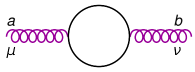
Introduction to QCD and Loop Calculations

Exercise 4.1 One-loop beta-function

We have

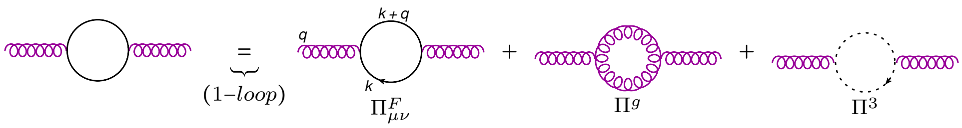
$$\beta(\alpha_s) = -b_0 \alpha^2{}_s + \mathcal{O}(\alpha_s^3, \varepsilon)$$
$$Z_g = 1 - \frac{1}{\varepsilon}\frac{b_0}{2}\alpha_s + \mathcal{O}(\alpha_s^2) \qquad (*)$$

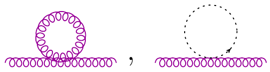
⇒ b_0 can be obtained from the pole part of Z_g
⇒ we need to calculate the gluon self energy

$$\Pi^{ab}_{\mu\nu}(q)$$ 

$$= -i\delta^{ab}\left(q_\mu q_\nu - q^2 g_{\mu\nu}\right)\Pi(q^2)$$

This form is impossible by current conservation



Note that massless *KANN ICH NICHT LESEN!*  are zero in dim-reg.

$$\Pi^F_{\mu\nu} = Trace(t^a t^b) \cdot \underbrace{(-1)}_{fermionloop} \cdot (-ig)^2 i^2 \int \frac{d^D k}{(2\pi)^D} \frac{Tr[\gamma_\mu(\not{k} + q - m)\gamma_\nu(\not{k} - m)]}{(k^2 - m^2 + i\delta)((k+p)^2 - m^2 + i\delta)}$$

Use:

$$Trace[\gamma_\mu \neq \gamma_\nu \not{k}] = 4\left(k_\mu r_\nu + k_\nu r_\mu - g_{\mu\nu} k \cdot r\right)$$
$$Trace[\gamma_\mu \gamma_\nu] = 4 g_{\mu\nu}$$
$$k^2 + k \cdot q - m^2 = (k+q)^2 - m^2 - k \cdot q - q^2$$

$$\Pi^F_{\mu\nu} = -4g^2 T_R \delta^{\alpha\beta} \int \frac{d^D k}{(2\pi)^D} \frac{k^\mu (k+q)^\nu + k^\nu (k+q)^\mu - g^{\mu\nu}(k^2 + k \cdot q - m^2)}{(k^2 - m^2 + i\delta)((k+p)^2 - m^2 + i\delta)}$$

$$\Pi^F_{\mu\nu} = -4g^2 T_R \delta^{\alpha\beta} \frac{i\pi^{D/2}}{(2\pi)^D}\Big\{2I_2^{\mu\nu}(q^2, m, m) + 2q^\nu I^\mu(q^2, m, m) - g^{\mu\nu}\left(I_1(m^2) - q_s I_2^s(q^2, m, m) - q^2 I_2(q^2, m, m)\right)\Big\}$$

We onlay need the pole coefficients of these integrals.

$$I_1(m^2) = \frac{1}{\varepsilon} m^2 + \mathcal{O}(\varepsilon^0)$$
$$I_2(q^2, m, m) = \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon^0)$$
$$I_2^\mu(q^2, m, m) = -\frac{1}{2} q^\mu I_2(m^2, m, m) = -\frac{1}{2}\, q^\mu\, \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon^0)$$
$$I_2^{\mu\nu}(q^2, m, m) = g^{\mu\nu}\frac{1}{\varepsilon}\left(-\frac{1}{12}q^2 + \frac{1}{2}m^2\right) + q^\mu q^\nu \cdot \frac{1}{\varepsilon} \cdot \frac{1}{3} + \mathcal{O}(\varepsilon^0)$$

