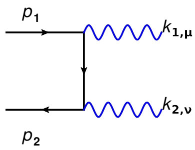
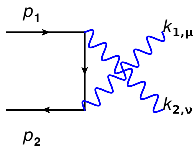
Solutions for sheet 3 *December 20, 2018*

Introduction to QCD and Loop Calculations

Exercise 3.1 QED Ward Identity

$\mathcal{M}_{\mu\nu} =$ 

$+$ 

$=: -ie^2(\mathcal{M}^{(1)}_{\mu\nu} + \mathcal{M}^{(2)}_{\mu\nu})$

$$p_1 + p_2 = k_1 + k_2 \qquad\qquad M = \epsilon^\mu(k_1)\epsilon^\nu(k_2)\mathcal{M}_{\mu\nu}$$

$$\mathcal{M}^{(1)}_{\mu\nu} = \bar{v}(p_2)\gamma_\nu \frac{\not{p}_1 - \not{k}_1}{(p_1-k_1)^2}\gamma_\mu u(p_1)$$

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$$k_1{}^\mu \mathcal{M}^{(1)}_{\mu\nu} \overset{\not{p}_1 u(p_1)=0}{=} \bar{v}(p_2)\gamma_\nu \frac{(\not{p}_1 - \not{k}_1)(\not{k}_1 - \not{p}_1)}{(p_1-k_1)^2} u(p_1) = -\bar{v}(p_2)\gamma_\nu u(p_1)$$

$$k_1{}^\mu \mathcal{M}^{(2)}_{\mu\nu} \overset{p_1-k_2=k_1-p_2}{=} \bar{v}(p_2)\not{k}_1 \frac{\not{k}_1 - \not{p}_2}{(k_1-p_2)^2}\gamma_\nu u(p_1) \overset{\bar{u}(p_2)\not{p}_2=0}{=} \bar{v}(p_2)\frac{(\not{k}_1 - \not{p}_2)^2}{(k_1-p_2)^2}\gamma_\nu u(p_1)$$

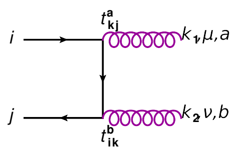
$$\Rightarrow k_1{}^\mu(\mathcal{M}^{(1)}_{\mu\nu} + \mathcal{M}^{(2)}_{\mu\nu}) = 0$$

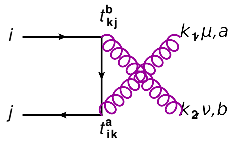
The analogous is true for contraction with k_2 because $\mathcal{M}_{\mu\nu}$is symmetric under exchange of the two photons.

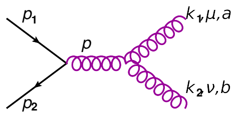
Exercise 3.2 Gauge invariance of QCD amplitudes

$$p = p_1 + p_2 = k_1 + k_2 \qquad k_1{}^\mu \mathcal{M}^{QED}_{\mu\nu} = 0 \qquad \bar{v}(p_2)(\not{p}_1 + \not{p}_2)u(p_1) = 0 \quad \text{(I)}$$

$$t^b t^a = t^a t^b - i f^{abc} t^c \qquad p^2 = (k_1+k_2)^2 = 2k_1k_2 \qquad k_1{}^\mu \mathcal{M}^{(1)}_{\mu\nu} = -\bar{v}(p_2)\gamma_\nu u(p_1) \quad \text{(II)}$$

$\mathcal{M}_{\mu\nu} =$ 

$+$ 

$+$ 

$$\mathcal{M}_{\mu\nu} = -ig_s{}^2\Big((t^b t^a)_{ij}\mathcal{M}^{(1)}_{\mu\nu} + (t^a t^b)_{ij}\mathcal{M}^{(2)}_{\mu\nu} + \mathcal{M}^{(3)}_{\mu\nu}\Big)$$

$$\mathcal{M}_{\mu\nu} = -ig_s{}^2\Big(\underbrace{(t^a t^b)_{ij}(\mathcal{M}^{(1)}_{\mu\nu} + \mathcal{M}^{(2)}_{\mu\nu})}_{\mathcal{M}^{QED}} - if^{abc}t^c\mathcal{M}^{(1)}_{\mu\nu} + \mathcal{M}^{(3)}_{\mu\nu}\Big)$$

$$\mathcal{M}^{(3)}_{\mu\nu} = if^{abc}t^c_{ij}\bar{v}(p_2)\gamma_\rho u(p_1)\frac{1}{p^2}V_{\mu\nu\rho}(k_1,k_2,-p)$$

$$V_{\mu\nu\rho}(k_1,k_2,-p) = (k_1-k_2)_\rho g_{\mu\nu} + (2k_2+k1)_\mu g_{\mu\rho}$$

$$k_1{}^\mu\, V_{\mu\nu\rho}(k_1,k_2,-(k_1+k_2)) = 2k_1\cdot k_2 g_{\nu\rho} - k_{1\nu}\underbrace{(k_{1\rho}+k_{2\rho})}_{use\ \text{(I)}} - k_{1\rho}k_{2\rho}$$

