Lectures on Gauge-Higgs Unification in extra dimensions

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Thanks to my collaborators on this subject

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Outline

1. Extra dimensions, KK, gauge theories in XD
2. Orbifolds, SU(5) breaking via orbifolds
3. Interval vs. orbifold approach
4. Generics of gauge-Higgs unification
5. Higgs potential calculation
6. Fermion masses
7. A semi-realistic model
8. Bounds on the model from EWPO’s
1. Extra dimensions, KK decomposition

- If XD’s exist, there has to be a reason why we have not seen them
  - Compactification
  - Localization of fields (will not use very much)
- Simplest example: scalar field on a circle

\[ S = \int d^4 x \int_0^{2\pi R} dy \partial_M \phi^* \partial^M \phi \]

- 5D EOM

\[ (\partial_\mu \partial^\mu - \partial_y^2) \phi = 0 \]
• But XD compact, $\phi$ has to periodic in $y$
• Fourier decomposition (=KK expansion)

$$
\phi(x, y) = \frac{1}{\sqrt{2\pi R}} \sum_{n=-\infty}^{\infty} \phi_n(x) e^{iny/R}
$$

• $\phi_n$: KK modes. One 5D field = an infinite tower of 4D fields $\rightarrow$ KK expansion
• Dim of $\phi_n$: 1 (like 4D). Dim. of $\phi$: 3/2 (like 5D)

$$
\partial_\mu \partial^\mu \phi_n(x) + \frac{n^2}{R^2} \phi_n(x) = 0
$$

• Different KK modes have different 4D masses, here $m_n^2 = n^2/R^2$
• Momentum along 5th dim $\sim$ mass along 4D
• Can get 4D effective action by integrating explicitly over the y coordinate in KK exp. just collection of massive 4D fields

\[ S_{\text{eff}} = \int d^4x \sum_{n=-\infty}^{\infty} \left( \partial_\mu \phi^*_n \partial^\mu \phi_n - m_n^2 |\phi_n|^2 \right) \]

• General KK expansion:
  • Take quadratic part of 5D action
  • Write fields as sum of ordinary 4D fields:

\[ \phi = \sum_n \phi_n(x) f_n(y), \quad (\partial_\mu \partial^\mu + m_n^2) \phi_n = 0 \]

• \( f_n(y) \): wave function of KK mode
• Higher powers will give interactions of KK modes
Gauge theories in an extra dimension

\[ S = \int d^5x \left( -\frac{1}{4} F^a_{MN} F^{MN \ a} \right) = \int d^5x \left( -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu \ a} - \frac{1}{2} F^a_{\mu5} F^{\mu5 \ a} \right) \]

\[ F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + g_5 f^{abc} A^b_M A^c_N \]

• Gauge coupling \( g_5 \) has dim. -1/2 (nonren.)
• Dim. of \( A_M \): 3/2 like a 5D scalar
• From 4D point of view:

\[ A_M \rightarrow A_\mu + A_5 \]

• \( A_5 \) is like a scalar, that can be like an eaten GB (or a higgs). Eaten GB: mixing of \( A_5 \) and \( A_\mu \)
• The mixing term

\[- \int d^4 x \int_0^{2\pi R} dy \frac{1}{2} F^a_{\mu 5} F^{a 5} = \]

\[- \int d^4 x \int_0^{2\pi R} dy \frac{1}{2} \left( \partial_\mu A^a_5 \partial_\mu A^a 5 + \partial_5 A^a_\mu \partial^5 A^a_\mu - 2 \partial_5 A^a_\mu \partial^5 A^5 a \right)\]

Kinetic+mass term
Mixing term

• Assuming periodic BC for ALL fields (on circle appropriate) can integrate by parts

\[S_{mix} = - \int_0^{2\pi R} \partial_\mu A^a_\mu \partial_5 A^a_5\]

• As usual add gauge fixing term to cancel the mixings in $R_\xi$ gauge
\[ S_{GF} = - \int d^4x \int_0^{2\pi R} \frac{1}{2\xi} (\partial_\mu A^{\mu a} - \xi \partial_5 A_5^a)^2 \]

- Chosen to reproduce normal gauge fixing piece for \( A_\mu \) and to cancel mixing
- Action decoupled

**Gauge bosons**

\[ \mathcal{L}_{A_\mu} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \frac{1}{2} \partial_5 A^a_\mu \partial_5 A^{\mu a} - \frac{1}{2\xi} (\partial_\mu A^{\mu a})^2 \]

- In unitary gauge \( \xi \to \infty \) a tower of massive gauge bosons \( m_n^2 = \frac{n^2}{R^2} \)
Scalars

\[ \mathcal{L}_{A_5} = \frac{1}{2} \partial_\mu A_5^a \partial^\mu A_5^a - \frac{\xi}{2} (\partial_5 A_5^a)^2 \]

• Tower of scalars with mass \( m_n^2 = \xi \frac{n^2}{R^2} \)

• Unless \( n=0 \) unphysical \((m \rightarrow \infty)\)

• \( A_5 \) provide longitudinal components of massive gauge fields. Only physical mode: massless zero mode

• Spectrum:
  1. Massive tower of GB’s
  2. Massless GB+\( A_5 \) scalar

This is what we want to eventually use for Higgs…
Fermions on a circle

• Somewhat tricky, 5D Dirac algebra contains $\gamma_5$
• Theory will NOT be chiral (only Dirac fermions)

$$\Psi = \begin{pmatrix} \chi \alpha \\ \bar{\psi} \dot{\alpha} \end{pmatrix}$$

• Action:

$$S = \int d^5x \left( \frac{i}{2} (\bar{\Psi} \Gamma^M \partial_M \Psi - \partial_M \bar{\Psi} \Gamma^M \Psi) - m \bar{\Psi} \Psi \right)$$

