1. Gauge transformations

(a) $$\partial_\mu q'(x) = \partial_\mu \left(U(x)\, q(x) \right) = U \partial_\mu q(x) + (\partial_\mu U)\, q(x)$$
$$= U \left(\partial_\mu q(x) + U^{-1} (\partial_\mu U)\, q(x) \right)$$

introduce the covariant derivative $\mathcal{D}_\mu (A) = \partial_\mu + ig \underbrace{T^a A^a_\mu}_{=: \boldsymbol{A}_\mu}$

where $$\boldsymbol{A}'_\mu = U (\boldsymbol{A}_\mu) U^{-1} + \frac{i}{g} (\partial_\mu U) U^{-1} \qquad (3)$$

then

$$\mathcal{D}_\mu (A')\, q'(x) = U \left(\partial_\mu q(x) + U^{-1} (\partial_\mu U)\, q(x) \right)$$
$$+ ig \left(U (\boldsymbol{A}_\mu) U^{-1} + \frac{i}{g} (\partial_\mu U) U^{-1} \right) U q(x)$$
$$= U \partial_\mu q(x) + ig\, U (\boldsymbol{A}_\mu)\, q(x)$$
$$= U \left(\mathcal{D}_\mu (A)\, q(x) \right)$$

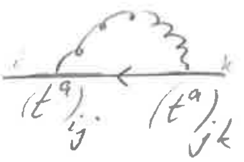
(b) second term in Eq. (3) generates extra terms in the bilinear $\frac{1}{2} m_A A_\mu A^\mu$

2. Colour factors

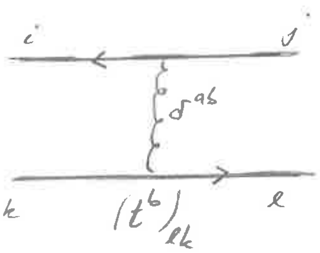
(a)

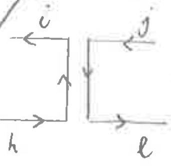
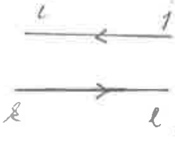
(1)  $= \mathrm{Trace}(t^a) = 0$

(2)  $= \mathrm{Trace}(t^a t^b) = \underset{=\frac{1}{2}}{T_R}\, \delta^{ab} = T_R$

(3)  $= C_F\, \delta_{ik} = C_F \cdot$ 

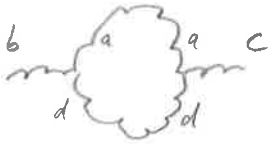
(4)  $f^{acd} f^{bcd} = C_A\, \delta^{ab} = C_A$ 

(5)  $t^a_{ij}\, t^a_{lk} = T_R \left(\delta_{ik}\delta_{lj} - \frac{1}{N_c}\delta_{ij}\delta_{lk} \right)$

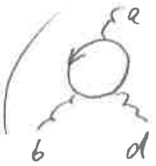
$= T_R \Big($  $- \frac{1}{N_c}$  $\Big)$

(b) $(2) \cdot \delta_{ab} =$  $= T_R \cdot \delta^{ab}\delta_{ab} = T_R \cdot \underbrace{(N_c^2 - 1)}_{=: N_A}$

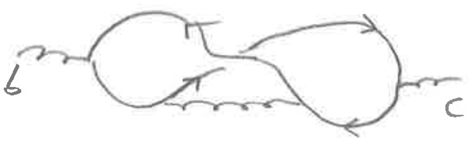
$=$  $= t^a_{ij}\, t^a_{jk}\, \delta_{ik} = C_F\, \delta_{ii} = C_F N_c \Rightarrow C_F = T_R \frac{N_c^2-1}{N_c}$

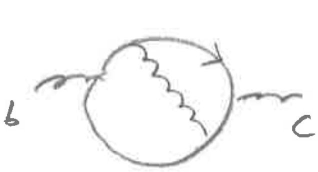
$T_R^2 C_A \delta_{bc} = T_R^2$  $=$

$= -T_R^2 f^{abd} f^{adc} = \mathrm{Trace}\left(t^a[t^b, t^d]\right) \cdot \mathrm{Trace}\left(t^a[t^d, t^c]\right)$

