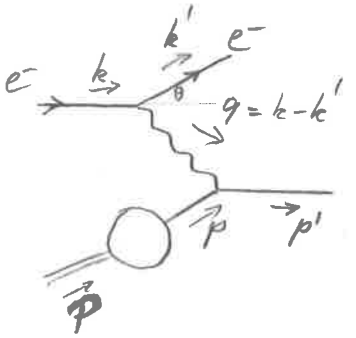
Exercise 5.1.



momentum conservation:

$$k + p = k' + p' \quad ; \quad \hat{s} = (k+p)^2 = 2k\cdot p$$

$$y = \frac{P\cdot q}{P\cdot k} = \frac{p\cdot q}{p\cdot k} = \frac{Q^2}{\hat{s}} = 1 - \frac{E'}{E} \quad ; \quad x = \frac{Q^2}{2qP}$$

$$p'^2 = 0 = (q+p)^2 = -Q^2 + 2q\cdot p \Rightarrow 2q\cdot p = Q^2$$

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = e_q^2 e^4 \frac{1}{Q^4} L^{\mu\nu} Q_{\mu\nu} = e_q^2 e^4 \frac{2\hat{s}^2}{Q^4}\left(1 + (1-y)^2\right)$$

partonic cross section: $$d\hat{\sigma} = \frac{1}{2\hat{s}}\, d\Phi_2\, \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2$$

2-particle phase space in 4 dimensions:

$$d\Phi_2 = \frac{d^4k'}{(2\pi)^3}\,\delta(k'^2)\,\frac{d^4p'}{(2\pi)^3}\,\delta(p'^2)\,(2\pi)^4\,\delta^{(4)}(k+p-k'-p')$$

$$= \frac{d^3k'}{(2\pi)^3\, 2E'}\,\frac{d^4p'}{(2\pi)^3}\,\delta(p'^2)\,(2\pi)^4\,\delta(k+p-k'-p')$$

$q = k - k'$

$p = \xi P$

$$= \frac{1}{(2\pi)^2}\,\frac{d^3k'}{2E'}\,\delta\left((q+p)^2\right)$$

Use $(q+p)^2 = -Q^2 + 2q\cdot p = -Q^2 + 2q\cdot P\,\xi = -Q^2\left(1 - \frac{\xi}{x}\right)$

$$= \frac{1}{8\pi^2}\, d\phi\, d\cos\theta\, d|\vec{k}'|\,\frac{|\vec{k}'|^2}{E'}\,\delta\left(Q^2\left(\frac{\xi}{x} - 1\right)\right)$$

use $|\vec{k}'| = E'$

$$= \frac{1}{4\pi}\,\frac{d\phi}{2\pi}\, d\cos\theta\, E'\, dE'\,\frac{x}{Q^2}\,\delta(\xi - x) \qquad (*)$$

Exercise 5.1.

Now use $E' = E(1-y) = \frac{\sqrt{\hat{s}}}{2}(1-y) \Rightarrow dE' = E\,dy$

and $q^2 = (k-k')^2 = -2EE'(1-\cos\theta) = -2E^2(1-y)(1-\cos\theta)$

$$\Rightarrow 1-\cos\theta = \frac{q^2}{-2E^2(1-y)} = \frac{-2MExy}{-2E^2(1-y)} = \frac{Mxy}{E(1-y)}$$

$$\Rightarrow d\cos\theta = \frac{My}{E(1-y)}\,dx$$

$$dE'\,d\cos\theta = \frac{My}{1-y}\,dx\,dy$$

insert into (*):

$$d\phi_2 = \frac{1}{4\pi}\frac{d\phi}{2\pi}E(1-y)\frac{x}{Q^2}\frac{My}{1-y}\,dx\,dy\,\delta(\xi-x)$$

Use $EMxy = \frac{Q^2}{2}$

$$\Rightarrow d\phi_2 = \frac{1}{8\pi}\frac{d\phi}{2\pi}\,dx\,dy\,\delta(\xi-x)$$

$$\Rightarrow \frac{d^2\hat{\sigma}}{dx\,dy} = \frac{1}{16\pi}\int\frac{d\phi}{2\pi}\frac{1}{\hat{s}}\delta(\xi-x)\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2$$

Use $\alpha = \frac{e^2}{4\pi}$

$$= \alpha^2 e_q^2\frac{\hat{s}}{Q^4}\int d\phi\,\delta(\xi-x)\left(1+(1-y)^2\right)$$

Use $\int d\phi = 2\pi$

$$= 4\pi\alpha^2\frac{1}{yQ^2}\left(1+(1-y)^2\right)\frac{1}{2}e_q^2\,\delta(\xi-x)$$

$Q^2 = \hat{s}y$

Exercise 5.2

Plus prescription (divergence at $x=1$):

$$\int_0^1 dx\, f(x)\,[g(x)]_+ = \int_0^1 dx\,(f(x)-f(1))\,g(x)$$

(a) $$\int_0^1 dx\left(\frac{1+x^2}{[1-x]_+} + K\,\delta(1-x)\right) \overset{!}{=} 0$$

Choose $f(x) = 1+x^2$

$$\int_0^1 dx\left(\frac{1}{1-x}\,(1+x^2-2) + K\,\delta(1-x)\right) =$$

$$\int_0^1 dx\left(\frac{-(1-x)(1+x)}{1-x} + K\,\delta(1-x)\right) =$$

$$-\int_0^1 dx\,(1+x) + K = -\frac{3}{2} + K \overset{!}{=} 0 \quad\rightarrow\quad K = \frac{3}{2}$$

(b) The parton distribution function giving the probability to find a quark in a quark can be written as

$$f_{q/q}(x,\mu_f) = \delta(1-x) + \frac{\alpha_s}{2\pi}\,P_{q\to qg}(x)\cdot\log\frac{Q^2}{\mu_f^2} + \ldots$$

so we must have $\int_0^1 dx\, f_{q/q}(x,\mu_f) = 1$ and therefore $\int_0^1 dx\, P_{q\to qg}(x) = 0$