$$\Rightarrow \{...\} = \frac{1}{\varepsilon}\left(g^{\mu\nu}\left(-\frac{1}{6}q^2 + m^2\right) + \frac{2}{3}q^\mu q^\nu - q^\mu q_\nu - g^{\mu\nu}\left(m^2 + \frac{1}{2}q^2 - q^2\right)\right)$$
$$= \frac{1}{\varepsilon}\left(g^{\mu\nu}(\frac{1}{3}q^2) - \frac{1}{3}q^\mu q^\nu\right)$$
$$= \frac{1}{\varepsilon}\frac{1}{3}\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)$$
$$\Rightarrow \Pi^F_{\mu\nu} = -\frac{4}{3}i\frac{\alpha_s}{4\pi}T_R\delta^{ab}\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\cdot\frac{1}{\varepsilon} + finite \qquad \text{with } \frac{g^2}{4\pi} = \alpha_s$$

$$\Pi^{ab,F}_{\mu\nu}(q) = -i\delta^{ab}\left(q_\mu q^\nu - q^2 g_{\mu\nu}\right)\Pi^F(q^2)$$
$$\Pi^F(q^2) = \frac{4}{3}\cdot\frac{\alpha_s}{4\pi}T_R\cdot\frac{1}{\varepsilon} + finite$$
$$\Pi^F_{ren}(q^2) = \Pi^F(q^2) + Z_3^{(F)} - 1$$
$$\Rightarrow Z_3^{(F)} = 1 + \underbrace{\frac{\alpha_s}{4\pi}T_R\cdot\frac{1}{\varepsilon}\left(-\frac{4}{3}\right)}_{\delta Z_3^{(F)}}$$

For one fermion flavour. For N_f flavours running in the fermion loop multiply by N_f
Now we can get $Z_3^{(F)}$ from the Slavnov-Taylor identities (eq. (3.43) in the script)

$$Z_g = \frac{Z_1}{Z_3{}^{3/2}} \text{ and } \frac{Z_1}{Z_3} = 1 + \text{terms} \sim C_A(\text{Ward identity}), \text{ wich do not contribute to the fermionic part}$$
$$\Rightarrow Z_g^{(F)} = \left(Z_3^{(F)}\right)^{-1/2} \approx 1 - \frac{1}{2}\delta Z_3^{(F)} = 1 + \frac{\alpha_s}{4\pi}\frac{2}{3}T_R\frac{1}{\varepsilon}$$

multiply by N_f for N_f flavours running in the loop and we use $(*)$

$$\Rightarrow b_0^{(F)} = \frac{1}{4\pi}\left(-\frac{4}{3}\right)T_R N_f \qquad \text{(see script eq. (1.28))}$$

Exercise 4.2: D-dimensional phase space integrals

Show that the D-dimensional three-particle phase space for $Q \to p_1 + p_2 + p_3$, with $p_i^2 = 0$, can be expressed in terms of the kinematic invariants $s_{ij} = (p_i + p_j)^2$ as

$$\mathrm{d}\Phi_3 = \frac{(2\pi)^{3-2D}}{2^{D+1}} (Q^2)^{\frac{2-D}{2}} \mathrm{d}\Omega_{D-2}\mathrm{d}\Omega_{D-3}\,\mathrm{d}s_{12}\,\mathrm{d}s_{13}\,\mathrm{d}s_{23}\,(s_{12}s_{13}s_{23})^{\frac{D-4}{2}}\,\delta\left(Q^2 - s_{12} - s_{13} - s_{23}\right)\,.$$

Solution:
We start from eq.(3.76) in the lecture notes:

$$\mathrm{d}\Phi_3 = \frac{1}{4}(2\pi)^{3-2\,D}\,\mathrm{d}E_1\mathrm{d}E_2\mathrm{d}\theta_1[E_1E_2\sin\theta]^{D-3}\mathrm{d}\Omega_{D-2}\,\mathrm{d}\Omega_{D-3}\,\delta((Q-p_1-p_2)^2)\,, \qquad (1)$$

based on the following parametrisation of the momenta:

$$\begin{aligned} Q &= (E, \vec{0}^{(D-1)}) \\ p_1 &= E_1\,(1, \vec{0}^{(D-2)}, 1) \\ p_2 &= E_2\,(1, \vec{0}^{(D-3)}, \sin\theta, \cos\theta) \\ p_3 &= Q - p_2 - p_1\,. \end{aligned} \qquad (2)$$

