

3.1. QED Ward Identity

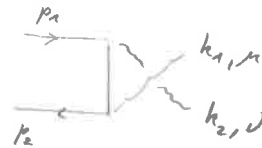
$$\mathcal{M}_{\mu\nu} = \begin{array}{c} p_1 \rightarrow \\ \rightarrow \\ \left[ \text{Feynman diagram: fermion loop with photon lines } k_1, \mu \text{ and } k_2, \nu \right] \\ \leftarrow \\ p_2 \rightarrow \end{array} + (k_1, \mu \leftrightarrow k_2, \nu) =: -i e^2 (\mathcal{M}_{\mu\nu}^{(1)} + \mathcal{M}_{\mu\nu}^{(2)})$$

$$p_1 + p_2 = k_1 + k_2$$

$$M = \epsilon^\mu(k_1) \epsilon^\nu(k_2) \mathcal{M}_{\mu\nu}$$

$$\mathcal{M}_{\mu\nu}^{(1)} = \bar{v}(p_2) \not{\epsilon}_\nu \frac{(\not{p}_1 - \not{k}_1)}{(p_1 - k_1)^2} \not{\epsilon}_\mu u(p_1)$$

$$\mathcal{M}_{\mu\nu}^{(2)} = \bar{v}(p_2) \not{\epsilon}_\mu \frac{(\not{p}_1 - \not{k}_2)}{(p_1 - k_2)^2} \not{\epsilon}_\nu u(p_1)$$



$$\not{k}_1^\mu \mathcal{M}_{\mu\nu}^{(1)} = \bar{v}(p_2) \not{\epsilon}_\nu \frac{(\not{p}_1 - \not{k}_1)(\not{k}_1 - \not{p}_1)}{(p_1 - k_1)^2} u(p_1) = -\bar{v}(p_2) \not{\epsilon}_\nu u(p_1)$$

use  $\not{p}_1 u(p_1) = 0$

$$\not{k}_1^\mu \mathcal{M}_{\mu\nu}^{(2)} = \bar{v}(p_2) \not{k}_1 \frac{\not{k}_1 - \not{p}_2}{(k_1 - p_2)^2} \not{\epsilon}_\nu u(p_1) = \bar{v}(p_2) \frac{(\not{k}_1 - \not{p}_2)^2}{(k_1 - p_2)^2} \not{\epsilon}_\nu u(p_1)$$

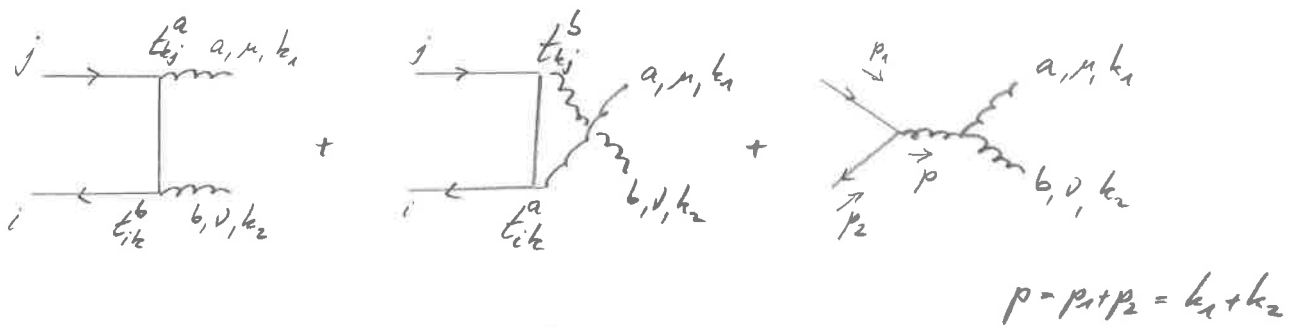
use  $p_1 - k_2 = k_1 - p_2$

use  $\bar{v}(p_2) \not{p}_2 = 0$

$$\Rightarrow \not{k}_1^\mu (\mathcal{M}_{\mu\nu}^{(1)} + \mathcal{M}_{\mu\nu}^{(2)}) = 0$$

The analogous is true for contraction with  $k_2$  because  $\mathcal{M}_{\mu\nu}$  is symmetric under exchange of the two photons.

### 3.2. Gauge invariance of QCD amplitudes



$$\mathcal{M}_{\mu\nu} = -ig_s^2 \left\{ (t^b t^a)_{ij} \mathcal{M}_{\mu\nu}^{(1)} + (t^a t^b)_{ij} \mathcal{M}_{\mu\nu}^{(2)} + \mathcal{M}_{\mu\nu}^{(3)} \right\}$$

use  $t^b t^a = t^a t^b - if^{abc} t^c$

$$\mathcal{M}_{\mu\nu} = -ig_s^2 \left\{ (t^a t^b)_{ij} \underbrace{(\mathcal{M}_{\mu\nu}^{(1)} + \mathcal{M}_{\mu\nu}^{(2)})}_{\mathcal{M}_{\mu\nu}^{QED}} - if^{abc} t_{ij}^c \mathcal{M}_{\mu\nu}^{(1)} + \mathcal{M}_{\mu\nu}^{(3)} \right\}$$

$$\mathcal{M}_{\mu\nu}^{(3)} = if^{abc} t_{ij}^c \bar{v}(p_2) \not{\epsilon}_\nu u(p_1) \frac{1}{p^2} V_{\mu\nu S}(k_1, k_2, -p)$$

$$V_{\mu\nu S}(k_1, k_2, -(k_1+k_2)) = (k_1 - k_2)_S g_{\mu\nu} + (2k_2 + k_1)_\mu g_{\nu S} - (2k_1 + k_2)_\nu g_{\mu S}$$

$$k_1^\mu V_{\mu\nu S}(k_1, k_2, -(k_1+k_2)) = 2k_1 \cdot k_2 g_{\nu S} - k_{1\nu} (k_{1S} + k_{2S}) - k_{1S} k_{2\nu}$$

$$\Rightarrow k_1^\mu \mathcal{M}_{\mu\nu}^{(3)} = if^{abc} t^c \bar{v}(p_2) \not{\epsilon}_\nu u(p_1) \left\{ g_{\nu S} - \frac{k_{1S} k_{2\nu}}{2k_1 \cdot k_2} \right\}$$

used:  $p^2 = (k_1 + k_2)^2 = 2k_1 \cdot k_2$

$$k_1^\mu \mathcal{M}_{\mu\nu}^{(3)} = i f^{abc} t^c \left\{ \underbrace{\bar{v}(p_2) \not{\epsilon}_\nu u(p_1)}_{(*)} - \bar{v}(p_2) \not{k}_1 u(p_1) \frac{k_{2\nu}}{2k_1 \cdot k_2} \right\}$$

we know  $k_1^\mu \mathcal{M}_{\mu\nu}^{\text{QED}} = 0$

$$k_1^\mu \mathcal{M}_{\mu\nu}^{(1)} = -\bar{v}(p_2) \not{\epsilon}_\nu u(p_1) \rightarrow \text{cancels with } (*)$$

$$\Rightarrow k_1^\mu \mathcal{M}_{\mu\nu} = -g_s^2 f^{abc} t^c \bar{v}(p_2) \frac{k_{1\nu}}{2k_1 \cdot k_2} u(p_1) \cdot k_{2\nu}$$

$$\Rightarrow k_1^\mu \mathcal{M}_{\mu\nu} \cdot \epsilon^\nu(k_2) = 0 \text{ only if } k_{2\nu} \cdot \epsilon^\nu(k_2) = 0$$

(on-shell, physical gluons)