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0.1 Conformal field theory in d dimensions

sec:CFT

0.1.1 Conformal coordinate transformations

Conformal coordinate transformations are defined as those local transformations $x^\mu \mapsto x'^\mu(x)$ that leave angles invariant. In a Euclidean d-dimensional space $\mathbb{R}^d$ we therefore can write

$$\mathrm{d}x_\mu \, \mathrm{d}x^\mu \; = \; \Omega^{-2}(x) \, \mathrm{d}x'_\mu \, \mathrm{d}x'^\mu . \tag{1}$$

eq:metri

The corresponding infinitesimal coordinate transformation from old coordinates x to new ones x' looks like

$$x'^\mu \; = \; x^\mu \; + \; v^\mu(x) \tag{2}$$

and we have

$$\Omega(x) \; = \; 1 \; - \; \sigma(x) \; , \qquad \sigma(x) \; = \; \frac{1}{d} \, \partial \cdot v(x) \; . \tag{3}$$

2,3

Equivalently to (1) we can formulate an equation for the vector v, the *conformal Killing equation*,

$$\partial_\mu v_\nu \; + \; \partial_\nu v_\mu \; = \; 2 \, \sigma(x) \, \eta_{\mu\nu} \; , \tag{4}$$

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taking its trace yields the expression (3) for $\sigma(x)$. We will work in d dimensional Euclidean space where $\eta_{\mu\nu} = \delta_{\mu\nu}$. Solutions v to (4) are referred to as *conformal Killing vectors*, the most general one reads

$$v_\mu \; = \; a_\mu \; + \; \omega_{\mu\nu} \, x^\nu \; + \; \lambda \, x_\mu \; + \; b_\mu x^2 \; - \; 2 \, (b \cdot x) \, x_\mu \; , \qquad \omega_{\mu\nu} = -\omega_{\nu\mu} \; . \tag{5}$$

eq:solut

This Killing vector leads to the scale factor $\sigma(x) = \lambda - 2(b \cdot x)$. Equation (5) is valid for any d. Note that in the special case of $d = 2$ the conformal Killing equation (4) is nothing but the *Cauchy-Riemann equations*

$$\partial_1 v_1 \; = \; \partial_2 v_2 \; , \qquad \partial_1 v_2 \; = \; -\partial_2 v_1 \; . \tag{6}$$

Thus, in $d = 2$ all holomorphic functions $v(x)$ are solutions and generate conformal coordinate transformations. In this case we have an infinite number of functions solving (5), accompanied by an infinite number of associated conserved quantities.

However, we will mostly consider theories in $d = 4$ dimensions, for example in Minkowski space or on the boundary of Ads_5. Here we have a finite amount of conserved quantities. Counting the independent components of the factors in the solutions (5) amounts to a total number of 15:

a_μ	4
$\omega_{\mu\nu}$	+ 6
λ	+ 1
b_μ	+ 4
total	15

The general conformal Killing vector (5) may be viewed as the combination of elementary transformations. The group of "large" conformal transformation is generated by infinitesimal elements of the conformal algebra. We define locally orthogonal tranformations $\mathcal{R}$ corresponding to a group element g of the conformal group as

$$\mathcal{R}^g_{\mu\alpha}(x) \quad := \quad \Omega^g(x) \, \frac{\partial x'_\mu}{\partial x^\alpha} \, . \tag{7}$$

One can easily show that $\mathcal{R} \in O(d)$, i.e. that $\mathcal{R}^g_{\mu\alpha}(x)\mathcal{R}^g_{\nu\alpha}(x) = \delta_{\mu\nu}$. The group multiplication and the inverse are given as follows:

$$\mathcal{R}^{g'}(gx)\,\mathcal{R}^g(x) \;=\; \mathcal{R}^{g'g}(x) \, , \qquad \big(\mathcal{R}^g(x)\big)^{-1} \;=\; \mathcal{R}^{g^{-1}}(gx) \tag{8}$$

With these we can construct translations and rotations as

$$x'_\mu \;=\; \mathcal{R}_{\mu\nu}\, x_\nu \;+\; a_\mu \, , \qquad \Omega(x) \;=\; 1 \, . \tag{9}$$

