y-x)^2}\right] \\
& +\frac{d}{32}(d-1)(d+2)\left[\frac{(y-x)^4}{(z-x)^4}+\frac{(z-x)^4}{(y-x)^4}\right] \\
& +\frac{d}{384}\left(d^3-3\,d^2-6\,d-8\right)\left[\frac{(y-x)^2}{(z-y)^2}+\frac{(z-x)^2}{(z-y)^2}\right] \\
& +\frac{(d+4)}{192}\left(d^3-6\,d^2+9\,d-8\right)\left[\frac{(y-x)^2}{(z-x)^2}+\frac{(z-x)^2}{(y-x)^2}\right] \\
& -\frac{d}{64}(d+4)(3d-4)\left[\frac{(z-y)^2\,(y-x)^2}{(z-x)^4}+\frac{(z-y)^2\,(z-x)^2}{(y-x)^4}\right] \\
& +\frac{(d-1)\,(d-4)}{16} \\
& -\frac{1}{384}(d^4+13d^3+6d^2+40d-128)\left[\frac{(z-y)^2}{(z-x)^2}+\frac{(z-y)^2}{(y-x)^2}\right] \\
& +\frac{d}{32}(d^2+6d-12)\left[\frac{(z-y)^4}{(z-x)^4}+\frac{(z-y)^4}{(y-x)^4}\right] \\
& +\frac{1}{192}(7d^3+39d^2-60d-32)\frac{(z-y)^4}{(y-x)^2\,(z-x)^2} \\
& -\frac{d}{128}\left(d^2+16\,d-32\right)\left[\frac{(z-y)^6}{(y-x)^2\,(z-x)^4}+\frac{(z-y)^6}{(y-x)^4\,(z-x)^2}\right] \\
& +\frac{d}{32}(d-2)\frac{(z-y)^8}{(y-x)^4\,(z-x)^4}
\end{aligned}
\tag{A.6.6}
$$

$$
\begin{aligned}
a_6^F(x,y,z) &= -\frac{d^2}{128}(d-1)\left[\frac{(y-x)^6}{(z-y)^2(z-x)^4}+\frac{(z-y)^6}{(y-x)^2(z-x)^4}\right] \\
&+\frac{d^2}{64}(d-2)\left[\frac{(y-x)^4}{(z-y)^2(z-x)^2}+\frac{(z-x)^4}{(z-y)^2(y-x)^2}\right] \\
&+\frac{d^2}{32}(d-1)\left[\frac{(y-x)^4}{(z-x)^4}+\frac{(z-y)^4}{(z-x)^4}\right] \\
&-\frac{d^2}{128}(d-3)\left[\frac{(y-x)^2}{(z-y)^2}+\frac{(z-x)^2}{(z-y)^2}\right] \\
&-\frac{d}{64}(d+1)(d-2)\left[\frac{(y-x)^2}{(z-x)^2}+\frac{(z-x)^2}{(y-x)^2}\right] \\
&-\frac{d^2}{64}3(d-1)\left[\frac{(z-y)^2(y-x)^2}{(z-x)^4}+\frac{(z-y)^2(z-x)^2}{(y-x)^4}\right] \\
&-\frac{d^2(d-3)}{32} \\
&-\frac{d}{128}\left(3d^2-9d+8\right)\left[\frac{(z-y)^2}{(z-x)^2}+\frac{(z-y)^2}{(y-x)^2}\right] \\
&+\frac{d}{32}(d-1)^2\frac{(z-y)^4}{(y-x)^2(z-x)^2} \\
&-\frac{d^2}{128}(d-1)\left[\frac{(z-x)^6}{(z-y)^2(y-x)^4}+\frac{(z-y)^6}{(y-x)^4(z-x)^2}\right] \\
&\frac{d^2}{32}(d-1)\left[\frac{(z-y)^4}{(y-x)^4}+\frac{(z-x)^4}{(y-x)^4}\right]
\end{aligned}
\tag{A.6.7}
$$

$$
\begin{aligned}
a_7^G(x,y,z) &= \frac{1}{128}(d^3-16d+8) \\
&+\frac{d^2}{256}\left[\frac{(y-x)^6}{(z-y)^2(z-x)^4}+\frac{(z-x)^6}{(z-y)^2(y-x)^4}\right] \\
&+\frac{d}{256}\left(d^2-2d-4\right)\left[\frac{(y-x)^4}{(z-x)^4}+\frac{(z-x)^4}{(y-x)^4}\right] \\
&-\frac{d}{128}\left(2d^2+d-8\right)\left[\frac{(z-y)^2(y-x)^2}{(z-x)^4}+\frac{(z-y)^2(z-x)^2}{(y-x)^4}\right] \\
&+\frac{d}{128}\left(3d^2+4d-12\right)\left[\frac{(z-y)^4}{(z-x)^4}+\frac{(z-y)^4}{(y-x)^4}\right] \\
&-\frac{d}{256}\left(4d^2+7d-16\right)\left[\frac{(z-y)^6}{(y-x)^2(z-x)^4}+\frac{(z-y)^6}{(y-x)^4(z-x)^2}\right] \\
&+\frac{d}{256}\left(d^2+2d-4\right)\frac{(z-y)^8}{(y-x)^4(z-x)^4}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{128}(2d^3+4d^2-3d-16)\left[\frac{(z-y)^2}{(y-x)^2}+\frac{(z-y)^2}{(z-x)^2}\right]\\
&+\frac{1}{128}(d^3-3d-8)\left[\frac{(y-x)^2}{(z-x)^2}+\frac{(z-x)^2}{(y-x)^2}\right]\\
&+\frac{d}{32}\left[\frac{(y-x)^4}{(z-y)^4}+\frac{(z-x)^4}{(z-y)^4}\right]\\
&-\frac{5d}{128}\left[\frac{(z-x)^2}{(z-y)^2}+\frac{(y-x)^2}{(z-y)^2}\right]\\
&+\frac{1}{128}(4d^3+7d^2-12d-8)\frac{(z-y)^4}{(y-x)^2(z-x)^2}\\
&-\frac{d}{256}(d-10)\left[\frac{(y-x)^4}{(z-y)^2(z-x)^2}+\frac{(z-x)^4}{(z-y)^2(y-x)^2}\right]\\
&-\frac{d}{128}\left[\frac{(y-x)^6}{(z-y)^4(z-x)^2}+\frac{(z-x)^6}{(z-y)^4(y-x)^2}\right]\\
&-\frac{3d}{64}\frac{(y-x)^2(z-x)^2}{(z-y)^4}
\end{aligned}
\qquad \text{(A.6.8)}
$$

