NRQCD

Aneesh Manohar

UC SAN DIEGO

Previous Work

with Iain Stewart and Andre Hoang.
Initial work with Michael Luke and Ira Rothstein

based on earlier work:
Caswell and Lepage (NRQED)
Bodwin, Braaten, Lepage (NRQCD)

Beneke, Griesshammer, Grinstein, Labelle, Pineda, Soto

Applications

Study non-relativistic dynamics in field theory.

- QED: Hydrogen, Positronium and Muonium. Can study energy levels (Lamb shift and hyperfine splitting), and decay widths.
- QCD
  - Υ ($\bar{b}b$)
  - $\bar{t}t$ near threshold

In QED, the expansion parameter is $\alpha \sim 1/137 \ll 1$. Nevertheless, one cannot always use perturbation theory in α.

Hydrogen Atom: One needs to solve the Schrödinger equation with the potential

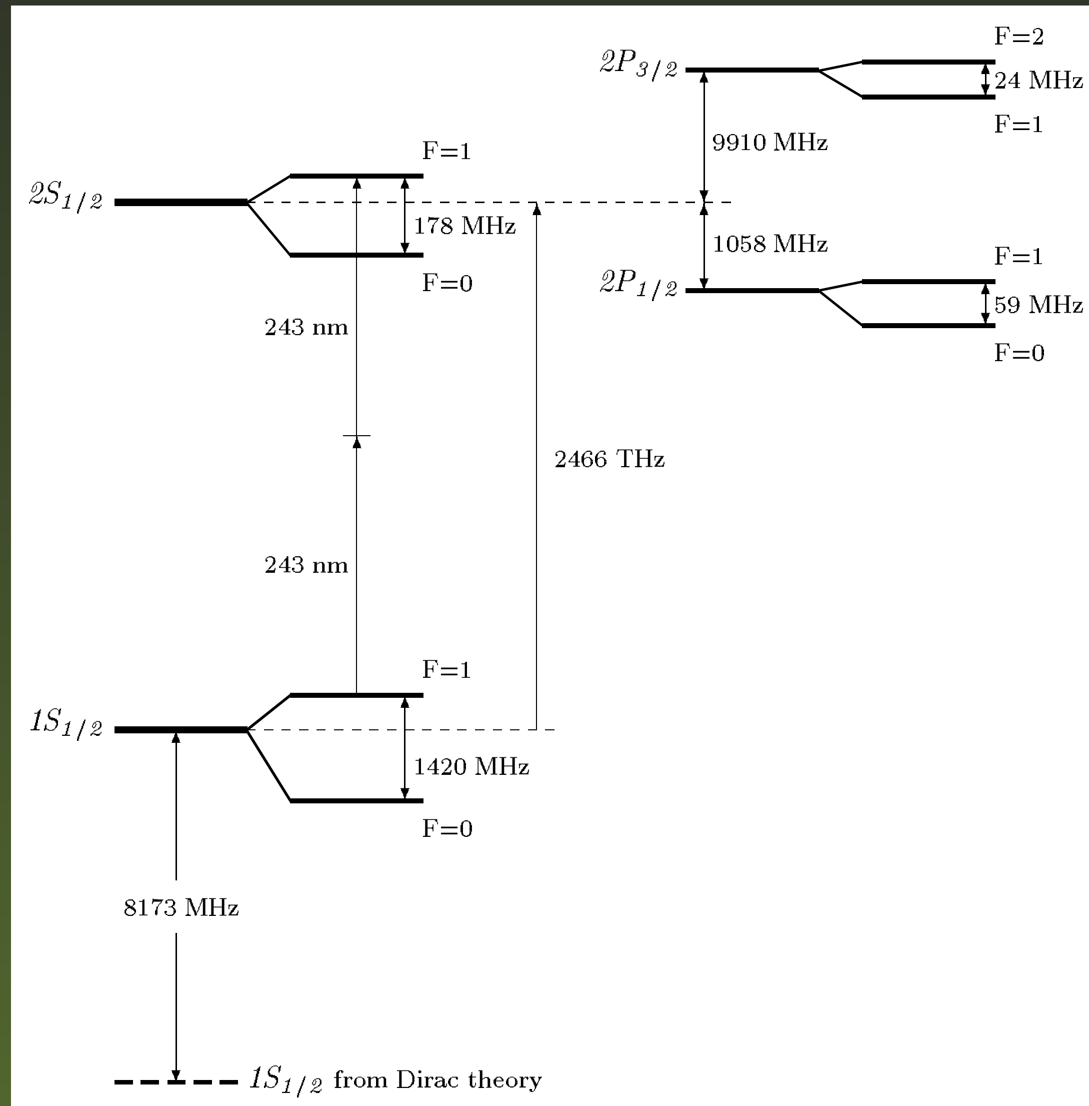
$$V = -\frac{4\pi\alpha}{r}$$

The Schrödinger equation sums up multiple iterations of the Coulomb potential. The energies can be expressed in a series in α [but the wavefunctions cannot].