Scale transformations ($\leftrightarrow \lambda$) and *special conformal transformations* ($\leftrightarrow b_\mu$) involve a non-trivial Ω factor:

$$x'_\mu \;=\; \lambda\, x_\mu \, , \qquad \Omega(x) \;=\; \lambda \tag{10}$$

$$x'_\mu \;=\; \frac{x_\mu + b_\mu x^2}{\Omega^g(x)} \, , \qquad \Omega^g(x) = 1 + 2b\cdot x + bx^2 . \tag{11}$$

Together, these transformations form a group isomorphic to $SO(d+1,1)$ (or $SO(d,2)$ in Minkowski spacetime). All transformations belonging to this group can be constructed by performing translations, rotations, and *inversions*; the latter are given by

$$x'_\mu \;=:\; (\mathrm{i}x)_\mu \;=\; \frac{x_\mu}{x^2} \, , \qquad \Omega^{\mathrm{i}}(x) \;=\; x^2 \tag{12}$$

$$\mathcal{R}^{\mathrm{i}}_{\mu\nu}(x) \;=:\; I_{\mu\nu}(x) \;=\; \delta_{\mu\nu} \;-\; 2\,\frac{x_\mu x_\nu}{x^2} \, . \tag{13}$$

Special conformal transformations can be composed by concatenating inversion + translation + inversion.

0.1.2 Conformal fields and correlation functions

So far we examined coordinate transformations. Now we will investigate the behaviour of fields. For instance, the $\mathcal{N} = 4$ *super Yang Mills theory* (SYM) mentioned in the introduction only contains fields transforming covariantly under the conformal group. In general QFTs (such as QED or QCD), conformal symmetry is generically broken by quantum effects (anomalies).
Necessary condition for a field theory to be conformally symmetric is a vanishing *β-function.* The latter describes the change of a coupling g with energy scales μ, i.e.

$$\beta(g) \quad = \quad \mu \, \frac{\partial g}{\partial \mu} \ , \tag{14}$$

so $\beta(g) = 0$ rephrases scale invariance.
A conformally covariant operator $\mathcal{O}$ of a conformal field theory (CFT) transforms as follows under infinitesimal conformal transformations (with Killing vector v and $\sigma = \partial \cdot v/d$):

$$\delta_v \mathcal{O} \quad = \quad -\left(L_v \, \mathcal{O}\right) \ , \qquad L_v \quad = \quad v(x) \cdot \partial \ + \ \Delta \, \sigma(x) - \ \frac{1}{2} \, \partial_{[\mu} v_{\nu]}(x) \, S_{\mu\nu} \tag{15}$$

Here, Δ denotes the *scaling dimension* of the operator $\mathcal{O}$ and $S_{\mu\nu}$ a generator of $O(d)$ in an appropriate representation. It only affects spinor-, vector- and tensor fields but no scalars φ:

$$\delta_v \varphi \quad = \quad -\big(v(x) \cdot \partial \ + \ \Delta \, \sigma(x)\big) \, \varphi \tag{16}$$

In general QFTs, correlation functions are defined as time ordered vacuum expectation values, e.g. a two point function of some field φ is given by

$$\langle \varphi(x) \, \varphi(y) \rangle \quad := \quad \langle 0 | \, \mathcal{T} \, \varphi(x) \, \varphi(y) \, | 0 \rangle \ , \tag{17}$$ 2,5

three-, four- and higher point functions by analogous expressions. Generically, their computation is quite involved and possible only in the framework of perturbation theory.
Let us also give the path integral analogue of the definition (17) in the operator approach. In a scalar field theory governed by action $\mathcal{S}[\varphi]$, the *partition function* $\mathcal{Z}$ and a general correlation function $\langle \mathcal{O} \rangle$ is defined by the path integrals

$$\mathcal{Z} \quad := \quad \int \mathcal{D}\varphi \ \mathrm{e}^{-\mathcal{S}[\varphi]} \ , \qquad \langle \mathcal{O} \rangle \quad := \quad \frac{1}{\mathcal{Z}} \int \mathcal{D}\varphi \ \mathcal{O} \, \mathrm{e}^{-\mathcal{S}[\varphi]} \ . \tag{18}$$