$$
\begin{aligned}
a_8^Q(x,y,z) \;=\; &-\frac{d}{8}(d-2)\left[\frac{(z-y)^6}{(y-x)^2(z-x)^4}+\frac{(z-y)^6}{(y-x)^4(z-x)^2}\right]\\
&+\frac{d}{32}(d-2)\frac{(z-y)^8}{(y-x)^4(z-x)^4}+\frac{(d-1)(d-4)}{16}\\
&-\frac{d^2}{16}\left[\frac{(y-x)^6}{(z-y)^2(z-x)^4}+\frac{(z-x)^6}{(z-y)^2(y-x)^4}\right]\\
&+\frac{1}{96}(d^3-11d^2+20d-16)\left[\frac{(z-x)^2}{(y-x)^2}+\frac{(y-x)^2}{(z-x)^2}\right]\\
&-\frac{1}{96}(d^3+3d^2+12d+32)\left[\frac{(z-y)^2}{(y-x)^2}+\frac{(z-y)^2}{(z-x)^2}\right]\\
&+\frac{1}{192}(d^3+39d^2-60d-32)\frac{(z-y)^4}{(y-x)^2(z-x)^2}\\
&+\frac{d}{192}\left(d^2-13\,d+4\right)\left[\frac{(z-x)^6}{(z-y)^4(y-x)^2}+\frac{(y-x)^6}{(z-y)^4(z-x)^2}\right]\\
&-\frac{d}{96}\left(d^2-13\,d+4\right)\left[\frac{(z-x)^4}{(z-y)^2(y-x)^2}+\frac{(y-x)^4}{(z-y)^2(z-x)^2}\right]\\
&-\frac{d}{16}(3\,d-4)\left[\frac{(z-y)^2(z-x)^2}{(y-x)^4}+\frac{(z-y)^2(y-x)^2}{(z-x)^4}\right]\\
&+\frac{d}{16}(2d-1)\left[\frac{(z-x)^4}{(y-x)^4}+\frac{(y-x)^4}{(z-x)^4}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{d^2}{64}\left[\frac{(z-x)^8}{(z-y)^4\,(y-x)^4}+\frac{(y-x)^8}{(z-x)^4\,(z-y)^4}\right]\\
&+\frac{d}{64}\,(13\,d-24)\left[\frac{(z-y)^4}{(y-x)^4}+\frac{(z-y)^4}{(z-x)^4}\right]\\
&-\frac{d}{192}\left(4\,d^2-25\,d+16\right)\left[\frac{(z-x)^4}{(z-y)^4}+\frac{(y-x)^4}{(z-y)^4}\right]\\
&+\frac{d}{96}\left(d^2-7\,d+4\right)\left[\frac{(z-x)^2}{(z-y)^2}+\frac{(y-x)^2}{(z-y)^2}\right]\\
&+\frac{d}{32}(d-4)(d-1)\left[\frac{(y-x)^2\,(z-x)^2}{(z-y)^4}\right] \qquad \text{(A.6.9)}
\end{aligned}
$$

$$
\begin{aligned}
a_9^R(x,y,z) \;=\;& \frac{1}{64}(d^2-6d+4)+\frac{d\,(d-2)\,(z-y)^8}{128\;(y-x)^4\,(z-x)^4}\\
&-\frac{d}{512}\left[\frac{(z-y)^6}{(y-x)^2\,(z-x)^4}-\frac{(z-y)^6}{(y-x)^4\,(z-x)^2}\right]\\
&+\frac{d^2}{256}\left[\frac{(z-x)^8}{(z-y)^4\,(y-x)^4}+\frac{(y-x)^8}{(z-x)^4\,(z-y)^4}\right]\\
&-\frac{1}{256}(2d^3-7d^2-6d-32)\left[\frac{(z-y)^2}{(y-x)^2}+\frac{(z-y)^2}{(z-x)^2}\right]\\
&-\frac{1}{128}(3d^2-5d+8)\left[\frac{(z-x)^2}{(y-x)^2}+\frac{(y-x)^2}{(z-x)^2}\right]\\
&-\frac{d}{128}(2d-1)\left[\frac{(z-x)^6}{(z-y)^4\,(y-x)^2}-\frac{(y-x)^6}{(z-y)^4\,(z-x)^2}\right]\\
&+\frac{d}{256}(7d-8)\left[\frac{(z-x)^4}{(z-y)^4}+\frac{(y-x)^4}{(z-y)^4}\right]\\
&-\frac{d}{256}(5d-2)\left[\frac{(y-x)^2}{(z-y)^2}+-\frac{(z-x)^2}{(z-y)^2}\right]\\
&+\frac{1}{128}(d^3+8d^2-10d-8)\frac{(z-y)^4}{(y-x)^2\,(z-x)^2}\\
&+\frac{d}{128}\left(d^2+4\,d-2\right)\left[\frac{(z-x)^4}{(y-x)^4}+\frac{(y-x)^4}{(z-x)^4}\right]\\
&-\frac{d}{64}(2d-3)\frac{(y-x)^2\,(z-x)^2}{(z-y)^4}\\
&-\frac{d}{256}(3d^2+12d-16)\left[\frac{(z-y)^2\,(y-x)^2}{(z-x)^4}+\frac{(z-y)^2\,(z-x)^2}{(y-x)^4}\right]\\
&+\frac{d}{512}(d^2+18d-4)\left[\frac{(z-x)^4}{(z-y)^2\,(y-x)^2}+\frac{(y-x)^4}{(z-y)^2\,(z-x)^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{d}{256}(d+8)(2d-3)\left[\frac{(z-y)^4}{(z-x)^4}+\frac{(z-y)^4}{(y-x)^4}\right] \\
&-\frac{d^2}{512}(d+8)\left[\frac{(z-x)^6}{(z-y)^2(y-x)^4}+\frac{(y-x)^6}{(z-y)^2(z-x)^4}\right]
\end{aligned}
\tag{A.6.10}
$$