Hydrogen Atom

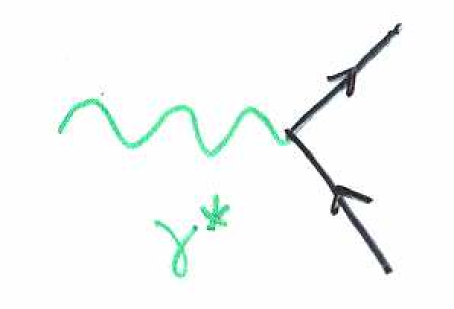
n-dependence: α^2
fine structure: α^4
Lamb shift: $\alpha^5 \ln \alpha$
hyperfine structure:
$\alpha^4 \; m_e/m_p$

$m_e \sim 7.5 \times 10^{14}$ MHz
$m_e \alpha^2 \sim 4 \times 10^{10}$ MHz

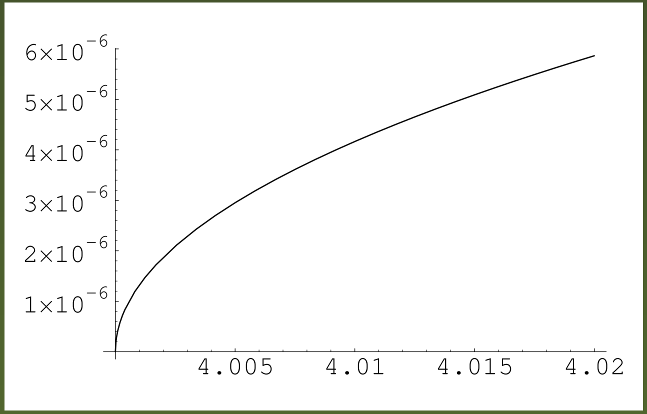


Threshold Pair Production

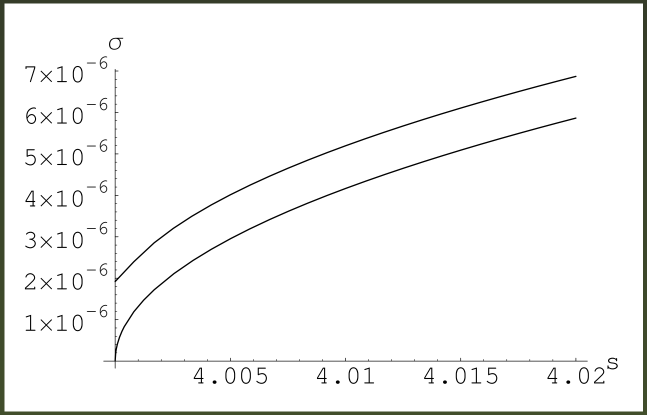
Cross-section for pairs ($\mu^+\mu^-$) near threshold.



$$\sigma = \frac{4\pi\alpha^2}{3E_{\mathrm{CM}}^2}\frac{\beta\left(3-\beta^2\right)}{2}$$



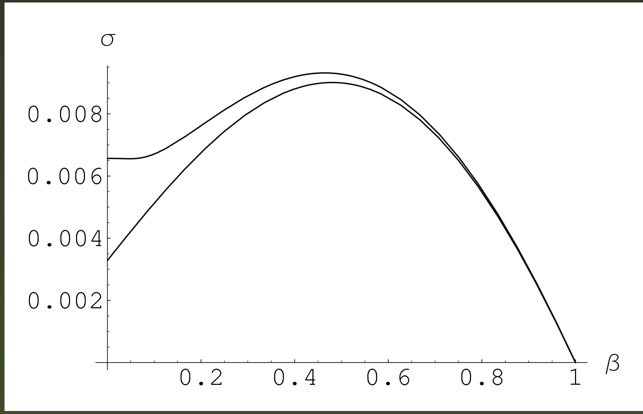
But when $\beta \sim \alpha$, there are big corrections above threshold



and bound state resonances below threshold.

$$1 + \frac{\pi\alpha^2}{2}$$

Using $\alpha \sim 0.11$



Scales

For a Coulomb system: $v \sim \alpha \ll 1$

3 important scales

- m (hard)
- $p \sim mv$ (soft)
- $E \sim mv^2$ (ultrasoft)

Field theory better for radiative corrections at m
Schrödinger equation better for non-relativistic dynamics and bound states.

