Introduction to QCD and Loop Calculations

Exercise 1.1 The Lagrangian for the quark field is:

$$\mathcal{L}_q(q_f, m_f) = \sum_{j,k=1}^{N_c} \bar{q}^j{}_f(x)(i\gamma_\mu \mathbf{D}^\mu[A] - m_f)_{jk} q^k{}_f(x) \tag{1}$$

where the covariant derivative $\mathbf{D}^\mu$ is given by

$$\mathbf{D}^\mu[A] = \partial^\mu + i g_s A^\mu \tag{2}$$

The quark and gluon fields transform under local gauge transformations $U(x)$ as

$$q_j(x) = U_{jk} q_k(x) \tag{3}$$

$$A_\mu(x) = U(x)A_\mu(x)U^{-1}(x) + \frac{i}{g_s}(\partial_\mu U(x))U^{-1}(x). \tag{4}$$

a)

$$\begin{aligned} & \mathbf{D}_\mu[\mathbf{A}']q'(x) \\ = \; & (\partial_\mu + \mathbf{A}'_\mu i g_s)q'(x) \\ = \; & \left(\partial_\mu + i g_s U \mathbf{A}_\mu U^{-1} - (\partial_\mu U)U^{-1}\right) U q \\ = \; & U\left(\partial_\mu q(x) + U^{-1}(\partial_\mu U)q(x) + i g_s \mathbf{A}_\mu U^{-1} - U^{-1}(\partial_\mu U)q(x)\right) \\ = \; & U\left(\partial_\mu q(x) + i g_s(\mathbf{A}_\mu)q(x)\right) \\ = \; & U(\mathbf{D}_\mu[\mathbf{A}]q(x)) \end{aligned} \tag{5}$$

Therefore we obtain for the lagrangian:

$$\begin{aligned} \mathcal{L}'_q &= \sum_{j,k=1}^{N_c} \bar{q}'^j{}_f(x)(i\gamma_\mu \mathbf{D}^\mu[A'] - m_f)_{jk} q'^k{}_f(x) \\ &= \sum_{j,k=1}^{N_c} \left(\bar{q}^j{}_f(x)\mathbf{U}^{-1} \; (i\gamma_\mu \mathbf{U}\cdot\mathbf{D}^\mu[A])_{jk} q^k{}_f(x) - \bar{q}^j{}_f(x)\mathbf{U}^{-1} \; (m_f)_{jk}\mathbf{U} q^k{}_f(x)\right) \\ &= \mathcal{L}_q \end{aligned} \tag{6}$$

b)

We have to add a term of the form $\frac{1}{2}m_A^2 A_\mu A^\mu$ to the Lagrangian. Then we set again $A_\mu \to A'_\mu$. The additional Terms are:

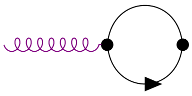
$$\begin{aligned} \frac{1}{2}m_A^2 \; \cdot \; & (U A_\mu A^\mu U^{-1} + \\ & \sim A_\mu(\partial_\mu U)U^{-1} \;\; + \\ & \sim ((\partial_\mu U)U)^2 \\ & \neq U \cdot \frac{1}{2}m_A^2 A_\mu A^\mu \cdot U^{-1} \end{aligned}$$

The 2nd and 3rd term will not cancel and therefore the lagrangian is not invariant anymore.

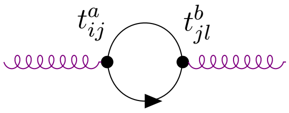
Exercise 1.2 There are two ways of calculating it.

<u>*Graphical way*</u>

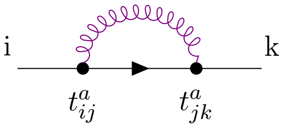
a)



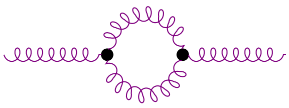
$= \text{Trace}\ (t^a) = 0$ (i)



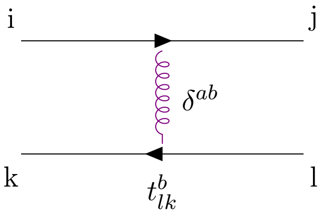
$= \text{Trace}\ (t^a t^b) = T_R \delta^{ab} = T_R \cdot$  (ii)



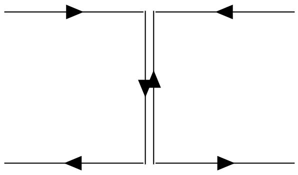
$= \mathrm{C}_F \ \ \delta_{ik} \ = \mathrm{C}_F \ \cdot$  (iii)



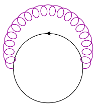
$= f^{acd} f^{bcd} = C_A \delta^{ab} = C_A \cdot$  (iv)

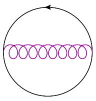


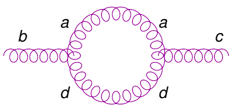
$= t^a_{ij} t^a_{lk} = T_R(\delta_{ik}\delta_{lj} - \frac{1}{N_c}\delta_{ik}\delta_{lj}) = T_R \cdot$ (v)



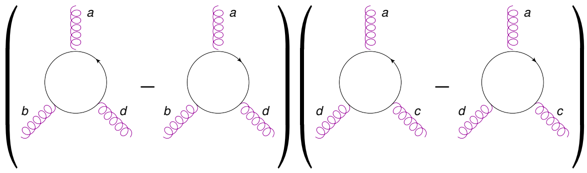
b)