$$
\begin{aligned}
a^S_{10}(x,y,z) &= \frac{1}{64}(d^2-6d+4) \\
&- \frac{d}{128}(d-2)\left[4\frac{(z-y)^6}{(y-x)^2(z-x)^4}-4\frac{(z-y)^6}{(y-x)^4(z-x)^2}\right. \\
&\qquad\left.-\frac{(z-y)^8}{(y-x)^4(z-x)^4}\right] \\
&- \frac{d^2}{128}\left[4\frac{(y-x)^6}{(z-y)^2(z-x)^4}+4\frac{(z-x)^6}{(z-y)^2(y-x)^4}\right. \\
&\qquad\left.-\frac{(z-x)^8}{(z-y)^4(y-x)^4}-\frac{(y-x)^8}{(z-y)^4(z-x)^4}\right] \\
&- \frac{d}{64}(2d-1)\left[\frac{(y-x)^6}{(z-y)^4(z-x)^2}+\frac{(z-x)^6}{(z-y)^4(y-x)^2}\right. \\
&\qquad\left.+2\frac{(y-x)^4}{(z-y)^2(z-x)^2}+2\frac{(z-x)^4}{(z-y)^2(y-x)^2}\right] \\
&- \frac{d}{32}(d-1)\left[\frac{(y-x)^2}{(z-y)^2}+\frac{(z-x)^2}{(z-y)^2}\right. \\
&\qquad\left.+2\frac{(z-y)^2(y-x)^2}{(z-x)^4}+2\frac{(z-y)^2(z-x)^2}{(y-x)^4}\right] \\
&+ \frac{d}{128}(7d-8)\left[\frac{(y-x)^4}{(z-y)^4}+\frac{(z-x)^4}{(z-y)^4}\right] \\
&- \frac{d}{32}(2d-3)\frac{(y-x)^2(z-x)^2}{(z-y)^4} \\
&+ \frac{d}{128}(7d-2)\left[\frac{(y-x)^4}{(z-x)^4}+\frac{(z-x)^4}{(y-x)^4}\right] \\
&+ \frac{d}{128}(7d-12)\left[\frac{(z-y)^4}{(y-x)^4}+\frac{(z-y)^4}{(z-x)^4}\right] \\
&- \frac{1}{32}(d^2+d-4)\left[\frac{(z-y)^2}{(z-x)^2}+\frac{(z-y)^2}{(y-x)^2}\right] \\
&- \frac{1}{32}(d^2-d+2)\left[\frac{(z-x)^2}{(y-x)^2}+\frac{(y-x)^2}{(z-x)^2}\right] \\
&+ \frac{1}{64}(4d^2-5d-4)\frac{(z-y)^4}{(y-x)^2(z-x)^2} \quad .
\end{aligned}
\tag{A.6.11}
$$

A.7 Functions Required for Calculating the Non-Integrable Singular Terms in $\langle TCC \rangle$

Here we list the the functions $b_i^{A_i}(\omega_1, \omega_2, \omega_3 = 0)$ in equation (4.3.30) after setting $\omega_3 = 0$ and $\varepsilon = \omega_1 + \omega_2$.

$$\begin{aligned} b_1^A(\omega_1,\omega_2) &= -\frac{(\omega_1+\omega_2)^2(\omega_1+\omega_2-1)(\omega_1+\omega_2-3)(\omega_1+\omega_2-5)}{384(\omega_2-6)(\omega_1-4)(\omega_1-6)(\omega_1+\omega_2-4)(\omega_2-4)} \times \\ & (24\omega_1^5 - 10\omega_2\omega_1^5 - 54\omega_1^4\omega_2^2 - 88\omega_2^3\omega_1^3 + \omega_1^5\omega_2^2 + 3\omega_2^3\omega_1^4 - 768\omega_1^2 \\ & +864\omega_1^3 - 288\omega_1^4 - 768\omega_2^2 + 864\omega_2^3 - 288\omega_2^4 - 10\omega_2^5\omega_1 + \omega_2^5\omega_1^2 \\ & +3\omega_1^3\omega_2^4 - 54\omega_2^4\omega_1^2 + 24\omega_2^5 + 264\omega_2\omega_1^4 - 768\omega_1\omega_2 + 3200\omega_1\omega_2^2 \\ & -1800\omega_1\omega_2^3 + 264\omega_1\omega_2^4 - 1800\omega_1^3\omega_2 - 3088\omega_1^2\omega_2^2 \\ & +3200\omega_1^2\omega_2 + 728\omega_1^3\omega_2^2 + 728\omega_2^3\omega_1^2) \end{aligned} \quad (A.7.1)$$

$$\begin{aligned} b_2^B(\omega_1,\omega_2) &= -\frac{(\omega_1+\omega_2-2)(\omega_1+\omega_2-1)(\omega_1+\omega_2)^2(\omega_1+\omega_2-5)(\omega_1+\omega_2-3)}{6(\omega_2-6)(\omega_2-4)(\omega_1+\omega_2-4)(\omega_1-4)(\omega_1-6)} \\ & \times(-6\omega_1^2 + \omega_1^2\omega_2 + 12\omega_1 - 20\omega_1\omega_2 + \omega_1\omega_2^2 - 6\omega_2^2 + 12\omega_2) \end{aligned} \quad (A.7.2)$$