Goal for the Effective Theory

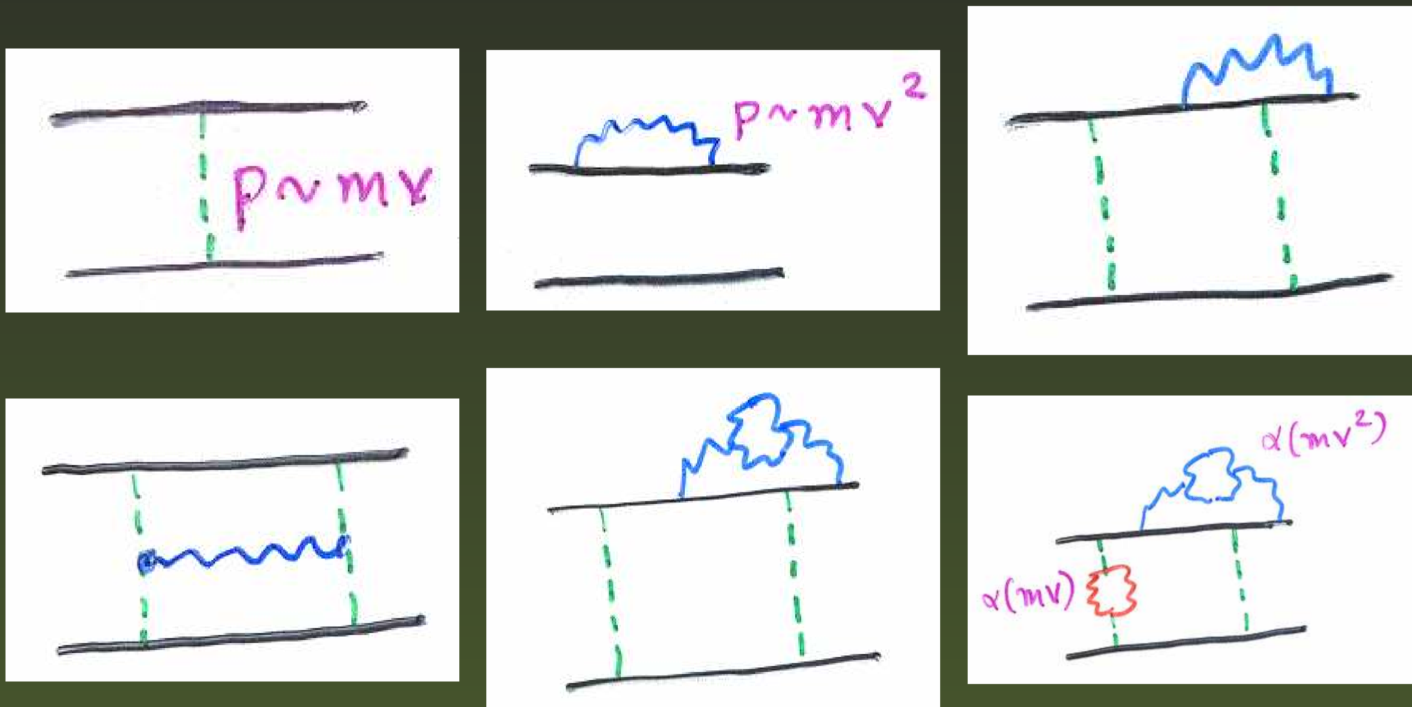
- Have a systematic expansion in some small parameter (v or α)
- Separate scales consistently
- Sum large logarithms (from ratios of scales) using the renormalization group

$$\ln \frac{p}{m}, \qquad \frac{1}{2} \ln \frac{E}{m}, \qquad \ln \frac{E}{p} \to \ln v \to \ln \alpha$$

- Determine scale for α_s:

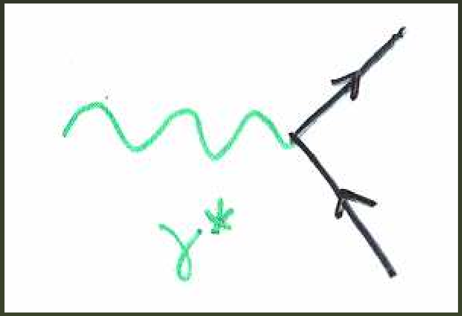
$$\alpha_s(m), \qquad \alpha_s(mv), \qquad \alpha_s(mv^2)$$

What is the problem?



Graphs involve $\alpha_s(mv)$ and $\alpha_s(mv^2)$ at the same time.
Usually, one has $\alpha_s(\mu)$.

$\bar{t}t$ production near threshold



Large ratio of scales for $\bar{t}t$ $(v \sim 0.14)$:

m	mv	mv^2
175 GeV	30 GeV	5 GeV
$\alpha_s(m) \sim 0.11$	$\alpha_s(mv) \sim 0.14$	$\alpha_s(mv^2) \sim 0.21$

$\alpha_s \ln v$ not small, resumming logarithms is important.

QED

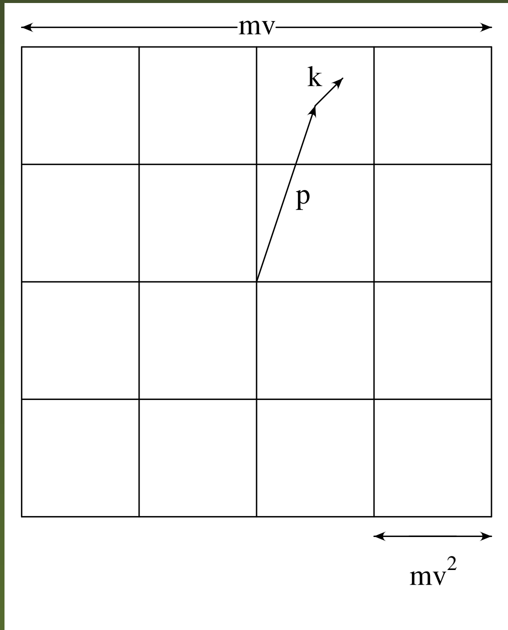
- The method is useful in QED, and provides new insights. New results on the nature of the QED perturbation series for bound states.
- $\alpha \ln \alpha \ll 1$, but we need to know the results to high precision.
- Interesting theoretical question as to how to formulate the effective theory.
- Complete calculation of the $\alpha^8 \ln^3 \alpha$ energies.
- Check of ideas before applying them to QCD.