• In terms of components

$$S = \int d^5x \left( -i \bar{\chi} \sigma^\mu \partial_\mu \chi - i \bar{\psi} \sigma^\mu \partial_\mu \bar{\psi} + \frac{1}{2} (\bar{\psi} \overleftrightarrow{\partial_5} \chi - \bar{\chi} \overleftrightarrow{\partial_5} \bar{\psi}) + m(\psi \chi + \bar{\chi} \bar{\psi}) \right)$$
• KK decomposition

\[ \chi = \sum_n g_n(y) \chi_n(x), \]
\[ \bar{\psi} = \sum_n f_n(y) \bar{\psi}_n(x) \]

• The KK modes are 4D Dirac fermions

\[ -i \bar{\sigma}^\mu \partial_\mu \chi^{(n)} + m_n \bar{\psi}^{(n)} = 0 \]
\[ -i \sigma^\mu \partial_\mu \bar{\psi}^{(n)} + m_n \chi^{(n)} = 0 \]

• Wave functions

\[ g''_n + (m_n^2 - m^2) g_n = 0 \]
\[ f''_n + (m_n^2 - m^2) f_n = 0 \]
• KK spectrum:
  
  Tower of massive KK modes
  
  \[ m_n^2 = m^2 + \frac{n^2}{R^2} \]

• On circle: no chiral zero mode even for \( m = 0 \)

• Clearly circle is too simple to
  • Reproduce SM
  • Give interesting possibilities for SSB
  • Give interesting zero mode spectra

• Look at next simplest possibility:
  • Orbifolds
  • Interval
2. Orbifolds

• Next simplest possibility: instead of circle compactify on a line segment $S^1/Z_2$. Will look in two slightly different approaches (orbifold vs. interval).

Geometric construction of $S^1/Z_2$
Effects on the fields

- $\tau, Z$ have to be symmetries of action
- Fields have to agree UP TO a symmetry transformation $T, Z$ ($T$ is SS-twist)

\[
\tau(2\pi R)\varphi(y) = T^{-1}\varphi(y + 2\pi R)
\]

\[
Z\varphi(y) = Z\varphi(-y)
\]

- Field identification will be

\[
\varphi(y + 2\pi R) = T\varphi(y)
\]

\[
\varphi(-y) = Z\varphi(y)
\]
A consistency condition

0 \quad 2\pi R \quad 4\pi R
A consistency condition

0 \quad 2\pi R \quad 4\pi R

T
A consistency condition

0 \to 2\pi R \to 4\pi R

\begin{align*}
T \\
Z
\end{align*}
A consistency condition

\[ 0 \quad 2\pi R \quad Z \quad 4\pi R \]

Diagram showing a line with marked points and arrows indicating transformations.
A consistency condition

\[
\begin{align*}
0 & \quad T^{-1} \quad 2\pi R \quad Z \quad 4\pi R \\
& \quad \quad T \quad Z
\end{align*}
\]
A consistency condition

\[ Z T = T^{-1} Z \]

• \( Z \) is a projection \( Z^2 = 1 \)

\[ Z T Z = T^{-1} \]

• \( ZT \) is also a projection

\[ (ZT)^2 = ZT TZT = T^{-1} T = 1 \]
The effect of $\mathbb{Z}T$

- $\mathbb{Z}T$ is a reflection around $\pi R$
- A generic $S^1/\mathbb{Z}_2$ is a combination of two (not necessarily commuting) parities $\mathbb{Z}$ and $\mathbb{Z}T$
A simple way to picture the orbifold BC’s

- Need to assign + or – parities to fields
- Parity assignments don’t have to be in same basis (eg. there could be a Scherk-Schwarz twist if parities don’t commute)
The orbifold BC’s

• Assign parities under two $Z_2$’s:
  - Scalars: $\phi(-y) = P\phi(y)$. $P = \pm 1$
  - Gauge fields: $A_\mu(-y) = PA_\mu(y)P^{-1}$
    - $A_5(-y) = -PA_5(y)P^{-1}$
  - Fermions: $\chi(-y) = P\chi(y)$
    - $\psi(-y) = -P\psi(y)$

• Reason:
  - $A_5$ opposite parity as $A_\mu$ (vector)
  - Term in fermion action: $\psi \partial_5 \chi$
• The KK spectrum

• Gauge bosons: If $A_\mu$ has zero mode, $A_5$ will NOT (and vice versa)
• LH ($\chi$) and RH ($\psi$) fermions have opposite BC’s: if one has zero mode, the other doesn’t $\rightarrow$ theory CHIRAL
A simple example: GUT breaking via orbifolds

(Altarelli, Feruglio; Hall, Nomura)

• Assume we have SUSY SU(5) in an extra D
• 5D fermions non-chiral, smallest SUSY in 5D: 8 supercharges (like N=2 in 4D)
• Need to use orbifold BC’s to
  • Break SUSY from N=2 to N=1
  • Break gauge SU(5) → SU(3)xSU(2)xU(1)
• BC to break SUSY (parities for VSF)

\[ Z_2 : \]

\[ + \rightarrow (\lambda_L, \lambda_R, \phi, A_\mu) \rightarrow - \]