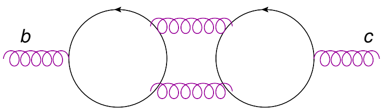
$(2)\cdot\delta_{ab} =$  $= T_R \cdot \delta^{ab}\delta_{ab} = T_R\left({N_c}^2 - 1\right) = T_R \cdot N_A$

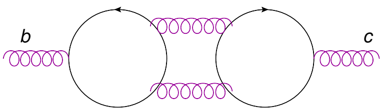
$=$  $= t^a_{ij}\, t^a_{ji}\, \delta_{ik} = C_F \delta_{ii} = C_F N_C \Rightarrow C_F = T_R \dfrac{{N_c}^2 - 1}{N_c}$

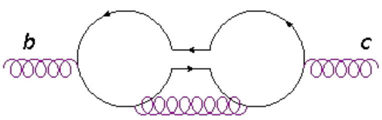
${T_R}^2 C_A \delta_{bc} = {T_R}^2 \cdot$ 

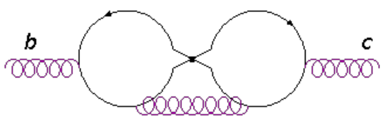
$= -{T_R}^2 f^{abd} f^{adc} = Trace\left(t^a[t^b, t^d]\right) Trace\left(t^a[t^d, t^c]\right)$

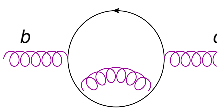
$=$ 

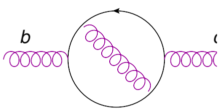
$= 2\cdot$ 

$- 2\cdot$ 

$= 2T_R\cdot$ 

$- 2T_R\cdot$ 

$= 2T_R\cdot$ 

$- 2T_R\cdot$ 

$= 2{T_R}^3 \delta_{bc} C_F - 2{T_R}^3 \delta_{bc}\left(-\frac{1}{N}\right) \overset{C_F = N - 1/N}{=} 2{T_R}^3 \delta_{bc} N$

$\Rightarrow C_A = 2T_R N$

Index notation way

Let T^a be the generators of a Lie Algebra with $[T^a, T^b] = i\, f^{abc}\, T^c$, and $\mathrm{Trace}(T^a T^b) = T_R\, \delta^{ab}$.

(a) The Casimir operator C_F of the fundamental representation of a Lie group is defined by

$$\sum_a (t^a t^a)_{ij} = C_F\, \delta_{ij}\ ,$$

where t^a are the generators in the fundamental representation.
Show that for $SU(N)$, $C_F = T_R\,(N^2-1)/N$.
Hint: Use the relation

$$\sum_a t^a_{ij} t^a_{kl} = T_R \left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl} \right)\ . \tag{7}$$

Solution:

$$\sum_a (t^a t^a)_{il} \;=\; \sum_a t^a_{ij} t^a_{jl} = T_R \left(\delta_{il}\,\delta_{jj} - \frac{1}{N}\delta_{il} \right) = T_R\, \delta_{il} \left(N - \frac{1}{N} \right)\ . \tag{8}$$

(b) The Casimir operator C_A of the adjoint representation of a Lie group is defined by

$$\sum_a (F^a F^a)_{bc} = C_A\, \delta_{bc}\ ,$$

where F^a are the generators in the adjoint representation, i.e. $(F^a)_{bc} = -i\, f^{abc}$.
Show that for $SU(N)$, $C_A = 2\, T_R\, N$.
Hint: Use the relations

$$\begin{aligned} T_R\, f^{abc} &= -i\, \mathrm{Trace}(t^a t^b t^c - t^b t^a t^c)\ , \\ \sum_a t^a_{ij} t^a_{kl} &= T_R \left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl} \right)\ , \\ \mathrm{Trace}(t^a t^b) &= T_R\, \delta^{ab}\ . \end{aligned} \tag{9}$$

Solution:

$$\begin{aligned} C_A\, \delta_{bc} &= \sum_a (F^a F^a)_{bc} = -\sum_a f^{abd} f^{adc} \\ &= 1/T_R^2\, \mathrm{Trace}(t^a[t^b,t^d])\mathrm{Trace}(t^a[t^d,t^c]) = 1/T_R^2\, t^a_{ij}[t^b,t^d]_{ji} t^a_{kl}[t^d,t^c]_{lk} \\ &= 1/T_R \left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl} \right) [t^b,t^d]_{ji}\, [t^d,t^c]_{lk} \\ &= 1/T_R\, [t^b,t^d]_{ji}\, [t^d,t^c]_{ij} \\ &= 1/T_R \left(t^b_{jk} t^d_{ki} t^d_{il} t^c_{lj} - t^b_{jk} t^d_{ki} t^c_{il} t^d_{lj} - t^d_{jk} t^b_{ki} t^d_{il} t^c_{lj} + t^d_{jk} t^b_{ki} t^c_{il} t^d_{lj} \right) \\ &= 1/T_R\, (\mathrm{term}\,1 + \mathrm{term}\,2 + \mathrm{term}\,3 + \mathrm{term}\,4)\ . \end{aligned} \tag{10}$$

Working these terms out using $\mathrm{Trace}(t^a t^b) = T_R\, \delta^{ab}$ and $\sum_a t^a_{ij} t^a_{kl} = T_R \left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl} \right)$ leads to

$$\begin{aligned} \mathrm{term}\,2 = \mathrm{term}\,3 &= T_R^2\, \frac{1}{N}\, \delta_{bc} \\ \mathrm{term}\,1 = \mathrm{term}\,4 &= T_R^2\, (N - \frac{1}{N}) \delta_{bc}\ . \end{aligned}$$