The Effective Theory

- Expansion in powers of velocity v.
- Non-relativistic fermions with propagator

$$\frac{1}{E - \mathbf{p}^2/2m}$$

- Fermions $\psi_{\mathbf{p}}(x)$.

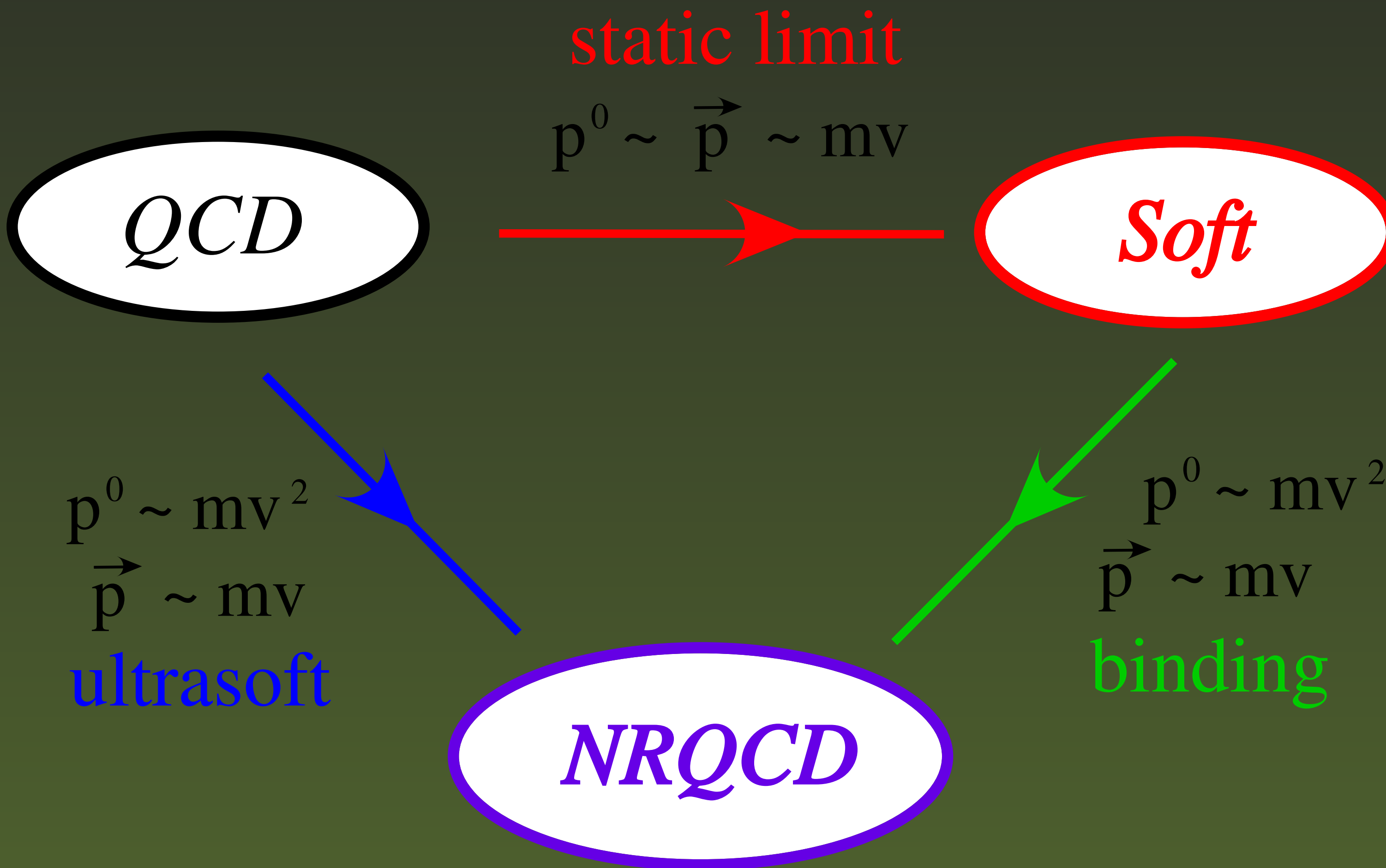


- Potentials $V_{\text{eff}}(\mathbf{p},\mathbf{p}')$ such as the Coulomb potential, hyperfine interaction, etc.

