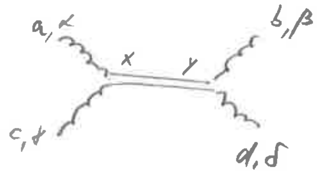
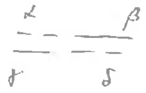
Exercise 2.1. 4-gluon vertex

(1)



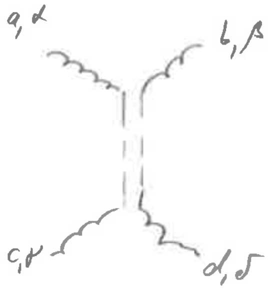
$$= +ig_s^2 f^{xac} f^{xdb} \left(g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\delta} g^{\beta\gamma}\right)$$

$$= -ig_s^2 f^{xac} f^{xbd} \left(g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\delta} g^{\beta\gamma}\right)$$



$$= -\frac{i}{2} \delta^{xy} \left(g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\delta} g^{\beta\gamma}\right)$$

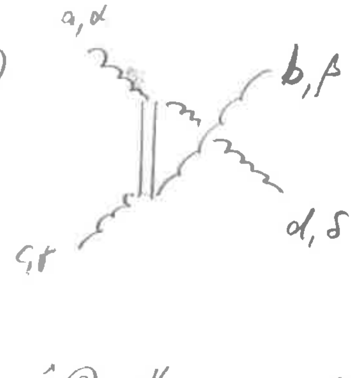
(2)



$$= +ig_s^2 f^{xba} f^{xcd} \left(g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma}\right)$$

$$= -ig_s^2 f^{xab} f^{xdc} \left(g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\gamma} g^{\beta\delta}\right)$$

(3)



$$= +ig_s^2 f^{xda} f^{xcb} \left(g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\beta} g^{\gamma\delta}\right)$$

$$= -ig_s^2 f^{xad} f^{xcb} \left(g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\beta} g^{\gamma\delta}\right)$$

$\hat{=}$ (2) with crossing $(b,\beta) \leftrightarrow (d,\delta)$

$$(1) + (2) + (3) = \Gamma^{abcd}_{\alpha\beta\gamma\delta}$$

2.2. Renormalisation schemes

$$\beta(\alpha_s^A) = -b_0^A(\alpha_s^A)^2\left[1+b_1^A\alpha_s^A + b_2^A(\alpha_s^A)^2 + \mathcal{O}(\alpha_s^A)^3\right] \qquad (1)$$

$$\beta(\alpha_s^B) = -b_0^B\cdot(\alpha_s^B)^2\left[1+b_1^B\alpha_s^B + b_2^B(\alpha_s^B)^2 + \mathcal{O}(\alpha_s^B)^3\right] \qquad (2)$$

Expand $\frac{1}{\alpha_s^B(\mu^2)}$: $\alpha_s^B = \alpha_s^A\left(1+\sum_{n=1}^{\infty} c_n(\alpha_s^A)^n\right)$

$$\Rightarrow \quad \frac{1}{\alpha_s^B} = \frac{1}{\alpha_s^A}\left(1 - c_1\alpha_s^A + c_1^2(\alpha_s^A)^2 - c_2(\alpha_s^A)^2 + \dots\right)$$

$$= \frac{1}{\alpha_s^A} - c_1 + (c_1^2 - c_2)\alpha_s^A + \mathcal{O}((\alpha_s^A)^3) \qquad (3)$$

$$\mu^2\frac{d}{d\mu^2}\left(\frac{1}{\alpha_s^B(\mu^2)}\right) = -\frac{1}{(\alpha_s^B)^2}\mu^2\frac{\partial\alpha_s^B(\mu^2)}{\partial\mu^2} = -\frac{1}{(\alpha_s^B)^2}\beta(\alpha_s^B)$$

$$\overset{(2)}{=} +b_0^B\left[1+b_1^B\alpha_s^B + b_2^B(\alpha_s^B)^2 + \mathcal{O}((\alpha_s^B)^3)\right] \qquad (4)$$

analogous

$$\mu^2\frac{d}{d\mu^2}\left(\frac{1}{\alpha_s^A(\mu^2)}\right) = b_0^A\left[1+b_1^A\alpha_s^A + b_2^A(\alpha_s^A)^2 + \mathcal{O}((\alpha_s^A)^3)\right]$$

act with $\mu^2\frac{d}{d\mu^2}$ on eq. (3)

$$\mu^2\frac{d}{d\mu^2}\left(\frac{1}{\alpha_s^B(\mu^2)}\right) = -\frac{1}{(\alpha_s^A)^2}\cdot\beta(\alpha_s^A) + (c_1^2-c_2)\overbrace{\mu^2\frac{\partial}{\partial\mu^2}\alpha_s^A}^{\beta(\alpha_s^A)}$$

$$= b_0^A\left[1+b_1^A\alpha_s^A + b_2^A(\alpha_s^A)^2 - (c_1^2-c_2)(\alpha_s^A)^2 + \mathcal{O}(\alpha_s^3)\right]$$

$$\mu^2 \frac{d}{d\mu^2}\left(\frac{1}{\alpha_s^B(\mu^2)}\right) = b_0^A\left[1 + b_1^A\left(\alpha_s^B - c_1 \alpha_s^2\right) + b_2^A \alpha_s^2 - (c_1^2 - c_2)\,\alpha_s^2 + \mathcal{O}(\alpha_s^3)\right]$$

$$= b_0^A\left[1 + b_1^A \alpha_s^B + \alpha_s^2\left(b_2^A - c_1 b_1^A + c_2 - c_1^2\right) + \mathcal{O}(\alpha_s^3)\right]$$

Comparing the above expression to eq. (4) leads to

$$b_0^A = b_0^B, \quad b_1^A = b_1^B, \quad b_2^B = b_2^A - c_1 b_1^A + c_2 - c_1^2 .$$