• Gauge breaking (on fundamental)

\[ H = (5, \bar{5}), \quad H' = (5', \bar{5}') \]
• Action of $Z_2'$ on adjoint:

\[ Z_2' : \begin{pmatrix} 24 \\ \end{pmatrix} \rightarrow \begin{pmatrix} + \\ - \\ + \end{pmatrix} \]

• Decomposition of SU(5) adjoint

\[ \begin{pmatrix} V_{SM}^a & X \\ Y & V_{SM}^a \end{pmatrix}, \quad \begin{pmatrix} \lambda^a & x \\ y & \lambda^a \end{pmatrix} \]

• Decomposition of bulk Higgses:

\[ H = (3 + 2, \bar{3} + \bar{2}), \quad H' = (3' + 2', \bar{3}' + \bar{2}') \]
The KK decomposition will be

<table>
<thead>
<tr>
<th>$(Z_2, Z_2')$</th>
<th>mode</th>
<th>KK mass</th>
<th>wave function</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(+ , +)$</td>
<td>$V^a_{SM}, 2, 2'$</td>
<td>$\frac{2n}{R}$</td>
<td>$\cos \frac{2ny}{R}$, $n = 0, 1, 2, \ldots$</td>
</tr>
<tr>
<td>$(+ , -)$</td>
<td>$\lambda^a, 3, \bar{3}'$</td>
<td>$\frac{2n+1}{R}$</td>
<td>$\sin \frac{(2n+1)y}{R}$</td>
</tr>
<tr>
<td>$(- , +)$</td>
<td>$x, y, \bar{3}, 3'$</td>
<td>$\frac{2n+1}{R}$</td>
<td>$\cos \frac{(2n+1)y}{R}$</td>
</tr>
<tr>
<td>$(- , -)$</td>
<td>$X, Y, \bar{2}, \bar{2}'$</td>
<td>$\frac{2n+2}{R}$</td>
<td>$\sin \frac{2ny}{R}$</td>
</tr>
</tbody>
</table>

**Diagram:**

- Mass
- $V^a_{SM}, 2, 2'$
- $\lambda^a, x, y, 3, \bar{3}, 3', \bar{3}'$
- $X, Y, \bar{2}, \bar{2}'$
• Important difference from 4D theory: **doublet-triplet** splitting automatically solved
• In 4D hard to understand why $m_3 > m_2$
• Here parity assignments solve it
• No dim 5 proton decay since no $3 \overline{3}$ mass
3. Interval vs. orbifold approach

• Could just start with field theory in line segment $[0, \pi R]$ and specify some BC’s
• For example scalar field

$$S_{bulk} = \int d^4 x \int_0^{\pi R} \left( \frac{1}{2} \partial^M \phi \partial_M \phi - V(\phi) \right) dy$$

• Variation of action (after integrating by parts)

$$\delta S = \int d^4 x \int_0^{\pi R} \left[ -\partial_M \partial^M \phi - \frac{\partial V}{\partial \phi} \right] \delta \phi - \left[ \int d^4 x \partial y \phi \delta \phi \right] \frac{\pi R}{0}$$

• Require $\delta S_{bulk}$ and $\delta S_{bound}$ vanish separately for arbitrary $\delta \phi$. Fixes BC!
\[ \partial_y \phi \bigg|_{y=0, \pi R} = 0 \]

- Neumann (flat) BC is natural
- How to interpret Dirichlet BC? Add a large localized mass on boundary!

\[ S = S_{bulk} - \int d^4 x \frac{1}{2} M_1^2 \phi^2 \bigg|_{y=0} - \int d^4 x \frac{1}{2} M_2^2 \phi^2 \bigg|_{y=\pi R} \]

- Boundary variation will be

\[ \delta S_{bound} = -\int \delta \phi (\partial_y \phi + M_2^2 \phi) \bigg|_{y=\pi R} + \int d^4 x \delta \phi (\partial_y \phi - M_1^2 \phi) \bigg|_{y=0} \]

- The BC will be:

\[ \partial_y \phi + M_2^2 \phi = 0 \quad \text{at} \quad y = \pi R, \]
\[ \partial_y \phi - M_1^2 \phi = 0 \quad \text{at} \quad y = 0 \]
• Interpretation of Dirichlet BC: limit of $M \to \infty$ limit of localized mass term.

• Can repeat procedure for gauge fields on interval. General BC with localized scalar VEV on boundaries:

• Gauge fields with $v_{1,2}$ boundary scalar VEV

$$\partial_y A_\mu \mp v_{1,2}^2 A_\mu = 0$$

• $A_5$ BC:

$$\partial_y A_5 - \frac{\xi_1}{\xi} \frac{m^2/\xi_1}{m^2/\xi_1 - v_1^2} A_5|_{y=0} = 0$$

$$\partial_y A_5 + \frac{\xi_2}{\xi} \frac{m^2/\xi_2}{m^2/\xi_2 - v_2^2} A_5|_{y=\pi R} = 0$$

• $\xi$: bulk gf., $\xi_{1,2}$: boundary gf. terms
• **Meaning:** if \( v=0 \) \[ \partial_y A_\mu | = 0, \quad A_5 | = 0 \]

• If \( v \to \infty \) \[ A_\mu | = 0, \quad \partial_y A_5 | = 0 \]

• Just like in the case of orbifolds. Except: symmetry used for orbifolding form a \( Z_2 \) subgroup \( \subset \) Cartan subalgebra.
• Can NOT reduce rank with a single orbifolding
• Can reduce rank with localized scalar
• Orbifold BC’s subset of possible BC’s on interval
• For example can break EWS via BC’s using the interval approach → HIGGSLESS EWSB (will not discuss here)
• Is there any advantage for using ONLY orbifolds?

