Exercise 1.2
Let t^a be the generators of a Lie Algebra with $[t^a, t^b] = i\, f^{abc}\, t^c$, and $\mathrm{Trace}(t^a t^b) = T_R\, \delta^{ab}$.

(a) The Casimir operator C_F of the fundamental representation of a Lie group is defined by

$$\sum_a (t^a t^a)_{ij} = C_F\, \delta_{ij}\ ,$$

where t^a are the generators in the fundamental representation.
Show that for $SU(N)$, $C_F = T_R\,(N^2-1)/N$.
Hint: Use the relation

$$\sum_a t^a_{ij} t^a_{kl} = T_R \left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right)\ . \tag{1}$$

Solution:

$$\sum_a (t^a t^a)_{il} = \sum_a t^a_{ij} t^a_{jl} = T_R \left(\delta_{il}\,\delta_{jj} - \frac{1}{N}\delta_{il}\right) = T_R\,\delta_{il}\left(N - \frac{1}{N}\right)\ . \tag{2}$$

(b) The Casimir operator C_A of the adjoint representation of a Lie group is defined by

$$\sum_a (F^a F^a)_{bc} = C_A\, \delta_{bc}\ ,$$

where F^a are the generators in the adjoint representation, i.e. $(F^a)_{bc} = -i\, f^{abc}$.
Show that for $SU(N)$, $C_A = 2\, T_R\, N$.
Hint: Use the relations

$$\begin{aligned}
T_R\, f^{abc} &= -i\,\mathrm{Trace}(t^a t^b t^c - t^b t^a t^c)\ , \qquad (3)\\
\sum_a t^a_{ij} t^a_{kl} &= T_R \left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right)\ ,\\
\mathrm{Trace}(t^a t^b) &= T_R\, \delta^{ab}\ .
\end{aligned}$$

Solution:

$$\begin{aligned}
C_A\,\delta_{bc} &= \sum_a (F^a F^a)_{bc} = -\sum_a f^{abd} f^{adc}\\
&= 1/T_R^2\, \mathrm{Trace}(t^a[t^b,t^d])\mathrm{Trace}(t^a[t^d,t^c]) = 1/T_R^2\, t^a_{ij}[t^b,t^d]_{ji} t^a_{kl}[t^d,t^c]_{lk}\\
&= 1/T_R \left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right) [t^b,t^d]_{ji}\,[t^d,t^c]_{lk}\\
&= 1/T_R\, [t^b,t^d]_{ji}\,[t^d,t^c]_{ij}\\
&= 1/T_R \left(t^b_{jk}t^d_{ki}t^d_{il}t^c_{lj} - t^b_{jk}t^d_{ki}t^c_{il}t^d_{lj} - t^d_{jk}t^b_{ki}t^d_{il}t^c_{lj} + t^d_{jk}t^b_{ki}t^c_{il}t^d_{lj}\right)\\
&= 1/T_R\,(\mathrm{term}\,1 + \mathrm{term}\,2 + \mathrm{term}\,3 + \mathrm{term}\,4)\ . \qquad (4)
\end{aligned}$$

Working these terms out using $\mathrm{Trace}(t^a t^b) = T_R\,\delta^{ab}$ and $\sum_a t^a_{ij} t^a_{kl} = T_R\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right)$ leads to

$$\begin{aligned}
\mathrm{term}\,2 = \mathrm{term}\,3 &= T_R^2\,\frac{1}{N}\,\delta_{bc}\\
\mathrm{term}\,1 = \mathrm{term}\,4 &= T_R^2\,(N - \frac{1}{N})\delta_{bc}\ .
\end{aligned}$$