

Introduction to Gauge/Gravity Duality

Examples III

To hand in Friday 15th November in the examples class

I. Irreducible representations of massless SUSY multiplets

The algebra for \mathcal{N} supercharge operators is given by

$$\{Q_\alpha^a, \bar{Q}_{\dot{\beta}b}\} = -2\sigma_{\alpha\dot{\beta}}^\mu P_\mu \delta_b^a \quad , \quad \{Q_\alpha^a, Q_\beta^b\} = 2\epsilon_{\alpha\beta} Z^{ab} = -2\epsilon_{\alpha\beta} Z^{ba},$$

with $\alpha, \beta \in \{1, 2\}$, $a, b \in \{1, 2, \dots, \mathcal{N}\}$ and σ^μ are components of the four vector $(-\mathbf{1}, \sigma^i)$ of 2×2 matrices with the standard Pauli matrices σ^i .

For studying massless representations, we choose the Lorentz frame with $P^\mu = (E, 0, 0, E)$ and $E > 0$. We use the mostly minus metric.

a) Show that the SUSY algebra relation reduces to

$$\{Q_\alpha^a, \bar{Q}_{\dot{\beta}b}\} = -2(\sigma^\mu P_\mu)_{\alpha\dot{\beta}} \delta_b^a = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\beta}} \delta_b^a.$$

(1 point)

b) Using the result in (a) and the second SUSY algebra relation given above, show that $Q_2^a = 0$ and $Z^{ab} = 0$.

(1 point)

c) The remaining operators Q_1^a and \bar{Q}_{1a} with $a = 1, 2, \dots, \mathcal{N}$ are used to represent field content of SUSY theories with \mathcal{N} supercharges as follows:

- Q_1^a lowers helicity λ by $\frac{1}{2}$, i.e. $Q_1^a |\lambda\rangle = |\lambda - \frac{1}{2}\rangle$;
- \bar{Q}_{1a} raises helicity λ by $\frac{1}{2}$, i.e. $\bar{Q}_{1a} |\lambda\rangle = |\lambda + \frac{1}{2}\rangle$.

Determine all the states of a gauge multiplet for $\mathcal{N} = 1, 2, 3, 4$ by starting from the highest helicity states $|\lambda\rangle = |1\rangle$ and applying products of Q_1^a operators on $|\lambda\rangle$ for all possible values of a .

(Hint: For $\mathcal{N} = D$, there are 2^D states in total.)

(3 points)

d) Additional part: Determine all the states of a gauge multiplet for $\mathcal{N} = 1, 2, 3, 4$ by starting from the lowest helicity states $|\lambda\rangle = |-1\rangle$ and applying products of \bar{Q}_{1a} operators on $|\lambda\rangle$ for all possible values of a . What is special about the $\mathcal{N} = 4$ case compared to those of $\mathcal{N} = 1, 2, 3$ according to CPT invariance? (3 additional points)

II. Scale invariance and β -functions

This question deals with a general d -dimensional quantum field theory. Consider the infinitesimal scale transformation $x^\mu \rightarrow x^\mu + \varepsilon x^\mu$, and assume that $\mathcal{O}_i(x)$ is a complete set of local operators that transform under this rescaling as

$$\delta \mathcal{O}_i(x) = -\varepsilon \Delta_i^j \mathcal{O}_j(x) .$$

a) Show that the scale transformation of a general expectation value is

$$\begin{aligned} \delta \left\langle \prod_m \mathcal{O}_{i_m}(x_m) \right\rangle &= -\varepsilon \int d^d x \left\langle T^\mu_\mu(x) \prod_m \mathcal{O}_{i_m}(x_m) \right\rangle \\ &\quad - \varepsilon \sum_n \Delta_{i_n}^j \left\langle \mathcal{O}_j(x_n) \prod_{m \neq n} \mathcal{O}_{i_m}(x_m) \right\rangle . \end{aligned}$$

The action and energy-momentum tensor of the theory can be rewritten in terms of the complete set,

$$S = \sum'_i \int d^d x g^i \mathcal{O}_i(x) , \quad \int d^d x T^\mu_\mu(x) = -\sum'_i \int d^d x \beta^i(g) \mathcal{O}_i(x) ,$$

where the prime on the sum means that it runs only over operators with dimensions less than or equal to d .

b) Show that the coefficients β^i are indeed β -functions by proving that the renormalization group equation holds,

$$\begin{aligned} \delta \left\langle \prod_m \mathcal{O}_{i_m}(x_m) \right\rangle &= -\varepsilon \sum'_i \beta^i(g) \frac{\partial}{\partial g^i} \left\langle \prod_m \mathcal{O}_{i_m}(x_m) \right\rangle \\ &\quad - \varepsilon \sum_n \Delta_{i_n}^j \left\langle \mathcal{O}_j(x_n) \prod_{m \neq n} \mathcal{O}_{i_m}(x_m) \right\rangle . \end{aligned}$$

What is the implication for the theory when all the β -functions vanish?

c) The one loop β -function of an $SU(N)$ gauge theory goes like

$$\beta(g) \sim -\frac{11}{3}N + \frac{2}{3} \sum_f T(f) + \frac{1}{3} \sum_s T(s) ,$$

where the sums are over the two-component Weyl fermions f and the complex scalars s coupled to the $SU(N)$ gauge field. T is the appropriate Dynkin index, which is 1/2 for the fundamental representation. Show that $\beta(g)$ vanishes in the case of $SU(N)$ $\mathcal{N} = 4$ Super Yang-Mills theory!

Hint: Think carefully about which representation the fields of this theory are in!

(2 points)