Yes! Localized VEVs vs. flat VEVs:

• **Orbifolds**: wave functions all orthogonal. If Higgs VEV of $A_5$: also flat, does NOT mix KK modes. No tree-level corrections to EWPO

• **Localized Higgs**: will mix KK modes, induce corrections to EWPO ($\Delta \rho, Z_{bb}, \ldots$)

• Flat Higgs VEV in flat orbifold theory~like T-parity in little Higgs (protects EWPO)…
4. Basics of gauge-Higgs unification (GHU)

- Idea: $A_5$ 4D scalar could be Higgs. How to find a setup where $A_5$ is a doublet of $SU(2)\times U(1)$ with correct hypercharge?
- Ideally, use flat space, and NO induced Scalars, just orbifold BCs

History:
1979 Manton, use 6D with monopole in sphere
1998 Hatanaka, Inami, Lim: revive idea, no concrete model
2001 Antoniadis, Benakli, Quiros: basic model, Higgs potential calculation
2002 C.C., Grojean, Murayama; von Gersdorff, Irges, Quiros: 6D problems, basics of flavor construction
2003 Scrucca, Serone, Silvestrini (+Wulzer): basic 5D model introduced and alayzed
2005 Cacciapaglia, C.C., Park; Panico, Serone, Wulzer: close to realistic model
• $A_5$ is in adjoint of gauge group, but Higgs is doublet: need to enlarge gauge group.
• If we want to use simplest orbifold (does not reduce rank): extended gauge group would be rank 2
• Simplest rank 2 group $SU(3)$

\[
\begin{pmatrix}
SU(2) & \vline & U(1)
\end{pmatrix}
\]

Forms one complex
$SU(2)$ doublet
The necessary BC’s

• The necessary projection (at both endpoints):

\[ P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \]

• Action on \( A_\mu \)

\[ PA_\mu(-y)P^{-1} \]

SU(2)xU(1) gauge zero modes
• Action on $A_5$:

$$-PA_5(-y)P^{-1}$$

Scalar doublet zero mode

• Picture:

$0 \quad \pi R$

SU(3)

SU(2)xU(1) orbifold fixed points

• For $(+,+)$ fields: $\cos(ny/R), \ m_n^2 = n^2/R^2$
• For $(-,-)$ fields: $\sin(ny/R), \ m_n^2 = n^2/R^2$
Why is this interesting? 5D gauge invariance:

\[
\begin{align*}
A_\mu & \rightarrow A_\mu + \partial_\mu \varepsilon(x, y) + i[\varepsilon(x, y), A_\mu] \\
A_5 & \rightarrow A_5 + \partial_5 \varepsilon(x, y) + i[\varepsilon(x, y), A_5]
\end{align*}
\]

**ε**: gauge transformation param., has its own KK expansion (same as \(A_\mu\)). For broken dir. \(\varepsilon(0, \pi R) = 0\), BUT \(\partial_5 \varepsilon \neq 0\).

**Shift symmetry** protects \(A_5\) from mass even at fixed points where gauge symmetry broken.

**Shift symmetry analog of broken global sym.** in little Higgs models protecting Higgs.
• Shift symmetry forbids tree-level potential for Higgs (formulation as SS theory)

• Local radiative potential for Higgs forbidden (formulation as SS theory)

• Non-local loop effects could still give a finite Higgs potential (loop has to stretch from one fixed point to other – does not shrink to zero – result must be finite…)

• Gauge-Higgs unification protects Higgs from divergences due to higher dim. gauge invar.
• Higgs potential only generated through finite loop effects
5. The calculation of the Higgs potential

• Need Coleman-Weinberg potential for Higgs
• Assume simplest SU(3) model for now
• Higgs VEV normalization:

\[ A_5 = \frac{1}{\sqrt{2}} \begin{pmatrix} - & H_5 \\ H_5^\dagger & - \end{pmatrix} \]

\[ \langle H_5 \rangle = \sqrt{2} \begin{pmatrix} 0 \\ \alpha / R \end{pmatrix} \]

\( \alpha \): VEV in units of radius. For realistic model needs to be \( \ll 1 \) (to separate KK modes from) SM particles
• For Coleman-Weinberg need $\alpha$-dependent Mass spectrum. For example gauge KK:

\[
\begin{pmatrix}
\frac{1}{\sqrt{2}} A^3 + \frac{1}{\sqrt{6}} A^8 \\
W^- \\
\tilde{W}^1 \\
\tilde{W}^2 \\
\end{pmatrix}
\begin{pmatrix}
W^+ \\
\tilde{W}^{1*} \\
\tilde{W}^{2*} \\
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\sqrt{2}} A^3 + \frac{1}{\sqrt{6}} A^8 \\
\frac{2}{\sqrt{6}} A^8 \\
\end{pmatrix}
\]

• Mass terms come from:

\[-\int_0^\pi R \frac{1}{2} \text{Tr} F_{5\mu}^2\]

\[-\frac{1}{2} \int_0^\pi R \text{Tr} \left( \partial_5 A_\mu - \partial_\mu A_5 + g_5 [\langle A_5 \rangle, A_\mu] \right)^2\]

