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Introduction to Gauge/Gravity Duality

Examples III

To hand in Friday 9th November in the examples class

I. **Coordinates of** AdS_{d+1}.

Lorentzian AdS_{d+1} can be defined by the locus

$$-L^2 = \eta_{ab} X^a X^b = -\left(X^{d+1}\right)^2 - \left(X^0\right)^2 + \sum_{i=1}^{d} \left(X^i\right)^2 , \tag{1}$$

where $X \in \mathbb{R}^{2,d}$ and $ds^2 = \eta_{ab} X^a X^b$ with $\eta = \mathrm{diag}(-1, 1, 1, \ldots, 1, -1)$. In the following we parametrize the locus (1) in different ways.

a) Draw a picture of AdS_2 embedded in $\mathbb{R}^{2,1}$. (2 points)

b) The *global coordinates* (ρ, τ, Ω_i) are defined by

$$\begin{aligned} X^{d+1} &= L \cosh\rho \sin\tau , \\ X^0 &= L \cosh\rho \cos\tau , \\ X^i &= L \sinh\rho\, \Omega_i , \end{aligned}$$

with $i = 1, ..., d$ and $\sum_{i=1}^{d} \Omega_i^2 = 1$. Using this parametrization calculate the induced metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ (where $x^\mu \in \{\rho, \tau, \Omega_i\}$) for AdS_{d+1} in global coordinates. (3 points)

c) Replace ρ by $r \equiv L \sinh\rho$ and show that the metric can be written in the form

$$ds^2 = -H(r) dt^2 + H(r)^{-1} dr^2 + r^2 d\Omega_{d-1}^2,$$

where $d\Omega_{d-1}^2 = \sum_{i=1}^{d} d\Omega_i d\Omega_i$. is the metric of the unit $(d-1)$–sphere, S^{d-1}. (2 points)

d) The *Poincare patch coordinates* (x^i, u) with $i = 1, ..., d$ are defined by

$$\begin{aligned} X^{d+1} + X^d &= u , \\ -X^{d+1} + X^d &= v , \\ X^i &= \frac{u}{L} x^i . \end{aligned}$$

Use the defining equation (1) to eliminate v in terms of u and x^i and show that the induced metric for (u, x^i) with $i = 1, ..., d$ takes the form

$$ds^2 = L^2 \frac{du^2}{u^2} + \frac{u^2}{L^2} dx^i dx_i .$$

Finally introduce $z = \frac{L^2}{u}$ and show that the metric is given by

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 + dx^i dx_i\right)$$

Which part of the AdS spacetime is not covered by these coordinates (Hint: z takes only positive values (Why?)). (3 points)

II. **Irreducible representations of massless SUSY multiplets**

The algebra for $\mathcal{N}$ supercharge operators is given by

$$\{Q^a_\alpha, \bar{Q}_{\dot\beta b}\} = -2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^a_b \qquad , \qquad \{Q^a_\alpha, Q^b_\beta\} = 2\epsilon_{\alpha\beta} Z^{ab} = -2\epsilon_{\alpha\beta} Z^{ba},$$

with $\alpha, \beta \in \{1,2\}$, $a, b \in \{1, 2, ...\mathcal{N}\}$ and σ^μ are components of the four vector $(-\mathbf{1}, \sigma^i)$ of 2×2 matrices with the standard Pauli matrices σ^i.

For studying massless representations, we choose the Lorentz frame with $P^\mu = (E, 0, 0, E)$ and $E > 0$. We use the mostly minus metric.

a) Show that the SUSY algebra relation reduces to

$$\{Q^a_\alpha, \bar{Q}_{\dot\beta b}\} = -2(\sigma^\mu P_\mu)_{\alpha\dot\beta}\delta^a_b = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\beta} \delta^a_b.$$

(1 point)

b) Using the result in (a) and the second SUSY algebra relation given above, show that $Q^a_2 = 0$ and $Z^{ab} = 0$. (1 point)

c) The remaining operators Q^a_1 and $\bar{Q}_{\dot 1 a}$ with $a = 1, 2, ..., \mathcal{N}$ are used to represent field content of SUSY theories with $\mathcal{N}$ supercharges as follows:

- Q^a_1 lowers helicity λ by $\frac{1}{2}$, i.e. $Q^a_1|\lambda\rangle \; = |\lambda - \frac{1}{2}\rangle$;
- $\bar{Q}_{\dot 1 a}$ raises helicity λ by $\frac{1}{2}$, i.e. $\bar{Q}_{\dot 1 a}|\lambda\rangle = |\lambda + \frac{1}{2}\rangle$.

Determine all the states of a gauge multiplet for $\mathcal{N} = 1, 2, 3, 4$ by starting from the highest helicity states $|\lambda\rangle = |1\rangle$ and applying products of Q^a_1 operators on $|\lambda\rangle$ for all possible values of a.

(Hint: For $\mathcal{N} = D$, there are 2^D states in total.) (3 points)

d) Additional part: Determine all the states of a gauge multiplet for $\mathcal{N} = 1, 2, 3, 4$ by starting from the lowest helicity states $|\lambda\rangle = |-1\rangle$ and applying products of $\bar{Q}_{\dot 1 a}$ operators on $|\lambda\rangle$ for all possible values of a. What is special about the $\mathcal{N} = 4$ case compared to those of $\mathcal{N} = 1, 2, 3$ according to CPT invariance? (3 additional points)

III. β **function of** $SU(N)$ $\mathcal{N} = 4$ **Super Yang-Mills theory**

The one loop β function of a $SU(N)$ gauge theory goes like

$$\beta(g) \sim -\frac{11}{3}N + \frac{2}{3}\sum_f T(f) + \frac{1}{3}\sum_s T(s)\,, \tag{2}$$

where the sums are over the two-component Weyl fermions f and the complex scalars s coupled to the $SU(N)$ gauge field. T is the appropriate Dynkin index, which is $1/2$ for the fundamental representation. Show that $\beta(g)$ vanishes in the case of $SU(N)$ $\mathcal{N} = 4$ Super Yang-Mills theory!
Hint: Think carefully about which representation the fields of this theory are in!
. (2 points)