**Introduction to QCD and Loop Calculations**

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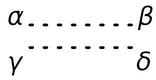
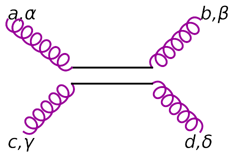
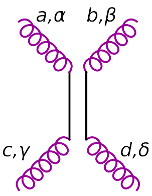
**Exercise 2.1** <u>4-gluon-vertex</u>



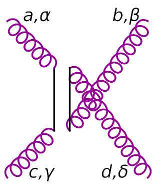

$= -\frac{i}{2}\delta^{xy}\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\beta\gamma}\right)$




$$\begin{aligned} &= +i{g_s}^2 f^{xac} f^{xdb}\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\beta\gamma}\right) \\ &= -i{g_s}^2 f^{xac} f^{xbd}\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\beta\gamma}\right) \end{aligned} \tag{i}$$




$$\begin{aligned} &= +i{g_s}^2 f^{xba} f^{xcd}\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}\right) \\ &= -i{g_s}^2 f^{xab} f^{xdc}\left(g^{\alpha\delta}g^{\beta\gamma} - g^{\alpha\gamma}g^{\beta\delta}\right) \end{aligned} \tag{ii}$$




$$\begin{aligned} &= +i{g_s}^2 f^{xda} f^{xcb}\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\beta}g^{\gamma\delta}\right) \\ &= -i{g_s}^2 f^{xad} f^{xcb}\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\beta}g^{\gamma\delta}\right) \end{aligned} \tag{iii}$$

$$\hat{=}\ (ii)\ withcrossing(b,\beta) \leftrightarrow (d\delta) \qquad (i) + (ii) + (iii) = \Gamma^{abcd}_{\alpha\beta\gamma\delta}$$

**Exercise 2.2** <u>Renormalisation schemes:</u>

$$\beta(\alpha_s^A) = -b_0^A(\alpha_s^A)^2\ \left[1 + b_1^A\alpha_s^A + b_2^A(\alpha_s^A)^2 + \mathcal{O}({\alpha_s}^3)\right] \tag{1}$$

$$\beta(\alpha_s^B) = -b_0^B(\alpha_s^A)^2\ \left[1 + b_1^B\alpha_s^B + b_2^B(\alpha_s^B)^2 + \mathcal{O}({\alpha_s}^3)\right] \tag{2}$$

Expand $\frac{1}{{\alpha_s}^B}$:

$$\begin{aligned} \alpha_s^B =& {\alpha_s}^A\left(1 + \sum_{n=1}^{\infty} c_n(\alpha_s^A)^n\right) \\ \Rightarrow \frac{1}{\alpha_s^B} =& \frac{1}{{\alpha_s}^A}\left(1 - c_1\alpha_s^A + {c_1}^2(\alpha_s^A)^2 - c_2(\alpha_s^A)^2 + \ldots\right) \\ =& \frac{1}{\alpha_s^A} - c_1 + ({c_1}^2 - c_2)\alpha_s^A + \mathcal{O}\left(\alpha_s^A\right)^3 \end{aligned} \tag{3}$$

$$\begin{aligned} \mu^2\frac{d}{d\mu^2}\frac{1}{\alpha_s^B(\mu^2)} &= -\frac{1}{(\alpha_s^B)^2}\,\mu^2\frac{\partial(\alpha_s^B)}{\partial\mu^2} = -\frac{1}{(\alpha_s^B)^2}\,\beta(\alpha_s^B) \\ &\overset{(2)}{=} +b_0^B\left[1 + b_1^B\alpha_s^B + b_2^B(\alpha_s^B)^2 + \mathcal{O}(\alpha_s^B)^3\right] \end{aligned} \tag{4}$$

analogous:

$$\mu^2 \frac{d}{d\mu^2}\left(\frac{1}{\alpha_s^A(\mu^2)}\right) = b_0^A\left[1 + b_1^A\alpha_s^A + b_2^A(\alpha_s^A)^2 + \mathcal{O}(\alpha_s^A)^3\right]$$

act with $\mu^2\frac{d}{d\mu^2}$ on $eq.(3)$

$$\begin{aligned}
\mu^2 \frac{d}{d\mu^2}\frac{1}{\alpha_s^B(\mu^2)} &= -\frac{1}{\alpha_s^{A2}}\cdot\beta(\alpha_s^A) + (c_1^2 - c_2)\mu^2\frac{\partial}{\partial\mu^2}(\alpha_s^B) \\
&= b_0^A\left[1 + b_1^A\alpha_s^A + b_2^A(\alpha_s^A)^2 - (c_1^2 - c_2)(\alpha_s^A)^2 + \mathcal{O}(\alpha_s^A)^3\right] \\
&= b_0^A\left[1 + b_1^A(\alpha_s^B - c_1\alpha_s^2) + b_2^A\alpha_s^2 - (c_1^2 - c_2)\alpha_s^2 + \mathcal{O}(\alpha_s^3)\right] \\
&= b_0^A\left[1 + b_1^A\alpha_s^B + \alpha_s^2(b_2^A - c_1 b_1^A + c_2 - c_1^2) + \mathcal{O}(\alpha_s^3)\right]
\end{aligned}$$

Comparing the above expression to $eq.(4)$ leads to:

$$b_0^A = b_0^B, b_1^A = b_1^B, b_2^B = b_2^A - c_1 b_1^A + c_2 - c_1^2 \tag{5}$$