• The mass matrix mixes various components In the 8x8 basis $A_1$-$A_8$ the mixing matrix is:
• **TeXForm on the Mathematica output:**

\[
\begin{pmatrix}
2(\alpha^2 + n^2) & 0 & 0 & 0 & 4\alpha n & 0 & 0 & 0 \\
0 & 2(\alpha^2 + n^2) & 0 & -4\alpha n & 0 & 0 & 0 & 0 \\
0 & 0 & 2(\alpha^2 + n^2) & 0 & 0 & 0 & -4\alpha n & -2\sqrt{3}\alpha^2 \\
4\alpha n & 0 & 0 & 0 & 2(\alpha^2 + n^2) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2n^2 & 0 & 0 & 0 \\
0 & 0 & -4\alpha n & 0 & 0 & 0 & 2(4\alpha^2 + n^2) & 4\sqrt{3}\alpha n \\
0 & 0 & -2\sqrt{3}\alpha^2 & 0 & 0 & 0 & 4\sqrt{3}\alpha n & 2(3\alpha^2 + n^2)
\end{pmatrix}
\]

• **Eigenvalues:**

\[
\frac{n^2}{R^2} \hspace{1cm} x2 \leftrightarrow \gamma
\]

\[
(\pm\alpha)^2/R^2 \hspace{1cm} x2 \leftrightarrow W^\pm
\]

\[
(\pm2\alpha)^2/R^2 \hspace{1cm} x1 \leftrightarrow Z
\]

• **Implies most problematic part of model:**

\[
M_Z^2/M_W^2 = 2
\]

• **Obviously due to wrong U(1) quantum number of Higgs**
• Unbroken U(1) after orbifolding: $T_8$
• Higgs quantum number:

**Usual normalization:**

• $g \rightarrow 1/2 \text{ diag } (1,-1), \text{ etc}$
• $g' \rightarrow \text{Higgs quantum number } 1/2$

• Here: for $\text{Tr } T_a T_b = 1/2$: $T_8 = 1/(2\sqrt{3}) \text{ diag}(1,1,-2)$
• Higgs quantum number $\sqrt{3}/2$. Rescale U(1):
  • $\sqrt{3}/2g = g'/2$

  $$\sin^2 \theta_w = g'^2/(g^2 + g'^2) = 3/(1+3) = 3/4$$

• Wrong U(1) normalization, need another U(1)
The Coleman-Weinberg potential

\[ V_{CW}(\phi) = \frac{1}{2} \sum I (-1)^{FI} \int \frac{d^4 p}{(2\pi)^4} \log(p^2 + M_I^2(\phi)) \]

• Can be rewritten in the form

\[ V_{CW} = -\frac{1}{32\pi^2} \sum I (-1)^{FI} \int_0^\infty d\ell e^{-\frac{M^2_I(\phi)}{\ell}} \]

• General form of KK mass spectrum (ABQ)

\[ M^2_{\tilde{m}} = \mu^2 + \sum_{i=1}^d \frac{(m_i + a_i(\phi))^2}{R_i^2} \]

\[ V_{CW} = -\sum I \frac{(-1)^{FI}}{32\pi^2} \int_0^\infty d\ell e^{-\frac{\mu^2}{\ell}} e^{-\frac{\sum_i (m_i + a_i)^2}{R_i^2 l}} \]
• Using a Poisson resummation

\[ \frac{1}{2\pi R} \sum m F(m/R) = \sum_n \tilde{F}(2\pi n R) \]

\[ \sum m e^{-\sum_i \frac{(m_i+a_i)^2}{2r_i^2}} = \pi^2 \prod_{i=1}^d R_i \sum m e^{2\pi i \sum j n_j a_j} e^{-\pi^2 \sum_j n_j^2 r_j^2} \]

\[ V_{CW}(\phi) = -\sum I \frac{(-1)^{FI}}{32\pi^2} \frac{d}{\pi^2} (\prod R_i) \sum m e^{2\pi i \sum j n_j a_j \int_0^\infty d l l^{1+d/2}} e^{-\mu/2} e^{-\pi^2 (\sum_j n_j^2 R_j^2)} l \]

• For example, if \( \mu=0 \) (no bulk mass)

\[ V_{CW} = -\sum I \frac{(-1)^{FI}}{32\pi^2} (\prod R_i) \frac{d}{\pi^2} \Gamma \left(2 + \frac{d}{2}\right) \sum m \neq 0 \frac{e^{2\pi i \vec{m} \cdot \vec{a}}}{(\pi^2 \sum_j n_j^2 R_j^2)^{2+d/2}} \]

• As expected potential finite (dropped a divergent constant piece…)
• Expression for potential in general case in 5D:

\[ V_{\text{eff}}(\beta) = \pm \frac{1}{32\pi^2} \frac{1}{(\pi R)^4} \mathcal{F}(\beta) \]

\[ \beta = k \alpha \]

• Where for no bulk mass term

\[ m_n^2 = (n+\beta)^2/R^2 \]

\[ \mathcal{F}(\beta) = \frac{3}{2} \sum_{n=1}^{\infty} \frac{\cos(2\pi \beta n)}{n^5} \]

• With bulk mass term

\[ m_n^2 = M^2 + (n+\beta)^2/R^2 \]

\[ \mathcal{F}_\kappa(\beta) = \frac{3}{2} \sum_{n=1}^{\infty} \frac{e^{-\kappa n} \cos(2\pi \beta n)}{n^3} \left( \frac{\kappa^2}{3} + \frac{\kappa}{n} + \frac{1}{n^2} \right) \]

