Exercise 4.2: D-dimensional phase space integrals

Show that the D-dimensional three-particle phase space for $Q \to p_1 + p_2 + p_3$, with $p_i^2 = 0$, can be expressed in terms of the kinematic invariants $s_{ij} = (p_i + p_j)^2$ as

$$d\Phi_3 = \frac{(2\pi)^{3-2D}}{2^{D+1}} (Q^2)^{\frac{2-D}{2}} d\Omega_{D-2} d\Omega_{D-3}\, ds_{12}\, ds_{13}\, ds_{23}\, (s_{12}s_{13}s_{23})^{\frac{D-4}{2}}\, \delta\left(Q^2 - s_{12} - s_{13} - s_{23}\right) .$$

Solution:
We start from eq.(3.76) in the lecture notes:

$$d\Phi_3 = \frac{1}{4}(2\pi)^{3-2\,D}\, dE_1 dE_2 d\theta_1 [E_1 E_2 \sin\theta]^{D-3} d\Omega_{D-2}\, d\Omega_{D-3}\, \delta((Q-p_1-p_2)^2)\,, \tag{1}$$

based on the following parametrisation of the momenta:

$$\begin{aligned} Q &= (E, \vec{0}^{(D-1)}) \\ p_1 &= E_1\,(1, \vec{0}^{(D-2)}, 1) \\ p_2 &= E_2\,(1, \vec{0}^{(D-3)}, \sin\theta, \cos\theta) \\ p_3 &= Q - p_2 - p_1\,. \end{aligned} \tag{2}$$

The kinematic invariants $s_{ij} = (p_i + p_j)^2$ are given by

$$\begin{aligned} s_{12} &= 2E_1E_2\,(1-\cos\theta) \\ s_{13} &= 2E_1(E - E_2\,(1-\cos\theta)) = E\,(E - 2E_2) \\ s_{23} &= 2E_2(E - E_1\,(1-\cos\theta)) = E\,(E - 2E_1)\,. \end{aligned} \tag{3}$$

The Jacobian for the transformation is

$$|\det J| = \left|\det\left(\frac{\partial(s_{12}, s_{13}, s_{23})}{\partial(E_1, E_2, \theta)}\right)\right| = 8E^2 E_1 E_2 \sin\theta\,. \tag{4}$$

We can save a lot of algebra if we relate $|\det J|$ to the determinant of the Gram matrix G_{ij}, which we can define either using the vectors Q, p_1, p_2, or the vectors p_1, p_2, p_3. The determinant will be the same in both cases. In the first case, the Gram matrix is given by

$$G^{(1)} = 2\begin{pmatrix} Q^2 & Qp_1 & Qp_2 \\ Qp_1 & 0 & p_1p_2 \\ Qp_2 & p_1p_2 & 0 \end{pmatrix} \tag{5}$$

On the other hand, using p_1, p_2, p_3 to form frame-independent variables, we have

$$G^{(2)} = \begin{pmatrix} 0 & s_{12} & s_{13} \\ s_{12} & 0 & s_{23} \\ s_{13} & s_{23} & 0 \end{pmatrix} \tag{6}$$

such that

$$\det G^{(1)} = \det G^{(2)} = \det G = 2s_{12}s_{13}s_{23} = 8E^2E_1^2E_2^2\sin^2\theta \tag{7}$$

and therefore

$$\det J = 2E\sqrt{2\det G} = 4E\,\sqrt{s_{12}s_{13}s_{23}}\,. \tag{8}$$

This leads to

$$\begin{aligned} \int d\theta_1 dE_1 dE_2\, [E_1E_2\sin\theta]^{D-3} &= \int d\theta_1 dE_1 dE_2 \left[\frac{\det J}{8E^2}\right]^{D-3} \\ &= \int ds_{12} ds_{13} ds_{23}\, (8E^2)^{3-D}\, (\det J)^{D-4} \\ &= E^{2-D} 2^{1-D} \int ds_{12} ds_{13} ds_{23} \left(\frac{\det G}{2}\right)^{\frac{D-4}{2}} . \end{aligned}$$

Combining everything one obtains

$$\int \mathrm{d}\Phi_3 = \frac{(2\pi)^{3-2\,D}}{2^{D-1}} (Q^2)^{1-D/2} \int \mathrm{d}\Omega_{D-2}\, \mathrm{d}\Omega_{D-3} \mathrm{d}s_{12} \mathrm{d}s_{13} \mathrm{d}s_{23}\ (s_{12}s_{13}s_{23})^{\frac{D-4}{2}}\ \delta(Q^2 - s_{12} - s_{13} - s_{23})\ .$$

Defining dimensionless variables by

$$y_1 = s_{12}/Q^2, y_2 = s_{13}/Q^2, y_3 = s_{23}/Q^2$$

we arrive at

$$\int \mathrm{d}\Phi_3 = \frac{(2\pi)^{3-2\,D}}{2^{D-1}} (Q^2)^{D-3} \int \mathrm{d}\Omega_{D-2}\, \mathrm{d}\Omega_{D-3} \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3\ (y_1y_2y_3)^{\frac{D-4}{2}}\ \delta(1 - \sum_{i=1}^{3} y_i)\ . \qquad (9)$$