$$\begin{aligned} b_3^C(\omega_1,\omega_2) &= \frac{(\omega_1+\omega_2-1)(\omega_1+\omega_2)\,(4-\omega_1-\omega_2)^{-1}}{1536(\omega_2+4)(\omega_2+2)(6-\omega_1)(4-\omega_1)(\omega_1+2)(4-\omega_2)(\omega_1+4)(6-\omega_2)} \\ & \times(8760\omega_1^6\omega_2^4 + 143488\omega_2\omega_1^6 - 290752\omega_1^4\omega_2^4 - 143200\omega_1^2\omega_2^6 + 609600\omega_1^2\omega_2^5 \\ & +1301568\omega_1^4\omega_2^3 - 90192\omega_1^6\omega_2^3 + 8760\omega_1^4\omega_2^6 + 6300\omega_1^6\omega_2^5 + 48\omega_1^5\omega_2^7 \\ & +6300\omega_1^5\omega_2^6 + 7184\omega_1^5\omega_2^5 - 146176\omega_1^5\omega_2^4 - 238016\omega_1^5\omega_2^3 + 1136\omega_1^3\omega_2^8 \\ & +8464\omega_1^3\omega_2^7 - 143200\omega_1^6\omega_2^2 - 24\omega_1^4\omega_2^8 - 90192\omega_1^3\omega_2^6 - 238016\omega_1^3\omega_2^5 \\ & +3732\omega_1^4\omega_2^7 - 20\omega_1^5\omega_2^8 - 16\omega_1^3\omega_2^9 - 4\omega_1^4\omega_2^9 - 27056\omega_1^2\omega_2^7 + 112\omega_1^2\omega_2^9 \\ & +3936\omega_1^2\omega_2^8 - 414720\omega_2^5 - 40\omega_1^6\omega_2^7 + 187392\omega_1^6 - 414720\omega_1^5 - 55936\omega_1\omega_2^7 \\ & +1474560\omega_1^3 - 602112\omega_1^4 + 640\omega_1\omega_2^9 - 1152\omega_1\omega_2^8 + 1474560\omega_2^3 - 602112\omega_2^4 \\ & +112\omega_1^6\omega_2^6 + 3732\omega_1^7\omega_2^4 + 768\omega_1^9 - 9216\omega_1^8 + 8448\omega_1^7 - 16\omega_1^9\omega_2^3 + 112\omega_1^9\omega_2^2 \\ & +640\omega_1^9\omega_2 - 5898240\omega_1\omega_2 + 6094848\omega_1\omega_2^2 - 1615872\omega_1\omega_2^3 \\ & -2300416\omega_1\omega_2^4 + 187392\omega_2^6 - 4\omega_1^9\omega_2^4 - 2300416\omega_1^4\omega_2 \\ & +609600\omega_1^5\omega_2^2 + 875520\omega_1^5\omega_2 - 4362240\omega_1^2\omega_2^2 \\ & +6094848\omega_1^2\omega_2 - 3582464\omega_1^3\omega_2^2 - 1615872\omega_1^3\omega_2 + 875520\omega_1\omega_2^5 \\ & +143488\omega_1\omega_2^6 + 3936\omega_1^8\omega_2^2 - 1152\omega_1^8\omega_2 + 1577344\omega_1^4\omega_2^2 - 3582464\omega_1^2\omega_2^3 \\ & +1577344\omega_1^2\omega_2^4 + 1532672\omega_1^3\omega_2^3 + 1301568\omega_1^3\omega_2^4 - 24\omega_1^8\omega_2^4 \\ & +8448\omega_2^7 + 1136\omega_1^8\omega_2^3 - 20\omega_1^8\omega_2^5 + 8464\omega_1^7\omega_2^3 - 27056\omega_1^7\omega_2^2 + 768\omega_2^9 \\ & -40\omega_1^7\omega_2^6 + 48\omega_1^7\omega_2^5 - 146176\omega_1^4\omega_2^5 - 9216\omega_2^8 - 55936\omega_1^7\omega_2) \end{aligned} \quad (A.7.3)$$

$$
\begin{aligned}
b_4^D(\omega_1,\omega_2) &= \frac{-(1-\omega_1-\omega_2)(\omega_1+\omega_2)\ (4-\omega_1-\omega_2)^{-1}}{3072(\omega_2+4)(\omega_2+2)(6-\omega_1)(4-\omega_1)(\omega_1+2)(4-\omega_2)(\omega_1+4)(6-\omega_2)} \\
&\times(2949120\omega_1^3 - 1204224\omega_1^4 + 2949120\omega_2^3 - 1204224\omega_2^4 \\
&-3384320\omega_1\omega_2^4 + 2030336\omega_1^2\omega_2^4 + 1969536\omega_1^3\omega_2^4 - 288896\omega_2^4\omega_1^4 \\
&-8847360\omega_2\omega_1 + 7667712\omega_2\omega_1^2 - 2383872\omega_2\omega_1^3 - 3384320\omega_2\omega_1^4 \\
&-5369856\omega_2^2\omega_1^2 + 7667712\omega_2^2\omega_1 - 5367808\omega_2^3\omega_1^2 - 2383872\omega_2^3\omega_1 \\
&-5367808\omega_2^2\omega_1^3 + 2030336\omega_2^2\omega_1^4 + 2106880\omega_2^3\omega_1^3 + 1969536\omega_2^3\omega_1^4 \\
&-829440\omega_2^5 + 374784\omega_2^6 + 374784\omega_1^6 - 829440\omega_1^5 \\
&-223424\omega_2^4\omega_1^5 + 1244160\omega_2\omega_1^5 + 1064832\omega_2^2\omega_1^5 + 5352\omega_2^4\omega_1^7 \\
&+277760\omega_2\omega_1^6 - 191936\omega_2^2\omega_1^6 + 3696\omega_2^4\omega_1^6 + 277760\omega_1\omega_2^6 \\
&-191936\omega_1^2\omega_2^6 + 1244160\omega_1\omega_2^5 + 1064832\omega_1^2\omega_2^5 - 48\omega_2^4\omega_1^8 + 1536\omega_1^9 \\
&-8\omega_2^4\omega_1^9 - 8\omega_2^9\omega_1^4 - 32\omega_2^9\omega_1^3 + 224\omega_2^9\omega_1^2 + 1280\omega_2^9\omega_1 - 40\omega_2^8\omega_1^5 \\
&+1888\omega_2^8\omega_1^3 + 5568\omega_2^8\omega_1^2 - 5376\omega_2^8\omega_1 - 48\omega_2^8\omega_1^4 + 1536\omega_2^9 - 18432\omega_2^8 \\
&+16896\omega_2^7 - 80\omega_2^7\omega_1^6 + 96\omega_2^7\omega_1^5 + 5352\omega_2^7\omega_1^4 - 140832\omega_2^3\omega_1^6 \\
&-268672\omega_2^3\omega_1^5 + 7712\omega_2^7\omega_1^3 - 50272\omega_2^7\omega_1^2 - 84224\omega_2^7\omega_1 + 1888\omega_2^3\omega_1^8 \\
&+7712\omega_2^3\omega_1^7 - 80\omega_2^6\omega_1^7 + 224\omega_2^6\omega_1^6 + 8184\omega_2^6\omega_1^5 + 3696\omega_2^6\omega_1^4 \\
&-140832\omega_2^6\omega_1^3 - 18432\omega_1^8 + 16896\omega_1^7 + 8184\omega_2^5\omega_1^6 + 96\omega_2^5\omega_1^7 \\
&+544\omega_2^5\omega_1^5 - 223424\omega_2^5\omega_1^4 - 268672\omega_2^5\omega_1^3 - 32\omega_2^3\omega_1^9 - 40\omega_2^5\omega_1^8 \\
&+5568\omega_2^2\omega_1^8 - 50272\omega_2^2\omega_1^7 + 224\omega_2^2\omega_1^9 - 5376\omega_2\omega_1^8 \\
&-84224\omega_2\omega_1^7 + 1280\omega_2\omega_1^9)
\end{aligned} \tag{A.7.4}
$$

