Prof. Dr. Johanna Erdmenger
Daniel Fernández

Introduction to Gauge/Gravity Duality

Examples II

To hand in Friday 24th October in the examples class

I. **Gross-Neveu Model**

The Gross-Neveu model is defined by the Lagrangian

$$\mathcal{L} = \bar{\psi}^a i\gamma^\mu \partial_\mu \psi^a + \frac{g_0}{2N}\left(\bar{\psi}^a\psi^a\right)^2 \text{, with } a = 1...N \tag{1}$$

in $d = 2$ spacetime dimensions. We choose the following representation for the γ^μ matrices

$$\gamma^0 = \sigma_z\,, \quad \gamma^1 = i\sigma_y \text{ and } \gamma_5 = \gamma^0\gamma^1 = \sigma_x\,, \tag{2}$$

with σ_i the standard Pauli matrices.

a) Show that this Lagrangian is symmetric under the discrete chiral transformation

$$\psi^a \to \gamma_5\psi^a\,,\, \bar{\psi}^a \to -\bar{\psi}^a\gamma_5\,. \tag{3}$$

Which terms are excluded by this symmetry? (2 points)

b) Show that this theory is renormalisable (at the level of dimensional analysis).
. (1 point)

c) Show that the Lagrangian

$$\tilde{\mathcal{L}} = \bar{\psi}^a i\gamma^\mu\partial_\mu\psi^a - \frac{N}{2g_0}\sigma^2 + \sigma\bar{\psi}^a\psi^a \tag{4}$$

is equivalent to $\mathcal{L}$ by integrating out the field $\sigma(x)$. (2 points)

d) In the large N limit the potential of this theory takes the form

$$V = \frac{N}{2g_0}\sigma^2 - \sum_{n=1}^{\infty}\frac{N}{2n}\mathrm{Tr}\left(\int\frac{\mathrm{d}^2p}{(2\pi)^2}\left(\frac{-\not{p}\sigma}{p^2+i\epsilon}\right)^{2n}\right)\,. \tag{5}$$

Explain the origin of each term. Which diagrams contribute to the second term?
. (2 points)

e) Compute the integral in equation (5) by rotating to Euclidean two-momentum p_E and integrating up to a cut off ${p_E}^2 \leq \Lambda^2$.
Result:

$$V = N\left[\frac{\sigma^2}{2g_0} + \frac{1}{4\pi}\sigma^2\left(\ln\left(\frac{\sigma^2}{\Lambda^2}\right) - 1\right)\right]\,. \tag{6}$$

Use this result to compute the renormalised coupling constant g at an arbitrary renormalisation mass M in terms of the bare coupling g_0 and the cut off:

$$\frac{1}{g} = \frac{1}{g_0} + \frac{1}{2\pi} \ln\left(\frac{M^2}{\Lambda^2}\right) + \frac{1}{\pi} \,. \tag{7}$$

Explain why this theory is asymptotically free. (3 points)

f) Show that the true vacuum of this theory spontaneously breaks the discrete chiral symmetry. (2 points)

References:
S. Coleman – 1/N, (1980)
D. J. Gross and A. Neveu – Dynamical symmetry breaking in asymptotically free field theories, Phys. Rev. D10:3235 (1974)

II. **Coordinates of** AdS_{d+1}.

Lorentzian AdS_{d+1} can be defined by the locus

$$-L^2 = \eta_{ab} X^a X^b = -\left(X^{d+1}\right)^2 - \left(X^0\right)^2 + \sum_{i=1}^{d} \left(X^i\right)^2 \,, \tag{8}$$

where $X \in \mathbb{R}^{2,d}$ and $ds^2 = \eta_{ab} X^a X^b$ with $\eta = \mathrm{diag}(-1, 1, 1, \ldots, 1, -1)$. In the following we parametrize the locus (8) in different ways.

a) Draw a picture of AdS_2 embedded in $\mathbb{R}^{2,1}$. (2 points)

b) The *global coordinates* (ρ, τ, Ω_i) are defined by

$$\begin{aligned} X^{d+1} &= L \cosh\rho \, \sin\tau \,, \\ X^0 &= L \cosh\rho \, \cos\tau \,, \\ X^i &= L \sinh\rho \, \Omega_i \,, \end{aligned}$$

with $i = 1, ..., d$ and $\sum_{i=1}^{d} \Omega_i^2 = 1$. Using this parametrization calculate the induced metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ (where $x^\mu \in \{\rho, \tau, \Omega_i\}$) for AdS_{d+1} in global coordinates. (3 points)

c) Replace ρ by $r \equiv L \sinh\rho$ and show that the metric can be written in the form

$$ds^2 = -H(r) dt^2 + H(r)^{-1} dr^2 + r^2 d\Omega_{d-1}^2,$$

where $d\Omega_{d-1}^2 = \sum_{i=1}^{d} d\Omega_i d\Omega_i$. is the metric of the unit $(d-1)$–sphere, S^{d-1}. (2 points)

d) The *Poincare patch coordinates* (x^i, u) with $i = 1, ..., d$ are defined by

$$\begin{aligned} X^{d+1} + X^d &= u \,, \\ -X^{d+1} + X^d &= v \,, \\ X^i &= \frac{u}{L} x^i \,. \end{aligned}$$

Use the defining equation (8) to eliminate v in terms of u and x^i and show that the induced metric for (u, x^i) with $i = 1, ..., d$ takes the form

$$ds^2 = L^2 \frac{du^2}{u^2} + \frac{u^2}{L^2} dx^i dx_i \,.$$

Finally introduce $z = \frac{L^2}{u}$ and show that the metric is given by

$$ds^2 = \frac{L^2}{z^2} \left(dz^2 + dx^i dx_i \right)$$

Which part of the AdS spacetime is not covered by these coordinates (Hint: z takes only positive values (Why?)). (3 points)