Exercise 3.3: Tensor Integrals

In the representation of tensor integrals, for tensor ranks $r \geq 2$, higher dimensional integrals I_N^{D+2m} arise as coefficients of metric tensors $(g^{\mu\nu})^{\otimes m}$. To see how these integrals arise, start from the representation in eq. (3.26) of the script, in terms of Feynman parameters and quadratic forms in the loop momentum, but this time with loop momenta in the numerator:

$$L_N^{\mu_1\mu_2} \;=\; \Gamma(N)\int_0^\infty \prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\int_{-\infty}^\infty \frac{d^Dl}{i\pi^{\frac{D}{2}}}\,l^{\mu_1}l^{\mu_2}\left[l^2-R^2+i\delta\right]^{-N}.$$

Then make the ansatz

$$L_N^{\mu_1\mu_2} = K\,g^{\mu_1\mu_2}$$

and determine K in terms of I_N^{D+2}, using the functional form given in eq. (3.18) of the script.

Solution:
As there is no dimensionful object in the integral which could carry the Lorentz structure, it must be proportional to the metric tensor:

$$\begin{aligned} L_N^{\mu_1\mu_2} &= \Gamma(N)\int_0^\infty \prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\int_{-\infty}^\infty d\bar{l}\,l^{\mu_1}l^{\mu_2}\left[l^2-R^2\right]^{-N} \\ &= K\,g^{\mu_1\mu_2}\,, \end{aligned} \tag{1}$$

with the definition $d\bar{l} = d^Dl/(i\pi^{\frac{D}{2}})$ and $i\delta$ dropped for ease of notation. Contracting both sides of eq. (1) with $g_{\mu_1\mu_2}$, we obtain

$$\begin{aligned} g_{\mu_1\mu_2}L_N^{\mu_1\mu_2} &= \Gamma(N)\int_0^\infty \prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\int_{-\infty}^\infty d\bar{l}\,l^2\left[l^2-R^2\right]^{-N} = K\,D \\ &= \Gamma(N)\int_0^\infty \prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\int_{-\infty}^\infty d\bar{l}\,\left\{\left[l^2-R^2\right]^{-N+1}+R^2\left[l^2-R^2\right]^{-N}\right\}. \end{aligned} \tag{2}$$

Now remember the formula for the scalar case:

$$\begin{aligned} I_N^D &= \Gamma(N)\int_0^\infty \prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\int_{-\infty}^\infty d\bar{l}\,\left[l^2-R^2\right]^{-N} \\ &= (-1)^N\Gamma(N-\frac{D}{2})\int_0^\infty \prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\,\left[R^2\right]^{D/2-N}. \end{aligned} \tag{3}$$

We see that it can be applied as well to the first term in eq. (2) with $N \to N-1$. We obtain:

$$\begin{aligned} g_{\mu_1\mu_2}L_N^{\mu_1\mu_2} &= (-1)^{N-1}\frac{\Gamma(N)}{\Gamma(N-1)}\Gamma(N-1-D/2)\int_0^\infty \prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\,\left[R^2\right]^{D/2-N+1} \\ &+ (-1)^N\Gamma(N-\frac{D}{2})\int_0^\infty \prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\,\left[R^2\right]^{D/2-N+1} \\ &= (-1)^N\Gamma(N-\frac{D+2}{2})\int \prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\,\left[R^2\right]^{(D+2)/2-N}\,\{-(N-1)+N-1-D/2\} \\ &= -\frac{D}{2}\,I_N^{D+2}\,. \end{aligned} \tag{4}$$

Hence we find $K = -\frac{1}{2}\,I_N^{D+2}$.