In CFTs, conformal symmetry is so strong that it determines the form of the two- and three point correlation functions up to a managable number of parameters. In the notation $(x-y)^2 = (x-y)_\mu (x-y)^\mu$, the two- and three point functions of scalars φ_i with scale dimensions Δ_i are given by

$$\langle \varphi_1(x) \, \varphi_2(y) \rangle \quad := \quad \frac{c \, \delta_{\Delta_1, \Delta_2}}{(x-y)^{2\Delta_1}} \tag{19}$$

$$\langle \varphi_1(x)\, \varphi_2(y)\, \varphi_3(z) \rangle \quad := \quad \frac{k}{(x-y)^{\Delta_1+\Delta_2-\Delta_3}\,(y-z)^{-\Delta_1+\Delta_2+\Delta_3}\,(x-z)^{\Delta_1-\Delta_2+\Delta_3}} \tag{20}$$

with constants c, k determined by the field content.

Four point correlators $\langle \varphi_1(x)\varphi_2(y)\varphi_3(z)\varphi_4(w)\rangle$ are less constraint by the symmetry since they involve dimensionless *cross ratios* $\frac{(x-y)^2}{(z-w)^2}$ and $\frac{(x-z)^2}{(y-w)^2}$.

0.1.3 The energy momentum tensor in a CFT

The symmetric *energy momentum tensor* $T_{\mu\nu}$ subject to the conservation law $\partial_\mu T^{\mu\nu} = 0$ (or rather $\nabla_\mu T^{\mu\nu} = 0$ in curved spacetime) generates the Noether currents associated with conformal symmetry. The infinitesimal transformations with conformal Killing vector v^μ gives rise to the conserved current

$$j^\mu \quad = \quad T^{\mu\nu}\, v_\nu \ . \tag{21}$$

In this subsection, we will now show an important property of the energy momentum tensor in a conformal field theory, namely its tracelessness $T^\mu{}_\mu = 0$.

It is a common method in QFT to introduce sources for operators in a QFT's action, and then express the operator (in correlation functions) as the functional derivative of the generating functional. To do so, the action $\mathcal{S}_0$ of our theory is modified by an additive term which couples the operator to its source. For instance consider some scalar operator φ and its source J,

$$\mathcal{S}[\varphi, J] \quad = \quad \mathcal{S}_0[\varphi] \ + \int \mathrm{d}^d x\ \varphi(x)\, J(x)\ . \tag{22}$$

Correlation function of that operator φ may now be calculated as the functional derivative of the generating functional $W[J] := -\ln \mathcal{Z}[J]$ of the theory with respect to the source J, e.g.

$$\langle \varphi(x) \rangle \quad \propto \quad \frac{\delta W[J]}{\delta J(x)}\ . \tag{23}$$

One can also apply this procedure to vector- and tensor operators,

$$\mathcal{S} \quad = \quad \mathcal{S}_0 \ + \int \mathrm{d}^d x\ \left(\varphi\, J \ + \ V_\mu\, A^\mu \ + \ T_{\mu\nu}\, g^{\mu\nu}\right)\ . \tag{24}$$

It can be shown that the source of the energy momentum tensor is exactly the quantity that has the properties of the metric. So the energy momentum tensor is obtained by calculating

$$T_{\mu\nu}(x) \quad = \quad -\frac{2}{\sqrt{|\det g|}}\, \frac{\delta W[g]}{\delta g^{\mu\nu}(x)}\ . \tag{25}$$

eq:defEM

The metric transforms under conformal coordinate transformations induced by a vector field v as $\delta_v g^{\mu\nu} = 2\sigma g^{\mu\nu}$, so requiring invariance of W implies

$$0 \quad = \quad \delta_v W[g] \quad = \quad \int \mathrm{d}^d x\ \frac{\delta W[g]}{\delta g^{\mu\nu}(x)}\, \delta_v g^{\mu\nu}(x) \quad = \quad \int \mathrm{d}^d x\ \left(-\frac{\sqrt{|\det g|}\, T_{\mu\nu}}{2}\right)\cdot \left(2\,\sigma\, g^{\mu\nu}\right)$$

$$= \; - \int \mathrm{d}^d x \; \sqrt{|\det g|}\, T_\mu{}^\mu \cdot \sigma \; . \tag{26}$$

Since $T_\mu{}^\mu$ vanishes upon integration against an arbitrary function σ, one can conclude the announced tracelessness of the energy momentum tensor

$$T_\mu{}^\mu \;=\; 0 \; . \tag{27}$$