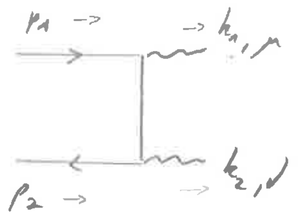
3.1. QED Ward Identity

$$\mathcal{M}_{\mu\nu} =$$



$$+ \left(k_{1,\mu} \leftrightarrow k_{2,\nu}\right) =: -i\, e^2 \left(\mathcal{M}^{(1)}_{\mu\nu} + \mathcal{M}^{(2)}_{\mu\nu}\right)$$

$$p_1 + p_2 = k_1 + k_2 \qquad\qquad M = \varepsilon^{\mu}(k_1)\, \varepsilon^{\nu}(k_2)\, \mathcal{M}_{\mu\nu}$$

$$\mathcal{M}^{(1)}_{\mu\nu} = \bar{v}(p_2)\, \gamma_\nu\, \frac{(\not{p}_1 - \not{k}_1)}{(p_1 - k_1)^2}\, \gamma_\mu\, u(p_1)$$

$$\mathcal{M}^{(2)}_{\mu\nu} = \bar{v}(p_2)\, \gamma_\mu\, \frac{(\not{p}_1 - \not{k}_2)}{(p_1 - k_2)^2}\, \gamma_\nu\, u(p_1)$$

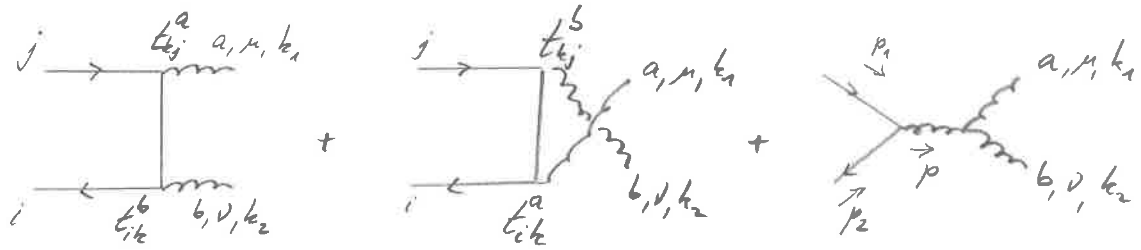
$$k_1^{\mu}\, \mathcal{M}^{(1)}_{\mu\nu} \underset{\substack{\text{use} \\ \not{p}_1 u(p_1) = 0}}{=} \bar{v}(p_2)\, \gamma_\nu\, \frac{(\not{p}_1 - \not{k}_1)(\not{k}_1 - \not{p}_1)}{(p_1 - k_1)^2}\, u(p_1) = -\,\bar{v}(p_2)\, \gamma_\nu\, u(p_1)$$

$$k_1^{\mu}\, \mathcal{M}^{(2)}_{\mu\nu} \underset{\substack{\text{use} \\ p_1 - k_2 = k_1 - p_2}}{=} \bar{v}(p_2)\, \not{k}_1\, \frac{\not{k}_1 - \not{p}_2}{(k_1 - p_2)^2}\, \gamma_\nu\, u(p_1) \underset{\substack{\text{use} \\ \bar{v}(p_2)\not{p}_2 = 0}}{=} \bar{v}(p_2)\, \frac{(\not{k}_1 - \not{p}_2)^2}{(k_1 - p_2)^2}\, \gamma_\nu\, u(p_1)$$

$$\Rightarrow k_1^{\mu}\left(\mathcal{M}^{(1)}_{\mu\nu} + \mathcal{M}^{(2)}_{\mu\nu}\right) = 0$$

The analogous is true for contraction with k_2 because $\mathcal{M}_{\mu\nu}$ is symmetric under exchange of the two photons.

3.2. Gauge invariance of QCD amplitudes



$$p = p_1 + p_2 = k_1 + k_2$$

$$\mathcal{M}_{\mu\nu} = -i g_s^2 \left\{ (t^b t^a)_{ij}\, \mathcal{M}^{(1)}_{\mu\nu} + (t^a t^b)_{ij}\, \mathcal{M}^{(2)}_{\mu\nu} + \mathcal{M}^{(3)}_{\mu\nu} \right\}$$

use $t^b t^a = t^a t^b - i f^{abc} t^c$

$$\mathcal{M}_{\mu\nu} = -i g_s^2 \left\{ (t^a t^b)_{ij} \cdot \underbrace{\left(\mathcal{M}^{(1)}_{\mu\nu} + \mathcal{M}^{(2)}_{\mu\nu}\right)}_{\mathcal{M}^{QED}} - i f^{abc} t^c_{ij}\, \mathcal{M}^{(1)}_{\mu\nu} + \mathcal{M}^{(3)}_{\mu\nu} \right\}$$

$$\mathcal{M}^{(3)}_{\mu\nu} = i f^{abc} t^c_{ij}\, \bar{v}(p_2) \gamma_\rho u(p_1) \frac{1}{p^2} V_{\mu\nu\rho}(k_1, k_2, -p)$$

$$V_{\mu\nu\rho}(k_1, k_2, -(k_1+k_2)) = (k_1 - k_2)_\rho g_{\mu\nu} + (2k_2 + k_1)_\mu g_{\nu\rho} - (2k_1 + k_2)_\nu g_{\mu\rho}$$

$$k_1^\mu V_{\mu\nu\rho}(k_1, k_2, -(k_1+k_2)) = 2 k_1 \cdot k_2\, g_{\nu\rho} - k_{1\nu} \underbrace{(k_{1\rho} + k_{2\rho})}_{=(p_1+p_2)_\rho} - k_{1\rho} k_{2\nu}$$

$\rightarrow$ use $\bar{v}(p_2)(\not{p}_1 + \not{p}_2) u(p_1) = 0$

$$\Rightarrow k_1^\mu \mathcal{M}^{(3)}_{\mu\nu} = i f^{abc} t^c\, \bar{v}(p_2) \gamma_\rho u(p_1) \left\{ g_{\nu\rho} - \frac{k_{1\rho} k_{2\nu}}{2 k_1 \cdot k_2} \right\}$$

used: $p^2 = (k_1 + k_2)^2 = 2 k_1 \cdot k_2$

$$k_1^\mu \mathcal{M}^{(3)}_{\mu\nu} = i f^{abc} t^c \left\{ \underbrace{\bar{v}(p_2)\gamma_\nu u(p_1)}_{(*)} - \bar{v}(p_2)\not{k}_1 u(p_1) \frac{k_{2\nu}}{2k_1\cdot k_2} \right\}$$

we know $k_1^\mu \mathcal{M}^{QED}_{\mu\nu} = 0$

$$k_1^\mu \mathcal{M}^{(1)}_{\mu\nu} = -\bar{v}(p_2)\gamma_\nu u(p_1) \rightarrow \text{cancels with } (*)$$

$$\Rightarrow k_1^\mu \mathcal{M}_{\mu\nu} = -g_s^2 f^{abc} t^c \frac{\bar{v}(p_2)\not{k}_1 u(p_1)}{2k_1\cdot k_2} \cdot k_{2\nu}$$

$$\Rightarrow k_1^\mu \mathcal{M}_{\mu\nu} \cdot \varepsilon^\nu(k_2) = 0 \quad \text{only if} \quad k_{2\nu}\cdot\varepsilon^\nu(k_2) = 0$$

(on-shell, physical gluons)