$$
\begin{aligned}
V^{(-1)} &= \frac{U_c}{\mathbf{k}^2}, \\
V^{(0)} &= \frac{U_k}{|\mathbf{k}|}, \\
V^{(1)} &= U_2 + U_s\,\mathbf{S^2} + \frac{U_r(\mathbf{p}^2+\mathbf{p}'^2)}{2\mathbf{k}^2} - \frac{i\mathbf{U}_\Lambda\cdot(\mathbf{p}'\times\mathbf{p})}{\mathbf{k}^2} \\
&\quad + U_t\Big(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 - \frac{3\,\mathbf{k}\cdot\boldsymbol{\sigma}_1\,\mathbf{k}\cdot\boldsymbol{\sigma}_2}{\mathbf{k}^2}\Big),
\end{aligned}
$$

- Ultrasoft gluons (multipole expanded)
  have $E \sim mv^2$, $p \sim mv^2$.
  Act on x
- Soft gluons
  have $E \sim mv$, $p \sim mv$.
  local in time, non-local in space
  Act on $\mathbf{p}$

- Static theory not the $v \to 0$ limit or $m \to \infty$ limit



all at the scale $\nu = 1$ $(\mu = m)$

$$L = \psi^\dagger \left(i\partial_t + \frac{\mathbf{D}^2}{2m} \right) \psi + \psi^\dagger \chi^\dagger \psi \chi V + \dots$$

Different forms for higher order terms: Choose to eliminate ∂_t by field redefinitions.

Nice example of how different fields lead to same measurable quantities.

Power Counting

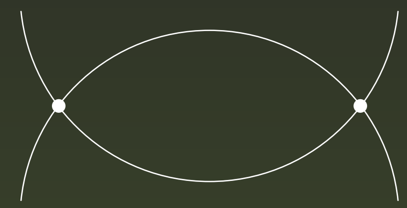
Kinetic Energy:

E	$\frac{\mathbf{p}^2}{2m}$	$\frac{\mathbf{p}^4}{8m^3}$
v^2	v^2	v^4
1	1	v^2

Potentials:

$\frac{1}{\mathbf{k}^2} \sim \frac{1}{r}$	$\frac{1}{m\mathbf{k}} \sim \frac{1}{mr^2}$	$\frac{1}{m^2}$	$\frac{\sigma_1 \cdot \sigma_2}{m^2}$
$1/v$	1	v	v

Loops



Loop graph with two insertions of the potential:

$$T\left(V^{(a)}V^{(b)}\right) \sim V^{(a+b)}$$

$$T\left(\frac{1}{\mathbf{k}^2} \quad \frac{1}{m\mathbf{k}}\right) \sim \frac{1}{\mathbf{k}^2}$$

Shows that the static potential is not the one to use for bound states. [Lattice computation using static sources should not be used for Υ.]

Computation method: Use dim reg. No double counting of gluon modes.

Power counting arguments imply

$$[A_S] = (mv)^{1-\epsilon}, \quad [A_U] = (mv^2)^{1-\epsilon}, \quad [\psi] = (mv)^{1-\epsilon},$$

Introduce $\mu^\epsilon \to \mu_S^\epsilon, \mu_U^\epsilon$, so that

$$g_S \mu_S^\epsilon, \quad g_U \mu_U^\epsilon, \quad V \mu_S^{2\epsilon}$$

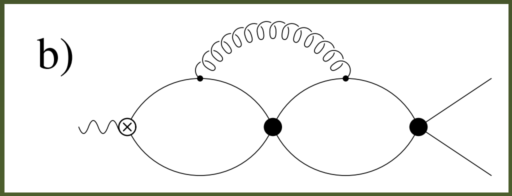
and *must have the correlation*, $\mu_S = m\nu$, $\mu_U = m\nu^2$. Power counting for operators implies BBL power counting for states.

Theory has a subtraction velocity, rather than a subtraction momentum.

Renormalization consistent: The running of $c_1(\nu)$, the coefficient of the $\bar{q}q$ production current, H energy levels.

$$\frac{1}{\epsilon}\left(\frac{1}{\epsilon}+\ln\frac{\mu_U m}{\mu_S^2}\right)$$

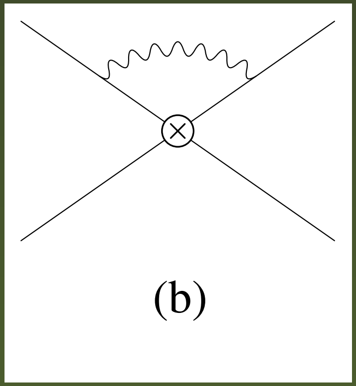
Can consistently renormalize if $\mu_U m = \mu_S^2$.

