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**Introduction to Gauge/Gravity Duality**

**Examples III**

**To hand in Friday 31st October in the examples class**

I. **Irreducible representations of massless SUSY multiplets**

The algebra for $\mathcal{N}$ supercharge operators is given by

$$\{Q^a_\alpha, \bar{Q}_{\dot{\beta}b}\} = -2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^a_b \qquad , \qquad \{Q^a_\alpha, Q^b_\beta\} = 2\epsilon_{\alpha\beta} Z^{ab} = -2\epsilon_{\alpha\beta} Z^{ba},$$

with $\alpha, \beta \in \{1,2\}$, $a, b \in \{1, 2, ...\mathcal{N}\}$ and $\sigma^\mu$ are components of the four vector $(-\mathbf{1}, \sigma^i)$ of $2 \times 2$ matrices with the standard Pauli matrices $\sigma^i$.

For studying massless representations, we choose the Lorentz frame with $P^\mu = (E, 0, 0, E)$ and $E > 0$. We use the mostly minus metric.

a) Show that the SUSY algebra relation reduces to

$$\{Q^a_\alpha, \bar{Q}_{\dot{\beta}b}\} = -2(\sigma^\mu P_\mu)_{\alpha\dot{\beta}} \delta^a_b = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\beta}} \delta^a_b.$$

(1 point)

b) Using the result in (a) and the second SUSY algebra relation given above, show that $Q^a_2 = 0$ and $Z^{ab} = 0$. (1 point)

c) The remaining operators $Q^a_1$ and $\bar{Q}_{\dot{1}a}$ with $a = 1, 2, ..., \mathcal{N}$ are used to represent field content of SUSY theories with $\mathcal{N}$ supercharges as follows:

- $Q^a_1$ lowers helicity $\lambda$ by $\frac{1}{2}$, i.e. $Q^a_1|\lambda\rangle = |\lambda - \frac{1}{2}\rangle$;
- $\bar{Q}_{\dot{1}a}$ raises helicity $\lambda$ by $\frac{1}{2}$, i.e. $\bar{Q}_{\dot{1}a}|\lambda\rangle = |\lambda + \frac{1}{2}\rangle$.

Determine all the states of a gauge multiplet for $\mathcal{N} = 1, 2, 3, 4$ by starting from the highest helicity states $|\lambda\rangle = |1\rangle$ and applying products of $Q^a_1$ operators on $|\lambda\rangle$ for all possible values of $a$.

(Hint: For $\mathcal{N} = D$, there are $2^D$ states in total.) (3 points)

d) Additional part: Determine all the states of a gauge multiplet for $\mathcal{N} = 1, 2, 3, 4$ by starting from the lowest helicity states $|\lambda\rangle = |-1\rangle$ and applying products of $\bar{Q}_{\dot{1}a}$ operators on $|\lambda\rangle$ for all possible values of $a$. What is special about the $\mathcal{N} = 4$ case compared to those of $\mathcal{N} = 1, 2, 3$ according to CPT invariance? (3 additional points)

II. **Scale invariance and $\beta$-functions**

This question deals with a general $d$-dimensional quantum field theory. Consider the infinitesimal scale transformation $x^\mu \to x^\mu + \varepsilon x^\mu$, and assume that $\mathcal{O}_i(x)$ is a complete set of local operators that transform under this rescaling as

$$\delta \mathcal{O}_i(x) = -\varepsilon \Delta_i^{\ j} \mathcal{O}_j(x) \ .$$

a) Show that the scale transformation of a general expectation value is

$$\begin{aligned} \delta \left\langle \prod_m \mathcal{O}_{i_m}(x_m) \right\rangle = & - \varepsilon \int d^d x \left\langle T^\mu_{\ \mu}(x) \prod_m \mathcal{O}_{i_m}(x_m) \right\rangle \\ & - \varepsilon \sum_n \Delta_{i_n}^{\ j} \left\langle \mathcal{O}_j(x_n) \prod_{m \neq n} \mathcal{O}_{i_m}(x_m) \right\rangle \ . \end{aligned}$$

The action and energy-momentum tensor of the theory can be rewritten in terms of the complete set,

$$S = \sum_i{}' \int d^d x \ g^i \mathcal{O}_i(x) \ , \qquad \int d^d x \ T^\mu_{\ \mu}(x) = -\sum_i{}' \int d^d x \ \beta^i(g) \mathcal{O}_i(x) \ ,$$

where the prime on the sum means that it runs only over operators with dimensions less than or equal to $d$.

b) Show that the coefficients $\beta^i$ are indeed $\beta$-functions by proving that the renormalization group equation holds,

$$\begin{aligned} \delta \left\langle \prod_m \mathcal{O}_{i_m}(x_m) \right\rangle = & - \varepsilon \sum_i{}' \beta^i(g) \frac{\partial}{\partial g^i} \left\langle \prod_m \mathcal{O}_{i_m}(x_m) \right\rangle \\ & - \varepsilon \sum_n \Delta_{i_n}^{\ j} \left\langle \mathcal{O}_j(x_n) \prod_{m \neq n} \mathcal{O}_{i_m}(x_m) \right\rangle \ . \end{aligned}$$

What is the implication for the theory when all the $\beta$-functions vanish?

c) The one loop $\beta$-function of an $SU(N)$ gauge theory goes like

$$\beta(g) \sim -\frac{11}{3} N + \frac{2}{3} \sum_f T(f) + \frac{1}{3} \sum_s T(s) \, ,$$

where the sums are over the two-component Weyl fermions $f$ and the complex scalars $s$ coupled to the $SU(N)$ gauge field. $T$ is the appropriate Dynkin index, which is $1/2$ for the fundamental representation. Show that $\beta(g)$ vanishes in the case of $SU(N)$ $\mathcal{N} = 4$ Super Yang-Mills theory!

*Hint:* Think carefully about which representation the fields of this theory are in!

. (2 points)