Exercise 4.1.

We have $\beta(\alpha_s) = -b_0\alpha_s^2 + \mathcal{O}(\alpha_s^3, \varepsilon)$

$$Z_g = 1 - \frac{1}{\varepsilon}\frac{b_0}{2}\alpha_s + \mathcal{O}(\alpha_s^2) \qquad (*)$$

$\Rightarrow$ b_0 can be obtained from the pole part of Z_g

$\Rightarrow$ we need to calculate the gluon self-energy

$\Pi^{ab}_{\mu\nu}(q) =$ 

$= -i\delta^{ab}(q_\mu q_\nu - q^2 g_{\mu\nu})\,\Pi(q^2)$

(this form is imposed by current conservation)

 $\underset{(1\text{-loop})}{=}$ 

note that massless tadpoles  are zero in dim. reg

$$\Pi^F_{\mu\nu} = \mathrm{Trace}(t^a t^b)\underset{\substack{\uparrow \\ \text{fermion loop}}}{(-1)}(-ig)^2 i^2 \int\frac{d^Dk}{(2\pi)^D}\frac{\mathrm{Tr}[\gamma_\mu(\not{k}+\not{q}-m)\gamma_\nu(\not{k}-m)]}{(k^2-m^2+i\delta)((k+q)^2-m^2+i\delta)}$$

Use $\mathrm{Trace}[\gamma_\mu \not{r} \gamma_\nu \not{k}] = 4(k_\mu r_\nu + k_\nu r_\mu - g_{\mu\nu}k\cdot r)$

$\mathrm{Trace}[\gamma_\mu\gamma_\nu] = 4g_{\mu\nu}$

$$\Pi^F_{\mu\nu} = -4g^2 T_R\delta^{ab}\int\frac{d^Dk}{(2\pi)^D}\frac{k^\mu(k+q)^\nu + k^\nu(k+q)^\mu - g^{\mu\nu}(k^2+k\cdot q-m^2)}{(k^2-m^2+i\delta)((k+q)^2-m^2+i\delta)}$$

$k^2+k\cdot q-m^2 = (k+q)^2 - m^2 - k\cdot q - q^2$

Exercise 4.1.

$$\Pi_{\mu\nu}^{F} = -4g^2 T_R \delta^{ab} \frac{i\pi^{\frac{D}{2}}}{(2\pi)^D} \Big\{ 2 I_2^{\mu\nu}(q^2, m, m) + 2 q^\nu I^\mu(q^2, m, m)$$

$$- g^{\mu\nu} \left(I_1(m^2) - q_\rho I_2^\rho(q^2, m, m) - q^2 I_2(q^2, m, m) \right) \Big\}$$

We only need the pole coefficients of these integrals.

$$I_1(m^2) = \frac{1}{\varepsilon} m^2 + \mathcal{O}(\varepsilon^0)$$

$$I_2(q^2, m, m) = \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon^0)$$

$$I_2^\mu(q^2, m, m) = -\frac{1}{2} q^\mu I_2(q^2, m, m) = -\frac{1}{2} q^\mu \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon^0)$$

$$I_2^{\mu\nu}(q^2, m, m) = g^{\mu\nu} \frac{1}{\varepsilon} \left(-\frac{1}{12} q^2 + \frac{1}{2} m^2 \right) + q^\mu q^\nu \cdot \frac{1}{\varepsilon} \cdot \frac{1}{3}$$

$$\Rightarrow \{\cdot\} = \frac{1}{\varepsilon} \Big(g^{\mu\nu} \left(-\frac{1}{6} q^2 + m^2 \right) + \frac{2}{3} q^\mu q^\nu - q^\mu q^\nu$$

$$- g^{\mu\nu} \left(m^2 + \frac{1}{2} q^2 - q^2 \right) \Big)$$

$$= \frac{1}{\varepsilon} \left(g^{\mu\nu} \left(+\frac{1}{3} q^2 \right) - \frac{1}{3} q^\mu q^\nu \right) = \frac{1}{\varepsilon} \frac{1}{3} \left(g^{\mu\nu} q^2 - q^\mu q^\nu \right)$$

$$\Rightarrow \Pi_{\mu\nu}^{F} = -\frac{4}{3} i \frac{\alpha_s}{4\pi} T_R \delta^{ab} \left(g^{\mu\nu} q^2 - q^\mu q^\nu \right) \cdot \frac{1}{\varepsilon} + \text{finite}$$

$$\frac{g^2 \pi^2}{16 \pi^4} = \frac{g^2}{4\pi} \frac{1}{4\pi} = \frac{\alpha_s}{4\pi}$$

Exercise 4.1.

$$\Pi_{\mu\nu}^{ab,F}(q) = -i\,\delta^{ab}\left(q_\mu q_\nu - q^2 g_{\mu\nu}\right)\Pi^F(q^2)$$

$$\Pi^F(q^2) = +\frac{4}{3}\cdot\frac{\alpha_s}{4\pi}\,T_R\cdot\frac{1}{\varepsilon} + \text{finite}$$

$$\Pi_{ren}^F(q^2) = \Pi^F(q^2) + Z_3^{(F)} - 1$$

$$\Rightarrow\quad Z_3^{(F)} = 1 + \underbrace{\frac{\alpha_s}{4\pi}\left(-\frac{4}{3}\right)T_R\cdot\frac{1}{\varepsilon}}_{\delta Z_3^{(F)}}$$

for one fermion flavour

(for N_f flavours running in the fermion loop multiply by N_f)

Now we can get $Z_g^{(F)}$ from the Slavnov-Taylor identities (eq. (3.43) in the script)

$$Z_g = \frac{Z_1}{Z_3^{\frac{3}{2}}} \quad\text{and}\quad \frac{Z_1}{Z_3} = 1 + \text{terms} \sim C_A \quad \text{(Ward Id.)}$$

which do not contribute to the fermionic part

$$\Rightarrow\quad Z_g^{(F)} = \left(Z_3^{(F)}\right)^{-\frac{1}{2}} \approx 1 - \frac{1}{2}\,\delta Z_3^{(F)} = 1 + \frac{\alpha_s}{4\pi}\,\frac{2}{3}\,T_R\,\frac{1}{\varepsilon}$$

multiply by N_f for N_f flavours running in the loop and use (*) p.-1-

$$\Rightarrow\quad b_0^{(F)} = \frac{1}{4\pi}\left(-\frac{4}{3}\right)T_R\,N_f \qquad \text{(see script eq. (1.28))}$$