The kinematic invariants $s_{ij} = (p_i + p_j)^2$ are given by

$$\begin{aligned} s_{12} &= 2E_1E_2\,(1-\cos\theta) \\ s_{13} &= 2E_1(E - E_2\,(1-\cos\theta)) = E\,(E - 2E_2) \\ s_{23} &= 2E_2(E - E_1\,(1-\cos\theta)) = E\,(E - 2E_1)\,. \end{aligned} \qquad (3)$$

The Jacobian for the transformation is

$$|\det J| = \left|\det\left(\frac{\partial(s_{12}, s_{13}, s_{23})}{\partial(E_1, E_2, \theta)}\right)\right| = 8E^2E_1E_2\sin\theta\,. \qquad (4)$$

We can save a lot of algebra if we relate $|\det J|$ to the determinant of the Gram matrix G_{ij}, which we can define either using the vectors Q, p_1, p_2, or the vectors p_1, p_2, p_3. The determinant will be the same in both cases. In the first case, the Gram matrix is given by

$$G^{(1)} = 2\begin{pmatrix} Q^2 & Qp_1 & Qp_2 \\ Qp_1 & 0 & p_1p_2 \\ Qp_2 & p_1p_2 & 0 \end{pmatrix} \qquad (5)$$

On the other hand, using p_1, p_2, p_3 to form frame-independent variables, we have

$$G^{(2)} = \begin{pmatrix} 0 & s_{12} & s_{13} \\ s_{12} & 0 & s_{23} \\ s_{13} & s_{23} & 0 \end{pmatrix} \qquad (6)$$

such that

$$\det G^{(1)} = \det G^{(2)} = \det G = 2s_{12}s_{13}s_{23} = 8E^2E_1^2E_2^2\sin^2\theta \qquad (7)$$

and therefore

$$\det J = 2E\sqrt{2\det G} = 4E\,\sqrt{s_{12}s_{13}s_{23}}\,. \qquad (8)$$

This leads to

$$\begin{aligned} \int \mathrm{d}\theta_1\mathrm{d}E_1\mathrm{d}E_2\,[E_1E_2\sin\theta]^{D-3} &= \int \mathrm{d}\theta_1\mathrm{d}E_1\mathrm{d}E_2\left[\frac{\det J}{8E^2}\right]^{D-3} \\ &= \int \mathrm{d}s_{12}\mathrm{d}s_{13}\mathrm{d}s_{23}\,(8E^2)^{3-D}\,(\det J)^{D-4} \\ &= E^{2-D}2^{1-D}\int \mathrm{d}s_{12}\mathrm{d}s_{13}\mathrm{d}s_{23}\left(\frac{\det G}{2}\right)^{\frac{D-4}{2}}\,. \end{aligned}$$

Combining everything one obtains

$$\int \mathrm{d}\Phi_3 = \frac{(2\pi)^{3-2\,D}}{2^{D-1}}(Q^2)^{1-D/2}\int \mathrm{d}\Omega_{D-2}\;\mathrm{d}\Omega_{D-3}\mathrm{d}s_{12}\mathrm{d}s_{13}\mathrm{d}s_{23}\;(s_{12}s_{13}s_{23})^{\frac{D-4}{2}}\;\delta(Q^2 - s_{12} - s_{13} - s_{23})\;.$$

Defining dimensionless variables by

$$y_1 = s_{12}/Q^2, y_2 = s_{13}/Q^2, y_3 = s_{23}/Q^2$$

we arrive at

$$\int \mathrm{d}\Phi_3 = \frac{(2\pi)^{3-2\,D}}{2^{D-1}}(Q^2)^{D-3}\int \mathrm{d}\Omega_{D-2}\;\mathrm{d}\Omega_{D-3}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3\;(y_1y_2y_3)^{\frac{D-4}{2}}\;\delta(1-\sum_{i=1}^{3}y_i)\;. \qquad (9)$$