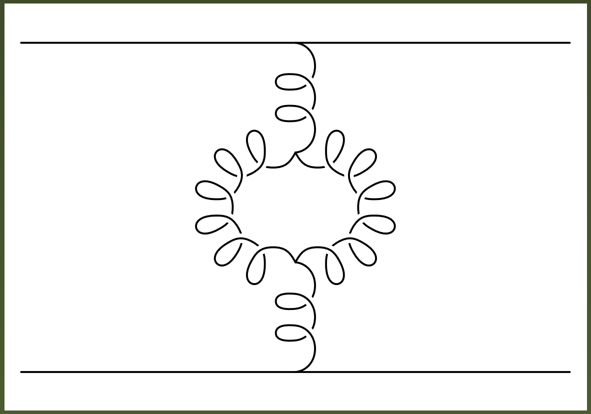


Can see this from the finite parts of the diagrams.

$$\ln \frac{E}{\mu_U}, \qquad \ln \frac{p}{\mu_S}$$

and logs minimized requires $\mu_U m = \mu_S^2$.



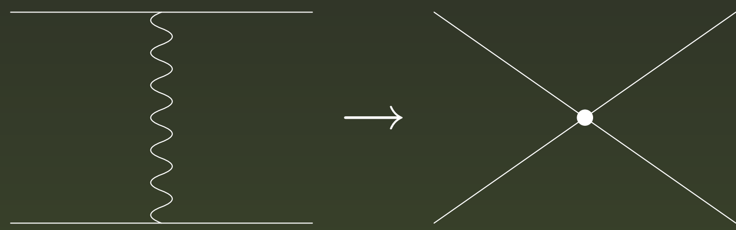


Potentials

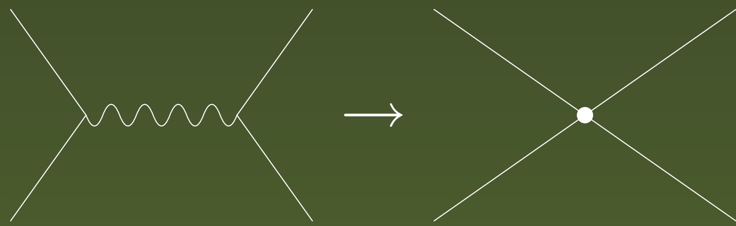
$$V(\mathbf{p},\mathbf{p}') = V^{(-1)} + V^{(0)} + V^{(1)} + V^{(2)} + \ldots$$

$$\begin{aligned}
V^{(-1)} &= \frac{U_c}{\mathbf{k}^2}, \\
V^{(0)} &= \frac{U_k}{|\mathbf{k}|}, \\
V^{(1)} &= U_2 + U_s\,\mathbf{S}^2 + \frac{U_r(\mathbf{p}^2+\mathbf{p}'^2)}{2\mathbf{k}^2} - \frac{i\mathbf{U}_\Lambda\cdot(\mathbf{p}'\times\mathbf{p})}{\mathbf{k}^2} \\
&\quad + U_t\left(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 - \frac{3\,\mathbf{k}\cdot\boldsymbol{\sigma}_1\,\mathbf{k}\cdot\boldsymbol{\sigma}_2}{\mathbf{k}^2}\right),
\end{aligned}$$

Matching Conditions



Only difference between Hydrogen and Positronium is annihilation contributions to the potentials, that first start at order $1/m^2$:

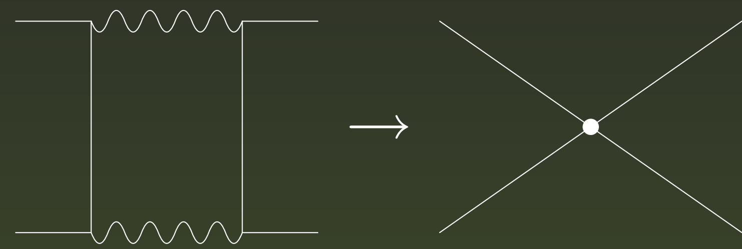


Coefficients

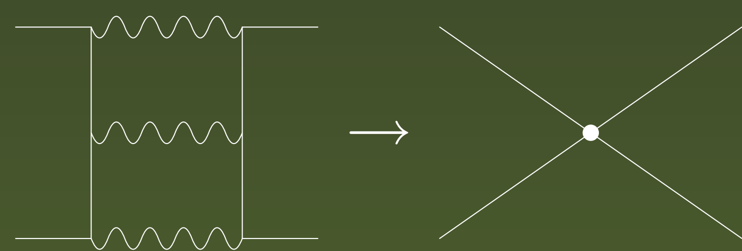
Particles of mass $m_{1,2}$ and charge $-e$, Ze

$$
\begin{aligned}
U_c &= -4\pi Z\alpha \\
U_2 &= \frac{\pi Z\alpha}{2}\left(\frac{1}{m_1}-\frac{1}{m_2}\right)^2 \\
U_s &= \frac{4\pi Z\alpha}{3m_1m_2}+\frac{\pi\alpha}{m_e^2} \\
U_r &= -\frac{4\pi Z\alpha}{m_1m_2} \\
U_k &= \frac{\pi^2Z^2m_r\alpha^2}{m_1m_2}
\end{aligned}
$$