• Where \( \kappa = 2\pi MR \). For large \( \kappa \) exponentially suppressed.
Comments

• $n=1$ term most important in series $\pm \cos 2\pi \beta$
• For fermions min. for $\beta = \frac{1}{2}$
• For bosons min. for $\beta = 0$
• For twisted fermions (will see later) spectrum

$$m_n^2 = M^2 + (n + \frac{1}{2} + \beta)^2/R^2$$

• Effect in potential $\beta \rightarrow \beta + \frac{1}{2}$

Summary:

Can calculate finite Higgs potential for arbitrary bulk fields. Need to know, what bulk fields...
6. The fermion fields & flavor structure

- Apparent problem: since Higgs=$A_5$, Yukawa coupling=gauge coupling. How to get fermion mass hierarchy?

1. Use Arkani-Hamed Schmaltz idea of localizing fermions at different parts of 5D
2. Use bulk fermions mixed with localized fermions at the fixed points (an X-D version of Frogatt-Nielsen)

- Will use second approach
• Every SM field $\rightarrow$ Dirac fermion in 5D $\Psi$
• Arrange BC’s such, that only one zero mode
• In order to avoid masses of order $M_W$ add a second bulk field with same quantum # but opposite parity assignments $\Psi'$
• Two fields will marry up with bulk mass $M\Psi\Psi'$

• At this point no chiral zero modes. We add them as fields localized at the fixed points and mix them with the bulk fields

$$\mathcal{L}_{loc} = \left[ -i \bar{Q}_L \bar{\sigma}^\mu \partial_\mu Q_L + \frac{\epsilon_L}{\sqrt{\pi R}} \psi^d Q_L + h.c. \right] \delta(y - y_L) + \left[ -i q_R \sigma^\mu \partial_\mu \bar{q}_R + \frac{\epsilon_R}{\sqrt{\pi R}} q_R \chi^s + h.c. \right] \delta(y - y_R),$$
Here $\psi_d$ is the doublet and $\chi_s$ is the singlet in the bulk field. Depending on choices of parity there is always a unique choice of which to add $\cdot y_L$ and $y_R$ could be either fixed points $\cdot 4$ distinct possibilities (same fp or opposite, fermions twisted or not...) $\cdot$In the $\alpha=0$ limit still a zero mode (odd number of chiral fermions), so light modes $m_0 \propto \alpha$ $\cdot$Mass spectrum will depend on $\alpha, \epsilon_L, \epsilon_R, M$
Example: down quark

• Use bulk triplets $\mathbf{3}$ and no twisting

Need to write down coupled bulk equations
Can diagonalize bulk equations
BC’s will provide equation for KK masses
• Equation for spectrum for $3$ with untwisted fermions:

\[
\gamma_3(w) = (\cos w - \cos(2\pi \alpha))^2 + 2\frac{\epsilon_L^2 + \epsilon_R^2}{w} \sin w (\cos w - \cos(2\pi \alpha)) + \\
-\frac{4\epsilon_L^2 \epsilon_R^2}{w^2} \cdot \begin{cases} \\
(\cos w + 1) \left( \cos w - 1 + 2\frac{w^2}{w^2 + \kappa^2} \sin^2(\pi \alpha) \right) & \text{different branes}, \\
\frac{1}{2} \left( \cos 2w - 1 + 2\frac{w^2}{w^2 + \kappa^2} \sin^2(2\pi \alpha) \right) & \text{same brane}.
\end{cases}
\]

• $y_3(w) = 0$ determines mass eigenmodes $m$

• $\kappa = 2\pi R M$, $w^2 = (2\pi R m)^2 - \kappa^2$

• Similar equation for the twisted case

\[
\tilde{\gamma}_3(w) = (\cos w + \cos(2\pi \alpha))^2 + 2\frac{\epsilon_L^2 + \epsilon_R^2}{w} \sin w (\cos w + \cos(2\pi \alpha)) + \\
-\frac{4\epsilon_L^2 \epsilon_R^2}{w^2} \cdot \begin{cases} \\
(\cos w - 1) \left( \cos w + 1 - 2\frac{\kappa^2}{w^2 + \kappa^2} \sin^2(\pi \alpha) \right) & \text{different branes}, \\
\frac{1}{2} \left( \cos 2w - 1 + 2\frac{w^2}{w^2 + \kappa^2} \sin^2(2\pi \alpha) \right) & \text{same brane}.
\end{cases}
\]
• Simple limits:

• No boundary mixings \((\epsilon_{L,R} \rightarrow 0)\)

\[
m_{n}^2 = M^2 + \begin{cases} \frac{(n+\alpha)^2}{R^2} & \text{untwisted} \\ \frac{(n+1/2+\alpha)^2}{R^2} & \text{twisted} \end{cases}
\]

• Small Higgs VEV \((\alpha \ll 1)\), large bulk mass \((\kappa \gg 1)\)

\[
\text{diff. branes} \rightarrow \frac{4\epsilon_L \epsilon_R}{\sqrt{(2\epsilon_L^2 + 1)(2\epsilon_R^2 + 1)}} \frac{\kappa}{2} e^{-\kappa/2}
\]
\[
\text{same brane} \rightarrow \frac{4\epsilon_L \epsilon_R}{\sqrt{(2\epsilon_L^2 + 1)(2\epsilon_R^2 + 1)}} \kappa e^{-\kappa}
\]