$$
\begin{aligned}
b_5^E(\omega_1,\omega_2) &= \frac{-(1-\omega_1-\omega_2)(\omega_1+\omega_2)\ (4-\omega_1-\omega_2)^{-1}}{3072(\omega_2+4)(\omega_2+2)(6-\omega_1)(4-\omega_1)(\omega_1+2)(4-\omega_2)(\omega_1+4)(6-\omega_2)} \\
&(472704\omega_1^4\omega_2^4 + 812160\omega_1^4\omega_2^3 + 1720\omega_1^5\omega_2^6 - 47136\omega_1^5\omega_2^5 \\
&-122560\omega_1^5\omega_2^4 + 288768\omega_1^5\omega_2^3 - 3072\omega_1^8 - 40\omega_1^5\omega_2^8 + 832\omega_1^5\omega_2^7 \\
&-1920\omega_1^7\omega_2 - 8\omega_1^9\omega_2^4 - 37984\omega_1^7\omega_2^2 + 208\omega_1^4\omega_2^8 + 2664\omega_1^4\omega_2^7 \\
&-38736\omega_1^4\omega_2^6 - 122560\omega_1^4\omega_2^5 - 40\omega_1^8\omega_2^5 + 208\omega_1^8\omega_2^4 + 1568\omega_1^8\omega_2^3 \\
&-1600\omega_1^8\omega_2^2 - 17152\omega_1^7\omega_2^3 - 1600\omega_1^2\omega_2^8 + 2664\omega_1^7\omega_2^4 + 384\omega_1\omega_2^9 \\
&-1920\omega_1\omega_2^7 - 8704\omega_1\omega_2^8 + 224\omega_1^2\omega_2^9 + 2949120\omega_1^2 \\
&-98304\omega_1^3 - 1142784\omega_1^4 + 2949120\omega_2^2 + 384\omega_1^9\omega_2 \\
&-98304\omega_2^3 - 1142784\omega_2^4 - 8\omega_1^4\omega_2^9 - 8704\omega_1^8\omega_2 \\
&+1568\omega_1^3\omega_2^8 + 832\omega_1^7\omega_2^5 - 17152\omega_1^3\omega_2^7 - 77280\omega_1^3\omega_2^6 \\
&+288768\omega_1^3\omega_2^5 - 80\omega_1^7\omega_2^6 - 37984\omega_1^2\omega_2^7 + 24576\omega_1^5 \\
&+660352\omega_1^5\omega_2^2 - 51712\omega_1^5\omega_2 + 196096\omega_1^6\omega_2 + 61248\omega_1^6\omega_2^2 \\
&+113664\omega_1^6 + 24576\omega_2^5 - 5898240\omega_1\omega_2 - 1867776\omega_1\omega_2^2 + 802816\omega_1\omega_2^3 \\
&-620544\omega_1\omega_2^4 + 61248\omega_1^2\omega_2^6 + 196096\omega_1\omega_2^6 + 113664\omega_2^6 \\
&+660352\omega_1^2\omega_2^5 - 51712\omega_1\omega_2^5 - 580864\omega_1^4\omega_2^2 - 620544\omega_1^4\omega_2
\end{aligned}
$$

$$
\begin{aligned}
&+1187840\omega_1^2\omega_2^2 - 1867776\omega_1^2\omega_2 - 1341440\omega_1^3\omega_2^2 + 802816\omega_1^3\omega_2 \\
&-1341440\omega_1^2\omega_2^3 - 580864\omega_1^2\omega_2^4 - 708096\omega_1^3\omega_2^3 + 812160\omega_1^3\omega_2^4 \\
&-77280\omega_1^6\omega_2^3 - 3072\omega_2^8 - 38736\omega_1^6\omega_2^4 - 80\omega_1^6\omega_2^7 \\
&+1248\omega_1^6\omega_2^6 + 1720\omega_1^6\omega_2^5 + 224\omega_1^9\omega_2^2) \qquad \text{(A.7.5)}
\end{aligned}
$$