Widths



$$U_2 + U_s\mathbf{S}^2 = -i\frac{\pi\alpha^2}{m_e^2}\left(2 - \mathbf{S}^2\right)$$



$$U_2 + U_s\mathbf{S}^2 = -i\frac{4\pi\alpha^3\left(\pi^2 - 9\right)}{9\pi m_e^2}\mathbf{S}^2$$

Renormalization Group Evolution

In B decay, one has M_W, m_b and $\Lambda_{\rm QCD}$

$M_W \to m_b$:

- Start with the electroweak theory
- At $\mu = M_W$, integrate out W and Z to get

$$\frac{4G_F}{\sqrt{2}} V_{cb}\ \bar{c}\,\gamma_\mu P_L\, b\ \bar{e}\,\gamma^\mu P_L\,\nu$$

- Integrate from $\mu = M_W$ to $\mu = m_b$ using RG equations. This sums powers of $\alpha_s \ln M_W/m_b$

$m_b \to \Lambda_{\rm QCD}$:

- Integrate out b modes at m_b (i.e. switch to HQET)
- Integrate m_b to some low scale using RG equations in HQET. This sums powers of $\alpha_s \ln m_b/\Lambda_{\rm QCD}$

At $\Lambda_{\rm QCD}$:

- Compute hadronic matrix elements at the low scale. No large logarithms are present in the matrix element.

Renormalization Group Evolution

Scales m, mv and mv^2 (analogous to M_W, m_b and $\Lambda_{\rm QCD}$):

Two-stage running: (Conventional method):

- Match to QED/QCD at $\mu = m$.
- Integrate μ from m to mv.
- Integrate out soft (mv) modes
- Integrate μ from mv to mv^2.
- Compute matrix elements

Velocity Renormalization Group

(Luke, Rothstein, A.M.; Stewart, A.M.)

One-stage running:

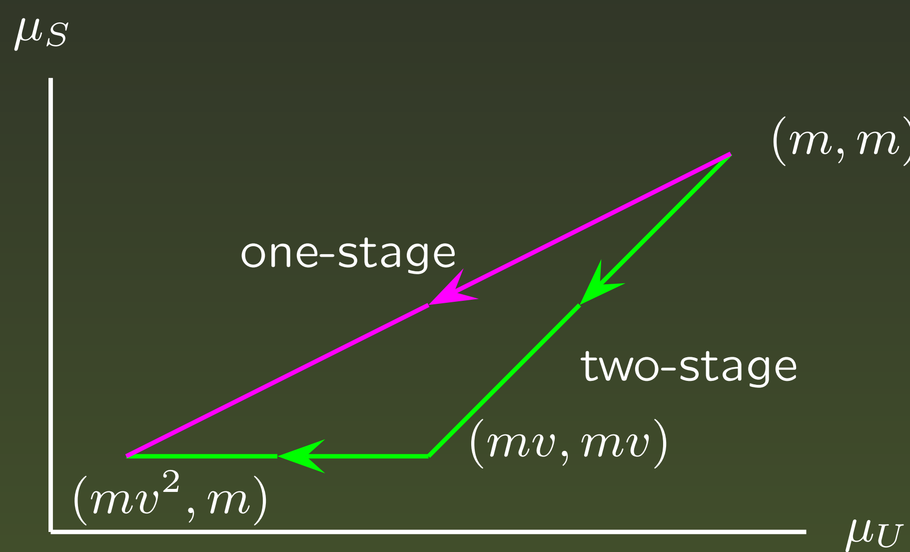
- Set (ν is a subtraction velocity)

$$\begin{aligned} \mu_S &= m\nu \\ \mu_U &= m\nu^2 \end{aligned}$$

- Match to QED/QCD at $\nu = 1$
- Integrate from $\nu = 1$ to $\nu = v$.
- Compute matrix elements

Run in velocity, not momentum

(with Soto, Stewart)



- Two methods give different answers, $\nabla \times \gamma \neq \mathbf{0}$.
- One-stage VRG agrees with explicit QED calculations.

Energy Series

Compute anomalous dimensions for the potentials:

$$\begin{aligned}
\gamma_{\rm LO} &: \quad \alpha^r \left(1 + \alpha \ln \alpha + \alpha^2 \ln^2 \alpha + \alpha^3 \ln^3 \alpha + \ldots\right) \\
\gamma_{\rm NLO} &: \quad \alpha^{r+1} \left(1 + \alpha \ln \alpha + \alpha^2 \ln^2 \alpha + \alpha^3 \ln^3 \alpha + \ldots\right) \\
\gamma_{\rm NNLO} &: \quad \alpha^{r+2} \left(1 + \alpha \ln \alpha + \alpha^2 \ln^2 \alpha + \alpha^3 \ln^3 \alpha + \ldots\right) \\
\gamma_{\rm N^3LO} &: \quad \alpha^{r+3} \left(1 + \alpha \ln \alpha + \alpha^2 \ln^2 \alpha + \alpha^3 \ln^3 \alpha + \ldots\right)
\end{aligned}$$