$$\Rightarrow k_1{}^\mu \mathcal{M}^{(3)}_{\mu\nu} = if^{abc}t^c\bar{v}(p_2)\gamma_\rho u(p_1)\,(g_{\nu\rho} - k_{1\rho}k_{2\nu}) = if^{abc}t^c\Big(\underbrace{\bar{v}(p_2)\gamma_\nu u(p_1)}_{use\ \text{(II)}}\Big)$$

$$\Rightarrow k_1{}^\mu \mathcal{M}_{\mu\nu} = sg_s{}^2 f^{abc}t^c\bar{v}(p_2)\frac{\not{k}_1}{2k_1k_2}u(p_1)\cdot k_{2\nu}$$

$$\Rightarrow k_1{}^\mu \mathcal{M}_{\mu\nu}\cdot\epsilon^\nu(k_2) = 0 \text{ only if } k_{2\nu}\cdot\epsilon^\nu(k_2) = 0 \qquad \text{(only physical gluon polarisations)}$$

Exercise 3.3: Tensor Integrals

In the representation of tensor integrals, for tensor ranks $r \geq 2$, higher dimensional integrals I_N^{D+2m} arise as coefficients of metric tensors $(g^{\mu\nu})^{\otimes m}$. To see how these integrals arise, start from the representation in eq. (3.26) of the script, in terms of Feynman parameters and quadratic forms in the loop momentum, but this time with loop momenta in the numerator:

$$L_N^{\mu_1\mu_2} = \Gamma(N) \int_0^\infty \prod_{i=1}^N dz_i\, \delta(1 - \sum_{l=1}^N z_l) \int_{-\infty}^\infty \frac{d^D l}{i\pi^{\frac{D}{2}}}\, l^{\mu_1} l^{\mu_2} \left[l^2 - R^2 + i\delta\right]^{-N} .$$

Then make the ansatz

$$L_N^{\mu_1\mu_2} = K\, g^{\mu_1\mu_2}$$

and determine K in terms of I_N^{D+2}, using the functional form given in eq. (3.18) of the script.

Solution:
As there is no dimensionful object in the integral which could carry the Lorentz structure, it must be proportional to the metric tensor:

$$\begin{aligned} L_N^{\mu_1\mu_2} &= \Gamma(N) \int_0^\infty \prod_{i=1}^N dz_i\, \delta(1 - \sum_{l=1}^N z_l) \int_{-\infty}^\infty d\bar{l}\, l^{\mu_1} l^{\mu_2} \left[l^2 - R^2\right]^{-N} \\ &= K\, g^{\mu_1\mu_2} \,, \end{aligned} \tag{1}$$

with the definition $d\bar{l} = d^D l/(i\pi^{\frac{D}{2}})$ and $i\delta$ dropped for ease of notation. Contracting both sides of eq. (1) with $g_{\mu_1\mu_2}$, we obtain

$$\begin{aligned} g_{\mu_1\mu_2} L_N^{\mu_1\mu_2} &= \Gamma(N) \int_0^\infty \prod_{i=1}^N dz_i\, \delta(1 - \sum_{l=1}^N z_l) \int_{-\infty}^\infty d\bar{l}\, l^2 \left[l^2 - R^2\right]^{-N} = K\, D \qquad (2) \\ &= \Gamma(N) \int_0^\infty \prod_{i=1}^N dz_i\, \delta(1 - \sum_{l=1}^N z_l) \int_{-\infty}^\infty d\bar{l} \left\{ \left[l^2 - R^2\right]^{-N+1} + R^2 \left[l^2 - R^2\right]^{-N} \right\} . \end{aligned}$$

Now remember the formula for the scalar case:

$$\begin{aligned} I_N^D &= \Gamma(N) \int_0^\infty \prod_{i=1}^N dz_i\, \delta(1 - \sum_{l=1}^N z_l) \int_{-\infty}^\infty d\bar{l} \left[l^2 - R^2\right]^{-N} \\ &= (-1)^N \Gamma(N - \frac{D}{2}) \int_0^\infty \prod_{i=1}^N dz_i\, \delta(1 - \sum_{l=1}^N z_l) \left[R^2\right]^{D/2-N} . \end{aligned} \tag{3}$$

We see that it can be applied as well to the first term in eq. (2) with $N \to N-1$. We obtain:

$$\begin{aligned} g_{\mu_1\mu_2} L_N^{\mu_1\mu_2} &= (-1)^{N-1} \frac{\Gamma(N)}{\Gamma(N-1)} \Gamma(N-1-D/2) \int_0^\infty \prod_{i=1}^N dz_i\, \delta(1 - \sum_{l=1}^N z_l) \left[R^2\right]^{D/2-N+1} \\ &+ (-1)^N \Gamma(N - \frac{D}{2}) \int_0^\infty \prod_{i=1}^N dz_i\, \delta(1 - \sum_{l=1}^N z_l) \left[R^2\right]^{D/2-N+1} \\ &= (-1)^N \Gamma(N - \frac{D+2}{2}) \int \prod_{i=1}^N dz_i\, \delta(1 - \sum_{l=1}^N z_l) \left[R^2\right]^{(D+2)/2-N} \{-(N-1) + N - 1 - D/2\} \\ &= -\frac{D}{2}\, I_N^{D+2} . \end{aligned} \tag{4}$$

Hence we find $K = -\frac{1}{2}\, I_N^{D+2}$.