$$
\begin{aligned}
b_6^F(\omega_1,\omega_2) \;=\; & \frac{(\omega_1+\omega_2-1)(\omega_1+\omega_2)}{1024(\omega_2+4)(\omega_2+2)(6-\omega_1)(4-\omega_1)(\omega_1+2)(4-\omega_2)(\omega_1+4)(6-\omega_2)} \\
&\times(-832\omega_1^5\omega_2^5 + 256\omega_1^5\omega_2^4 - 32256\omega_1^5 - 768\omega_1^7\omega_2^2 + 14848\omega_2\omega_1^6 \\
&+14848\omega_2^6\omega_1 + 9216\omega_2^6\omega_1^2 + 32000\omega_2^2\omega_1^5 + 9216\omega_2^2\omega_1^6 - 160\omega_1^7\omega_2^3 \\
&+256\omega_1^4\omega_2^5 - 32128\omega_1\omega_2^5 + 32000\omega_2^5\omega_1^2 - 368640\omega_1^2 \\
&+104448\omega_1^3 + 116736\omega_1^4 - 32128\omega_1^5\omega_2 - 576\omega_1^4\omega_2^6 \\
&-128\omega_1^7\omega_2 - 368640\omega_2^2 + 104448\omega_2^3 + 116736\omega_2^4 - 328192\omega_2^4\omega_1 \\
&-198912\omega_2^4\omega_1^2 + 51456\omega_2^4\omega_1^3 + 29952\omega_2^4\omega_1^4 + 1259520\omega_2^2\omega_1 \\
&+780288\omega_2^2\omega_1^2 - 410624\omega_2^2\omega_1^3 - 198912\omega_2^2\omega_1^4 + 591360\omega_2^3\omega_1 \\
&-410624\omega_2^3\omega_1^2 - 338176\omega_2^3\omega_1^3 + 51456\omega_2^3\omega_1^4 - 1474560\omega_1\omega_2 \\
&+1259520\omega_1^2\omega_2 - 6144\omega_2^6 + 591360\omega_1^3\omega_2 \\
&-328192\omega_1^4\omega_2 + 23840\omega_1^3\omega_2^5 - 1024\omega_1^3\omega_2^6 - 160\omega_1^3\omega_2^7 \\
&+1536\omega_1^7 - 32256\omega_2^5 - 768\omega_1^2\omega_2^7 + 1536\omega_2^7 - 1024\omega_1^6\omega_2^3 \\
&-576\omega_1^6\omega_2^4 - 128\omega_1\omega_2^7 - 6144\omega_1^6 + 23840\omega_1^5\omega_2^3) \qquad \text{(A.7.6)}
\end{aligned}
$$

$$
\begin{aligned}
b_7^G(\omega_1,\omega_2) \;=\; & \frac{-2(\omega_1+\omega_2)(\omega_1+\omega_2-1)(\omega_1+\omega_2-5)(\omega_1+\omega_2-3)}{512(\omega_1+\omega_2-2)(\omega_1-6)(\omega_1-4)(\omega_1+\omega_2-4)(\omega_2-4)(\omega_2-6)} \\
&\times(24\omega_2^5 - 768\omega_1^2 + 864\omega_1^3 - 288\omega_1^4 - 768\omega_2^2 + 864\omega_2^3 \\
&-288\omega_2^4 + 2144\omega_2^2\omega_1 - 1520\omega_2^2\omega_1^2 + 80\omega_2^2\omega_1^3 + 18\omega_2^2\omega_1^4 \\
&-1080\omega_2^3\omega_1 + 80\omega_2^3\omega_1^2 + 24\omega_2^3\omega_1^3 + 3\omega_2^3\omega_1^4 + 72\omega_2^4\omega_1 \\
&+18\omega_2^4\omega_1^2 + 3\omega_2^4\omega_1^3 + 24\omega_1^5 - 1080\omega_2\omega_1^3 + 72\omega_2\omega_1^4 \\
&-256\omega_2\omega_1 + 2144\omega_2\omega_1^2 + 6\omega_2\omega_1^5 + 6\omega_2^5\omega_1 \\
&+\omega_2^5\omega_1^2 + \omega_2^2\omega_1^5) \qquad \text{(A.7.7)}
\end{aligned}
$$

$$
\begin{aligned}
b_8^Q(\omega_1,\omega_2) \;=\; & \frac{-2(\omega_1+\omega_2)(\omega_1+\omega_2-1)(\omega_1+\omega_2-3)(\omega_1+\omega_2-5)}{384(\omega_1+\omega_2-4)(\omega_1+\omega_2-2)(\omega_1-4)(\omega_1-6)(\omega_2-4)(\omega_2-6)} \\
&\times(\omega_1^5\omega_2^2 + \omega_1^2\omega_2^5 + 768\omega_1\omega_2 + 736\omega_1\omega_2^2 + 736\omega_1^2\omega_2 \\
&-1008\omega_1^2\omega_2^2 - 56\omega_1^3\omega_2^3 + 3\omega_1^4\omega_2^3 + 3\omega_1^3\omega_2^4 - 312\omega_1^3\omega_2 \\
&-72\omega_1^4\omega_2 + 64\omega_1^3\omega_2^2 - 30\omega_1^4\omega_2^2 - 312\omega_1\omega_2^3 - 72\omega_1\omega_2^4 + 64\omega_1^2\omega_2^3 \\
&-30\omega_1^2\omega_2^4 - 384\omega_1^2 + 288\omega_1^3 - 384\omega_2^2 + 288\omega_2^3 \\
&-24\omega_2^5 - 24\omega_1^5 - 2\omega_1\omega_2^5 - 2\omega_1^5\omega_2) \qquad \text{(A.7.8)}
\end{aligned}
$$

$$
\begin{aligned}
b_9^R(\omega_1,\omega_2) \;=\; & \frac{-2(\omega_1+\omega_2)(\omega_1+\omega_2-1)(\omega_1+\omega_2-3)(\omega_1+\omega_2-5)}{512(\omega_1-4)(\omega_1-6)(\omega_1+\omega_2-2)(\omega_1+\omega_2-4)(\omega_2-4)(\omega_2-6)} \\
& \times(-16\omega_2^3\omega_1^3+3\omega_2^4\omega_1^3+1792\omega_2\omega_1-896\omega_2\omega_1^2 \\
& +168\omega_2\omega_1^3-72\omega_2\omega_1^4-152\omega_2^2\omega_1^3-6\omega_2^2\omega_1^4-6\omega_2^4\omega_1^2 \\
& -896\omega_2^2\omega_1+400\omega_2^2\omega_1^2+168\omega_2^3\omega_1-152\omega_2^3\omega_1^2 \\
& -72\omega_2^4\omega_1-96\omega_2^4-384\omega_2^2-384\omega_1^2+384\omega_1^3-96\omega_1^4 \\
& +384\omega_2^3+3\omega_2^3\omega_1^4+\omega_2^5\omega_1^2+\omega_2^2\omega_1^5+2\omega_1\omega_2^5+2\omega_2\omega_1^5) \qquad \text{(A.7.9)}
\end{aligned}
$$

$$
\begin{aligned}
b_{10}^S(\omega_1,\omega_2) \;=\; & \frac{2(\omega_1+\omega_2)(\omega_1+\omega_2-1)(\omega_1+\omega_2-3)(\omega_1+\omega_2-5)}{128(-2+\omega_1+\omega_2)(\omega_1-6)(\omega_1-4)(\omega_1+\omega_2-4)(\omega_2-4)(\omega_2-6)} \\
& (\omega_2^4\omega_1^2+4\omega_2^3\omega_1^2+6\omega_2^4\omega_1+2\omega_2^3\omega_1^3+4\omega_1^3\omega_2^2 \\
& -640\omega_1\omega_2+264\omega_1\omega_2^2+264\omega_1^2\omega_2+6\omega_1^4\omega_2+96\omega_2^2+24\omega_1^4 \\
& +96\omega_1^2-96\omega_2^3+24\omega_2^4-96\omega_1^3+\omega_1^4\omega_2^2) \qquad \text{(A.7.10)}
\end{aligned}
$$