$= \Big($  $-$  $\Big)\Big($  $-$  $\Big)$

$= 2$  $- 2$ 

$= 2 T_R$  $- 2 T_R$ 

$= 2 T_R$  $- 2 T_R$ 

$= 2 T_R^3 \delta_{bc} C_F - 2 T_R^3 \left(-\frac{1}{N}\right) \delta_{bc}$ $\qquad C_F = N - \frac{1}{N}$

$= 2 T_R^3 \delta_{bc} \cdot N$

$\Rightarrow C_A = 2 T_R \cdot N$

Exercise 1.2 (in index notation)

Let T^a be the generators of a Lie Algebra with $[T^a, T^b] = i\, f^{abc}\, T^c$, and $\mathrm{Trace}(T^a T^b) = T_R\, \delta^{ab}$.

(a) The Casimir operator C_F of the fundamental representation of a Lie group is defined by

$$\sum_a (t^a t^a)_{ij} = C_F\, \delta_{ij}\ ,$$

where t^a are the generators in the fundamental representation.
Show that for $SU(N)$, $C_F = T_R\,(N^2-1)/N$.
Hint: Use the relation

$$\sum_a t^a_{ij} t^a_{kl} = T_R \left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl} \right) . \tag{1}$$

Solution:

$$\sum_a (t^a t^a)_{il} \;=\; \sum_a t^a_{ij} t^a_{jl} = T_R \left(\delta_{il}\,\delta_{jj} - \frac{1}{N}\delta_{il} \right) = T_R\,\delta_{il} \left(N - \frac{1}{N} \right) . \tag{2}$$

(b) The Casimir operator C_A of the adjoint representation of a Lie group is defined by

$$\sum_a (F^a F^a)_{bc} = C_A\, \delta_{bc}\ ,$$

where F^a are the generators in the adjoint representation, i.e. $(F^a)_{bc} = -i\, f^{abc}$.
Show that for $SU(N)$, $C_A = 2\, T_R\, N$.
Hint: Use the relations

$$\begin{aligned} T_R\, f^{abc} &= -i\, \mathrm{Trace}(t^a t^b t^c - t^b t^a t^c)\ , \\ \sum_a t^a_{ij} t^a_{kl} &= T_R \left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl} \right) , \\ \mathrm{Trace}(t^a t^b) &= T_R\, \delta^{ab}\ . \end{aligned} \tag{3}$$

Solution:

$$\begin{aligned} C_A\, \delta_{bc} &= \sum_a (F^a F^a)_{bc} = -\sum_a f^{abd} f^{adc} \\ &= 1/T_R^2\, \mathrm{Trace}(t^a [t^b, t^d])\mathrm{Trace}(t^a [t^d, t^c]) = 1/T_R^2\, t^a_{ij} [t^b, t^d]_{ji} t^a_{kl} [t^d, t^c]_{lk} \\ &= 1/T_R \left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl} \right) [t^b, t^d]_{ji}\, [t^d, t^c]_{lk} \\ &= 1/T_R\, [t^b, t^d]_{ji}\, [t^d, t^c]_{ij} \\ &= 1/T_R \left(t^b_{jk} t^d_{ki} t^d_{il} t^c_{lj} - t^b_{jk} t^d_{ki} t^c_{il} t^d_{lj} - t^d_{jk} t^b_{ki} t^d_{il} t^c_{lj} + t^d_{jk} t^b_{ki} t^c_{il} t^d_{lj} \right) \\ &= 1/T_R\, (\mathrm{term}\,1 + \mathrm{term}\,2 + \mathrm{term}\,3 + \mathrm{term}\,4)\ . \end{aligned} \tag{4}$$

Working these terms out using $\mathrm{Trace}(t^a t^b) = T_R\, \delta^{ab}$ and $\sum_a t^a_{ij} t^a_{kl} = T_R \left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl} \right)$ leads to

$$\begin{aligned} \mathrm{term}\,2 = \mathrm{term}\,3 &= T_R^2\, \frac{1}{N}\, \delta_{bc} \\ \mathrm{term}\,1 = \mathrm{term}\,4 &= T_R^2\, (N - \frac{1}{N})\delta_{bc}\ . \end{aligned}$$