Exponentially suppressed by bulk mass…
• Small bulk mass ($\kappa \ll 1$): if untwisted there will be a mode with $m = M_W$. Reason: bulk mass couples two fermions, and only one mixes with localized fields. Other light:

\[
\begin{align*}
\text{diff. branes} & \rightarrow m_q \pi R = \frac{\epsilon_L \epsilon_R}{\sqrt{(1 + \epsilon_L^2)(1 + \epsilon_R^2) - \cos^2 \pi \alpha}} \\
\text{same brane} & \rightarrow m_q \pi R = \frac{\epsilon_L \epsilon_R \cos \pi \alpha}{\sqrt{(1 + \epsilon_L^2)(1 + \epsilon_R^2) - \cos^2 \pi \alpha}}
\end{align*}
\]

Lessons:

• Many ways to get suppression: large bulk mass, small boundary mixing
• Hard to get a large mass. Upper limit $M_W$.
• Upper limit achievable for vanishing bulk mass
• Upper limit can be relaxed for bigger reps due to non-trivial group-theory factors (Dynkin)
• Example: behavior of lowest eigenmode for different (d) or same (s) brane, untwisted or twisted (t) bulk fermions:

![Graph showing behavior of lowest eigenmode](image)

• Final remark: for large mixings spectrum can be deformed a lot. Need to modify formula for Higgs potential! Using result of Goldberger & Rothstein:
• Mass given by \( y(m) = 0 \), contribution to CW:

\[
V_{eff} = \frac{1}{2} \int_0^\infty \frac{d^4 p}{(2\pi)^4} \ln \mathcal{Y}(i p)
\]

• In our case

\[
\mathcal{F}_\epsilon(\kappa, \alpha) = \frac{1}{8} \int_\kappa^\infty d\zeta \zeta (\zeta^2 - \kappa^2) \ln \frac{\mathcal{Y}(i \zeta)}{K(\zeta)}
\]

• Function \( K \) to regulate divergent constant

• Contribution of various bulk fields to CW:

<table>
<thead>
<tr>
<th>bulk field</th>
<th>multiplicity</th>
<th>term</th>
</tr>
</thead>
<tbody>
<tr>
<td>gauge (adj.)</td>
<td>-3</td>
<td>( 2\mathcal{F}(\alpha) + \mathcal{F}(2\alpha) )</td>
</tr>
<tr>
<td>down (3)</td>
<td>3 x 8</td>
<td>( \mathcal{F}_{\kappa d}(\alpha) )</td>
</tr>
<tr>
<td>up (6)</td>
<td>3 x 8</td>
<td>( \mathcal{F}<em>{\kappa u}(\alpha) + \mathcal{F}</em>{\kappa u}(2\alpha) )</td>
</tr>
<tr>
<td>lepton (10)</td>
<td>8</td>
<td>( 2\mathcal{F}<em>{\kappa l}(\alpha) + \mathcal{F}</em>{\kappa l}(2\alpha) + \mathcal{F}_{\kappa l}(3\alpha) )</td>
</tr>
</tbody>
</table>
7. A semi-realistic model

• To fix $\sin^2 \theta_W$ we add an additional $U(1)_x$
• Gauge group $SU(3) \times U(1)_x$ broken by orbifold to $SU(2)_L \times U(1)_8 \times U(1)_x$, and $U(1)_8 \times U(1)_x \rightarrow U(1)_Y$ on the fixed point (localized Higgs or anomaly)
• This last breaking distorts wave functions, we’ll have to pay the price for that…

Two main problems: (Scrucca, Serone, Silvestrini)

• Higgs mass too small (& KK modes light)
• Top mass too small
• Reason: if assume (well motivated)
  • all mixings of same order
  • fermion hierarchy only from bulk masses

• Most bulk masses very large, contribution to CW very suppressed. Basically top dominates radiative potential, and minimum of top+gauge contribution gives

\[
\begin{align*}
\alpha &\sim 0.3, \quad m_h \sim 0.2 - 0.3m_W \\
\frac{1}{R} &\sim 3 - 5m_W \sim 250 - 400\text{GeV} \\
m_t &\leq m_W
\end{align*}
\]

• This is obviously bad
• **Fix Higgs mass and VEV**: assume that some light fermions light due to small mixing rather than due to large bulk mass
• These bulk fermions will also contribute
• Take different representations and twist some of fermions→get a much more versatile Higgs potential

**A successful example**

• Top: rep. $\bar{6}$, large mixing $\chi_{L,R}\sim 3$, $\kappa_t\sim 1$
• Bottom: twisted $3$, $\kappa_b=0$
• Tau: $10$, $\kappa_{\tau}=1$
• Light gens: twisted $3+6+10$, common $\kappa_l$
• The Higgs potential:

\[ V_{\text{eff}} \]

- Total
- top + tau
- bottom + light gen.
- gauge

• VEV and Higgs mass

\[ \alpha \]

- 1 light gen.
- 2 light gen.

\[ m_H \]

- 1 light gen.
- 2 light gen.
- LEP–II
• **Fix top mass**: upper bound on fermion mass actually depends on representation

\[ m_t \leq k m_W \]

• \( k^2 \): number of indices of rep. top is embedded

• For \( m_t = 2m_w \) need a 4-index irrep...