A.8 Green Function for the Differential Operator on k-Forms

In section 5.3 we use the flat space Green function for the conformal second order differential operator (5.3.30) acting on 2-forms to construct four-dimensional effective actions. Here we calculate the flat space Green function for the general operator (5.3.15) found by Branson [27] acting on k-forms in general d dimensions. On flat space this operator reduces to

$$
\begin{aligned}
D^{(k)}\omega_{\mu_1\cdots\mu_k} &= -k(\gamma+1)\partial^\lambda\partial_{[\lambda}\omega_{\mu_1\cdots\mu_k]}-(k-1)(\gamma-1)\partial_{[\mu_1}\partial^\lambda\omega_{|\lambda|\mu_2\cdots\mu_k]} \\
&= -(\gamma+1)\partial^2\omega_{\mu_1\cdots\mu_k}\;+\;k\,\partial_{[\mu_1}\partial^\lambda\omega_{|\lambda|\mu_2\cdots\mu_k]} \qquad \text{(A.8.1)}
\end{aligned}
$$

with $\gamma=(d-2k)/2$. Its Fourier transform is

$$
\begin{aligned}
P_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(p) &= (\gamma+1)\mathcal{E}^A_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}p^2 \\
&\quad -2k\mathcal{E}^A_{\mu_1\cdots\mu_k,\varepsilon\lambda_2\cdots\lambda_k}\mathcal{E}^A_{\eta\lambda_1\cdots\lambda_k,\nu_1\cdots\nu_k}p_\varepsilon p_\eta\,, \qquad \text{(A.8.2)}
\end{aligned}
$$

where $\mathcal{E}^A$ is the projector on totally antisymmetric tensors of rank k. The inverse of the Fourier transform is defined by

$$
P^{-1}{}_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(p)\,P_{\nu_1\cdots\nu_k,\lambda_1\cdots\lambda_k}(p)=\mathcal{E}^A_{\mu_1\cdots\mu_k,\lambda_1\cdots\lambda_k} \qquad \text{(A.8.3)}
$$

from which we obtain

$$P^{-1}{}_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(p) \;=\; \frac{1}{(\gamma+1)}\mathcal{E}^A_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}\frac{1}{p^2} + \frac{2k}{(\gamma+1)(\gamma-1)}\mathcal{E}^A_{\mu_1\cdots\mu_k,\varepsilon\lambda_2\cdots\lambda_k}\mathcal{E}^A_{\eta\lambda_1\cdots\lambda_k,\nu_1\cdots\nu_k}\frac{p_\varepsilon p_\eta}{p^4}\,. \quad (A.8.4)$$

When Fourier transforming back to d dimensional position space we have

$$\frac{1}{(2\pi)^d}\int \mathrm{d}^d p\, e^{-ip\cdot x}\frac{1}{p^2} = \frac{\Gamma(d/2-1)}{4\pi^{d/2}}\frac{1}{x^{d-2}}\,, \quad \frac{1}{(2\pi)^d}\int \mathrm{d}^d p\, e^{-ip\cdot x}\frac{1}{p^4} = \frac{\Gamma(d/2-2)}{16\pi^{d/2}}\frac{1}{x^{d-4}}\,,$$
$$\frac{1}{(2\pi)^d}\int \mathrm{d}^d p\, e^{-ip\cdot x}\frac{p_\varepsilon p_\eta}{p^4} = -\frac{\Gamma(d/2-2)}{16\pi^{d/2}}\partial_\varepsilon\partial_\eta\frac{1}{x^{d-4}}\,. \quad (A.8.5)$$

Using these relations to Fourier transform the inverse (A.8.4) we obtain for the Green function for the operator (A.8.1) in Euclidean position space

$$G^{(k)}_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(x) \;=\; \frac{\Gamma(\frac{1}{2}d-1)}{4\pi^{d/2}}\frac{2(d-2)}{(d-2k-2)(d-2k+2)} \times \Big[\mathcal{E}^A_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}\frac{1}{x^{d-2}} - 2k\mathcal{E}^A_{\mu_1\cdots\mu_k,\varepsilon\lambda_2\cdots\lambda_k}\mathcal{E}^A_{\eta\lambda_1\cdots\lambda_k,\nu_1\cdots\nu_k}\frac{x_\varepsilon x_\eta}{x^d}\Big] \quad (A.8.6)$$

and finally

$$G^{(k)}_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(x) = \frac{\Gamma(\frac{1}{2}d)}{\pi^{d/2}(d-2k-2)(d-2k+2)}\,\mathcal{I}^A_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(x)\frac{1}{x^{d-2}}\,, \quad (A.8.7)$$

where $\mathcal{I}^A$ is the inversion on k-forms. On functions the operator (5.3.15) reduces to $(d+2)/2\cdot\Delta_d$ with Δ_d the conformal scalar operator defined in (5.1.3). For the flat space Green function, (A.8.7) gives in this case

$$G(x) = \frac{\Gamma(\frac{1}{2}d)}{\pi^{d/2}(d-2)(d+2)}\frac{1}{x^{d-2}}\,. \quad (A.8.8)$$

Acting on 2-forms in four dimensions, (A.8.7) agrees with (5.3.39) in section 5.3. Moreover (A.8.8) agrees with (5.2.19) in four dimensions up to a factor of $(d+2)/2 = 3$ as expected.

A.9 Green Function for the Differential Operator Acting on Tensors with Weyl Symmetry

Here we calculate the flat space Green function for the operator Δ^C given by (5.3.63). All relevant calculations are performed using FORM.

