Prof. Dr. Johanna Erdmenger
Mario Flory, Charlotte Sleight

Introduction to Gauge/Gravity Duality

Examples IV

To hand in Friday 13th November in the examples class

Coordinates and curvature of AdS_{d+1}

Lorentzian AdS_{d+1} can be defined by the locus

$$-L^2 = \eta_{ab} X^a X^b = -\left(X^{d+1}\right)^2 - \left(X^0\right)^2 + \sum_{i=1}^{d} \left(X^i\right)^2 , \tag{1}$$

where $X \in \mathbb{R}^{2,d}$ and $ds^2 = \eta_{ab} X^a X^b$ with $\eta = \text{diag}(-1, 1, 1, \ldots, 1, -1)$. In the following we parametrize the locus (1) in different ways.

a) Draw a picture of AdS_2 embedded in $\mathbb{R}^{2,1}$. (2 points)

b) The *global coordinates* (ρ, τ, Ω_i) are defined by

$$\begin{aligned} X^{d+1} &= L \cosh\rho \, \sin\tau \, , \\ X^0 &= L \cosh\rho \, \cos\tau \, , \\ X^i &= L \sinh\rho \, \Omega_i \, , \end{aligned}$$

with $i = 1, ..., d$ and $\sum_{i=1}^{d} \Omega_i^2 = 1$. Using this parametrization calculate the induced metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ (where $x^\mu \in \{\rho, \tau, \Omega_i\}$) for AdS_{d+1} in global coordinates. (3 points)

c) Replace ρ by $r \equiv L \sinh\rho$ and show that the metric can be written in the form

$$ds^2 = -H(r)dt^2 + H(r)^{-1} dr^2 + r^2 d\Omega_{d-1}^2,$$

where $d\Omega_{d-1}^2 = \sum_{i=1}^{d} d\Omega_i d\Omega_i$. is the metric of the unit $(d-1)$–sphere, S^{d-1}. (2 points)

d) The *Poincaré patch coordinates* (x^i, u) with $i = 0, ..., d-1$ are defined by

$$\begin{aligned} X^{d+1} + X^d &= u \, , \\ -X^{d+1} + X^d &= v \, , \\ X^i &= \frac{u}{L} x^i \, . \end{aligned}$$

Use the defining equation (1) to eliminate v in terms of u and x^i and show that the induced metric for (u, x^i) with $i = 0, ..., d-1$ takes the form

$$ds^2 = L^2 \frac{du^2}{u^2} + \frac{u^2}{L^2} dx^i dx_i \, .$$

Now introduce $z = \frac{L^2}{u}$ and show that the metric is given by

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 + dx^i dx_i\right) \tag{2}$$

Which part of the AdS spacetime is not covered by these coordinates (Hint: z takes only positive values (Why?)). (3 points)

e) Finally, calculate the the Christoffel symbols $\Gamma^{\mu}_{\rho\sigma}$ of the metric (2) as well as the Riemann tensor $R^{\mu}{}_{\nu\rho\sigma}$ in the Poincaré patch of AdS_{d+1}. Check that the Ricci scalar R is given by $R = -\frac{d(d+1)}{L^2}$ and that Anti-de Sitter is *maximally symmetric* since

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d+1)}\left(g_{\nu\sigma}g_{\mu\rho} - g_{\nu\rho}g_{\mu\sigma}\right) . \tag{3}$$

Moreover, confirm that Anti-de Sitter space satisfies Einstein's equations with $T_{\mu\nu} = 0$, where the cosmological constant Λ is given by

$$\Lambda = -\frac{d(d+1)}{2L^2} . \tag{4}$$

(3 points)