• Simplest possibility \( 15 \) dim rep:

\[
(15)_{-2/3} \rightarrow (1, 2/3) + (2, 1/6) + (3, -1/3) + (4, -5/6) + (5, -4/3)
\]

• To get biggest top mass \( (2m_w) \) need top to be a bulk zero mode. So we only add a single \( 15 \) with usual orbifold projections. Remove ad’l zero modes via mixing with localized fields
• For EWSB third generation enough (twisted fermions for $b, \tau$). Possible reps (choose them as small as possible to not lower cutoff further)

<table>
<thead>
<tr>
<th>model a</th>
<th>bottom</th>
<th>tau</th>
</tr>
</thead>
<tbody>
<tr>
<td>model b</td>
<td>$(3, 6)_{1/3}$</td>
<td>$(1, 3)_{-2/3}$</td>
</tr>
</tbody>
</table>

• The Higgs potential
• Results for $\alpha$, $m_H$, $m_{\text{top}}$, and fine tuning ($f$)

• Fine tuning defined via the usual log derivative

\[ f = \frac{d \log \alpha(\epsilon)}{d \log \epsilon} \]
• Some particular model points:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$1/R$</th>
<th>$f$</th>
<th>$m_H$</th>
<th>$m_t$</th>
<th>$m'_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.08</td>
<td>1 TeV</td>
<td>31%</td>
<td>110</td>
<td>113</td>
<td>189</td>
</tr>
<tr>
<td></td>
<td></td>
<td>42%</td>
<td>125</td>
<td>110</td>
<td>186</td>
</tr>
<tr>
<td>0.05</td>
<td>1.6 TeV</td>
<td>11%</td>
<td>120</td>
<td>149</td>
<td>381</td>
</tr>
<tr>
<td></td>
<td></td>
<td>14%</td>
<td>133</td>
<td>149</td>
<td>375</td>
</tr>
<tr>
<td>0.04</td>
<td>2 TeV</td>
<td>7%</td>
<td>124</td>
<td>154</td>
<td>519</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9%</td>
<td>136</td>
<td>154</td>
<td>514</td>
</tr>
<tr>
<td>0.03</td>
<td>2.7 TeV</td>
<td>4%</td>
<td>128</td>
<td>157</td>
<td>753</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5%</td>
<td>140</td>
<td>157</td>
<td>746</td>
</tr>
<tr>
<td>0.02</td>
<td>4 TeV</td>
<td>2%</td>
<td>134</td>
<td>159</td>
<td>1224</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2%</td>
<td>144</td>
<td>159</td>
<td>1213</td>
</tr>
</tbody>
</table>

• Introducing a large representation dangerous for lowering cutoff scale: only a few x 1/R.
• Would need to check stability of results under loop corrections (two loop Higgs potential?)
8. Bounds on the model from EWPT

- $W, Z, \langle H \rangle$: flat: no mixing induced among KK modes, no correction to EWPO from these at tree level, and loop should be small.

- Only possible source: exotic zero modes that mix with SM fields and pick up mass via boundary terms (otherwise orthogonality OK).

- Two such sources:
  
  - Fermion zero modes needed to generate fermion masses: $Z_{bb}$ affected.
  - Additional $U(1)_X$ to fix $\sin^2 \theta_W$: will affect $\Delta \rho$. 

**Zbb from mixing with heavy quarks**

- Light fermions mixing negligible. Only 3\textsuperscript{rd} gen. problematic. Lowest order Yukawa by gauge inv.

\[ \mathcal{V}_{-1/3} Q_L H^T \bar{3}_{-1/3} + \mathcal{V}_{2/3} Q_L H \bar{3}_{2/3} \]

- General expression for corr. of Z-vertex:

\[ \Delta = \frac{\delta g}{g} = \frac{1}{1 - \frac{2}{3} \sin^2 \theta_W} \left( \mathcal{V}_{2/3}^2 - \mathcal{V}_{-1/3}^2 \right) \left( \frac{m_W}{m_3} \right)^2 \]

- For the 15 rep: \( \mathcal{V}_{-1/3} = \sqrt{3}, \ m_3^2 = \frac{3}{(R^2 \pi^2)} \)

\[ \Delta \sim -11 \alpha^2 \]

- Bound from LEP: \( \alpha < 0.021, \frac{1}{R} > 3.9 \text{ TeV} \)
Effects of additional $U(1)_X$

$$X_\mu = \frac{1}{\sqrt{3}g^2 + g_x^2} \left( \sqrt{3}g A_\mu^8 - g_x A_\mu^x \right)$$

- $X_\mu$ gets a localized mass. After EWSB mixing with $Z$ induced, correction to $T$:

$$T = \frac{4\pi}{e^2} \Delta \rho = \frac{4\pi}{e^2} \frac{\pi^2}{3} \frac{3 - 4\sin^2 \theta_W}{\cos^2 \theta_W} \alpha^2 \approx 1.2 \cdot 10^3 \alpha^2$$

- Strongest bound on model $1/R > 5$ TeV, $\alpha < 0.018$
- Also contributes to $\delta g/g$ bound $1/R > 4.5$ TeV
Summary

• In extra dim’s a possible solution to hierarchy problem is via gauge-Higgs unification
• Need to extend gauge group and orbifold it to \( SU(2) \times U(1) \)
• Simplest (and most realistic) example in 5D \( SU(3) \times U(1)_x \)
• Generically hard to get a large separation of Higgs VEV and KK modes, and heavy Higgs, top
• Can use many bulk fermions to generate a sufficiently generic Higgs pot.
• Top mass fixed via large bulk representation
• Constraints from \( Z_{bb}, \Delta \rho \): little hierarchy …