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Introduction to Gauge/Gravity Duality

Examples VIII

To hand in Friday 11th December in the examples class

I. **Penrose-Brown-Henneaux Transformation**

Show that, when transforming the metric

$$ds^2 = L^2 \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{\mu\nu} dx^\mu dx^\nu \tag{1}$$

according to the Penrose-Brown-Henneaux diffeomorphism

$$\rho = \rho'(1 - 2\sigma(x')) \qquad x^\mu = (x')^\mu + a^\mu(x', \rho') \tag{2}$$

and demanding that

$$g'_{55} = g_{55} \text{ and } g'_{\mu 5} = g_{\mu 5}\,, \tag{3}$$

one obtains

$$\partial_5 a^\mu = \frac{L^2}{2} g^{\mu\nu} \partial_\nu \sigma \tag{4}$$

and

$$g_{\mu\nu} \to g_{\mu\nu} + 2\sigma \left(1 - \rho \frac{\partial}{\partial \rho}\right) g_{\mu\nu} + \nabla_\mu a_\nu + \nabla_\mu a_\nu\,. \tag{5}$$

Note that the index 5 stands for the ρ direction. Explain why this transformation induces a Weyl transformation at the boundary. (5 points)

II. Holographic conformal anomaly

The starting point for the calculation of the holographic conformal anomaly is the action

$$S = \frac{1}{16\pi G} \left[\int d^{d+1}x \, \sqrt{g} \, \left(R + \frac{d(d-1)}{L^2} \right) + 2 \int d^d x \, \sqrt{\gamma} K \right] , \tag{6}$$

including a Gibbons-Hawking boundary term. As discussed in the lecture, for regularising this action when writing it in terms of Fefferman-Graham coordinates,

$$ds^2 = L^2 \left[\frac{d\rho^2}{4\,\rho^2} + \frac{1}{\rho} \, g_{\mu\nu}(\rho, x) \, dx^\mu \, dx^\nu \right] , \qquad \lim_{\rho \to 0} g_{\mu\nu}(\rho, x) = g^{(0)}_{\mu\nu}(x) \,, \tag{7}$$

it is necessary to add counterterms of the form

$$S_{\text{ct}} = -\frac{1}{16\pi G} \int d^{d+1}x \, \sqrt{\det g^{(0)}} \left(\epsilon^{-2} a_{(0)} + \epsilon^{-1} a_{(2)} + \cdots - \ln\epsilon \, a_{(d)} \right) , \tag{8}$$

with

$$a_{(0)} = \frac{2(d-1)}{L}\,, \quad a_{(2)} = \frac{L}{2(d-1)}R\,, \quad a_{(4)} = \frac{L^3}{2(d-2)^2}(R^{\mu\nu}R_{\mu\nu} - \frac{1}{(d-1)}R^2)\,. \tag{9}$$

For $d = 4$ dimensions, show that for a function $\sigma(x)$ corresponding to a conformal transformation, i.e. $\sigma(x) = (\lambda - 2b \cdot x)$, the terms involving $\sqrt{\det g^{(0)}}\epsilon^{-2}a_{(0)}$ and $\sqrt{\det g^{(0)}}\epsilon^{-1}a_{(2)}$ in $S_{\rm ct}$ are invariant under the residual PBH transformation

$$\delta S_{\rm ct} \;=\; 2\int d^4x\,\sigma(x)\left(\epsilon\frac{\delta}{\delta\epsilon} - g^{(0)\mu\nu}\frac{\delta}{\delta g^{(0)\mu\nu}}\right) S_{\rm ct}\,. \tag{10}$$

For the remaining term involving $\sqrt{\det g^{(0)}}\ln\epsilon\, a_{(4)}$, show for the transformation (10) that

$$\begin{aligned}\delta S_{\rm ct} &= 2\int d^4x\,\sigma(x)\left(\epsilon\frac{\delta}{\delta\epsilon} - g^{(0)\mu\nu}\frac{\delta}{\delta g^{(0)\mu\nu}}\right)\mathcal{S}_{\rm ct} = 2\int d^4x\,\sigma(x)\epsilon\,\frac{\delta}{\delta\epsilon}\,S_{\rm ct}\\ &= \frac{L^3}{64\pi G}\int d^4x\,\sqrt{\det g_{(0)}}\left(R^{\mu\nu}R_{\mu\nu} - \frac{R^2}{3}\right)\,.\end{aligned} \tag{11}$$

Hint: Use the conformal transformations

$$\delta E = 4\sigma E - G^{\mu\nu}\nabla_\mu\nabla_\nu\sigma\,, \qquad \delta R = 2\sigma R - \nabla^2\sigma \tag{12}$$

for the Euler density and the Ricci tensor, with $G_{\mu\nu}$ the Einstein tensor.

(5 points)