The flat space reduction of the operator Δ^C is

$$(\Delta^C C)_{\mu\sigma\rho\nu}\Big|_{g=\delta} = -\tfrac{d}{16}\partial^2 C_{\mu\sigma\rho\nu} + \mathcal{E}^C_{\mu\sigma\rho\nu,\mu'\sigma'\rho'\nu'}\partial_{\mu'}\partial_\alpha C_{\alpha\sigma'\rho'\nu'} \tag{A.9.9}$$

which has the Fourier transform

$$P^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(p) = \tfrac{1}{16}\, d\, p^2\, \mathcal{E}^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta} - p_\kappa p_\lambda\, \mathcal{E}^C_{\mu\sigma\rho\nu,\kappa,\sigma'\rho'\nu'}\mathcal{E}^C_{\alpha\gamma\delta\beta,\lambda\sigma'\rho'\nu'}\,. \tag{A.9.10}$$

The inverse of this Fourier transform is defined by

$$P^C_{\mu\sigma\rho\nu,\mu'\sigma'\rho'\nu'}(p) P^{C^{-1}}_{\mu'\sigma'\rho'\nu',\alpha\gamma\delta\beta}(p) = \mathcal{E}^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}\,, \tag{A.9.11}$$

from which we obtain

$$\begin{aligned} P^{C^{-1}}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(p) \;=\;& \frac{16}{d}\,\frac{1}{p^2}\,\mathcal{E}^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta} \\ & + \frac{256}{d(d-4)}\,\frac{p_\varepsilon p_\eta}{p^4}\,\mathcal{E}^C_{\mu\sigma\rho\nu,\varepsilon\varphi\theta\omega}\mathcal{E}^C_{\alpha\gamma\delta\beta,\eta\varphi\theta\omega} \\ & + \frac{2048}{d(d-4)(d-6)}\,\frac{p_\varepsilon p_\eta p_\kappa p_\lambda}{p^6}\,\mathcal{E}^C_{\mu\sigma\rho\nu,\varepsilon\varphi\eta\omega}\mathcal{E}^C_{\alpha\gamma\delta\beta,\kappa\varphi\lambda\omega}\,. \end{aligned} \tag{A.9.12}$$

To Fourier transform this expression back to position space, we use the equations (A.8.5) as well as

$$\begin{aligned} \frac{1}{(2\pi)^d}\int \mathrm{d}^d p\, e^{-ipx}\frac{1}{p^6} \;&=\; \frac{\Gamma(d/2-3)}{\pi^{d/2}2^6\Gamma(3)}\,\frac{1}{x^{d-6}}\,, \\ \frac{1}{(2\pi)^d}\int \mathrm{d}^d p\, e^{-ipx}\,\frac{p_\varepsilon p_\eta p_\kappa p_\lambda}{p^6} \;&=\; \partial^x_\varepsilon\partial^x_\eta\partial^x_\kappa\partial^x_\lambda\,\frac{1}{(2\pi)^d}\int \mathrm{d}^d p\, e^{-ipx}\frac{1}{p^6}\,, \end{aligned} \tag{A.9.13}$$

which gives

$$\begin{aligned} &\frac{1}{(2\pi^d)}\int \mathrm{d}^d p\, e^{-ipx} P^{C^{-1}}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(p) \\ &= \frac{4}{d}\,\frac{\Gamma(d/2-1)}{\pi^{d/2}}\Bigg[\;\frac{1}{x^{d-2}}\,\mathcal{E}^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta} \\ &\qquad + \frac{8}{(d-4)}\Big(\frac{1}{x^{d-2}}\,\mathcal{E}^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta} \;-\; (d-2)\frac{x_\varepsilon x_\eta}{x^d}\,\mathcal{E}^C_{\mu\sigma\rho\nu,\varepsilon\varphi\theta\omega}\mathcal{E}^C_{\alpha\gamma\delta\beta,\eta\varphi\theta\omega}\Big) \\ &\qquad + \frac{16}{(d-4)(d-6)}\Big(\tfrac{3}{2}\frac{1}{x^{d-2}}\,\mathcal{E}^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta} - 3(d-2)\frac{x_\varepsilon x_\eta}{x^d}\,\mathcal{E}^C_{\mu\sigma\rho\nu,\varepsilon\varphi\theta\omega}\mathcal{E}^C_{\alpha\gamma\delta\beta,\eta\varphi\theta\omega} \\ &\qquad\qquad + d(d-2)\frac{x_\varepsilon x_\eta x_\kappa x_\lambda}{x^{d+2}}\,\mathcal{E}^C_{\mu\sigma\rho\nu,\varepsilon\varphi\eta\omega}\mathcal{E}^C_{\alpha\gamma\delta\beta,\kappa\varphi\lambda\omega}\Big)\Bigg]\,. \end{aligned} \tag{A.9.14}$$

Now the inversion on the space of tensors with Weyl symmetry defined in (3.5.3) may be written as

$$\begin{aligned} \mathcal{I}^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(x) &= \mathcal{E}^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta} - 8\,\mathcal{E}^C_{\mu\sigma\rho\nu,\varepsilon\varphi\theta\omega}\mathcal{E}^C_{\alpha\gamma\delta\beta,\eta\varphi\theta\omega}\frac{x_\varepsilon x_\eta}{x^2} \\ &\quad + 16\,\mathcal{E}^C_{\mu\sigma\rho\nu,\varepsilon\varphi\eta\omega}\mathcal{E}^C_{\alpha\gamma\delta\beta,\kappa\varphi\lambda\omega}\frac{x_\varepsilon x_\eta x_\kappa x_\lambda}{x^4}\,, \end{aligned} \tag{A.9.15}$$

so that we obtain for the flat space Green function of Δ^C defined in (5.3.73)

$$\begin{aligned} G^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(x)\Big|_{g=\delta} &= \frac{1}{(2\pi^d)}\int \mathrm{d}^d p\, e^{-ipx}\, {P^C}^{-1}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(p) \\ &= \frac{8\,\Gamma(d/2)}{\pi^{d/2}(d-4)(d-6)}\frac{1}{x^{d-2}}\,\mathcal{I}^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(x)\,, \end{aligned} \tag{A.9.16}$$

which is in agreement with conformal invariance.

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