In QCD, the Coulomb potential (α_s^2) runs, so the N^3LO running of the Coulomb potential is as important as the leading order running of the hyperfine and spin-orbit potentials (α_s^4)

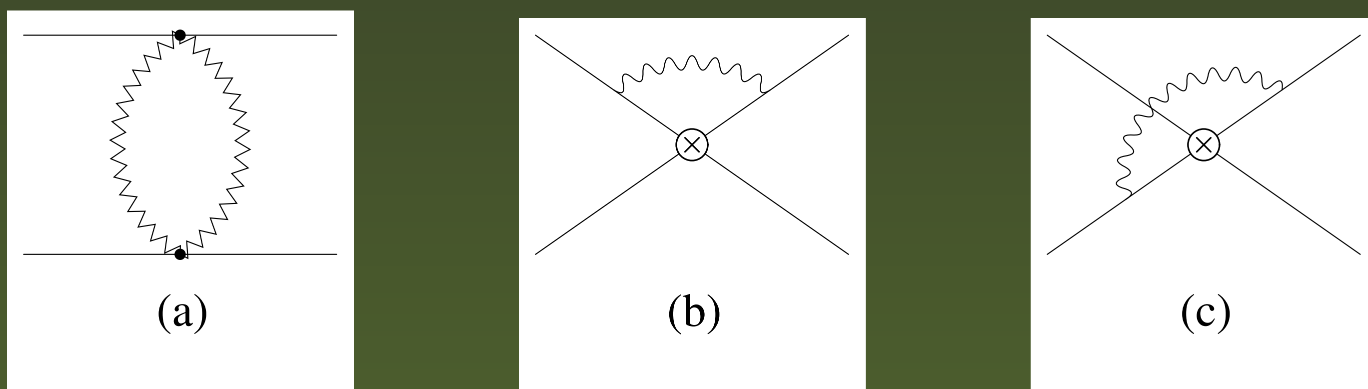
In QED, the Coulomb potential does not run, and there is no order α^3 contribution to the energy. So one can compute

$$\begin{array}{cccc} \alpha^5 \ln \alpha & \alpha^6 \ln^2 \alpha & \alpha^7 \ln^3 \alpha & \dots \\ \alpha^6 \ln \alpha & \alpha^7 \ln^2 \alpha & \alpha^8 \ln^3 \alpha & \dots \end{array}$$

energy series in QED by integrating the RG equations using the LO and NLO anomalous dimensions of the hyperfine and spin-orbit potentials (order αv potentials).

Compute γ

$$V^{(1)} = U_2 + U_s\,\mathbf{S^2} + \frac{U_r(\mathbf{p}^2 + \mathbf{p}'^2)}{2\mathbf{k}^2} - \frac{i\mathbf{U}_\Lambda \cdot (\mathbf{p}' \times \mathbf{p})}{\mathbf{k}^2}$$
$$+U_t\left(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 - \frac{3\,\mathbf{k}\cdot\boldsymbol{\sigma}_1\,\mathbf{k}\cdot\boldsymbol{\sigma}_2}{\mathbf{k}^2}\right),$$



$+ \ldots$

RG Equation

At NNLO,

$$\nu \frac{dU_k}{d\nu} = 0 \qquad \propto C_A \text{ in QCD}$$

$$\nu \frac{dU_2}{d\nu} = \frac{2\alpha}{3\pi}\left(\frac{1}{m_1} + \frac{Z}{m_2}\right)^2 U_c + \frac{14 Z^2 \alpha^2}{3 m_1 m_2} = \gamma_0 U_c$$

γ_0

$$\gamma_0 = \frac{2\alpha}{3\pi}\left(\frac{1}{m_1^2} + \frac{Z}{4m_1m_2} + \frac{Z^2}{m_2^2}\right)$$

γ_0 is a constant in QED, since α does not run

Integrate:

$$U_2(\nu) = U_2(1) + \gamma_0 U_c \ln \nu$$

Only a single term, so the NNLO series terminates

$\alpha^5 \ln \alpha$, $\alpha^6 \ln^2 \alpha$, $\alpha^7 \ln^3 \alpha$, $\alpha^8 \ln^4 \alpha$, $\ldots \alpha^{104} \ln^{100} \alpha$ $\ldots$

Bethe Lamb Shift

$$\begin{aligned}\Delta E &= \langle U_2 \rangle \\ &= \gamma_0 U_c \ln \nu \, |\psi(0)|^2 \\ &= -\frac{8 Z^4 \alpha^5 m_R^3}{3\pi n^3} \left(\frac{1}{m_1^2} + \frac{Z}{4 m_1 m_2} + \frac{Z^2}{m_2^2} \right) \ln Z\alpha ,\end{aligned}$$

H—(Bethe 1947)

$$|\psi(0)|^2 = \frac{(m_R Z\alpha)^3}{\pi n^3} \qquad n\text{S state}$$

N^3LO

$$
\begin{aligned}
\nu \frac{dU_{2+s}}{d\nu}\bigg|_{\mathrm{N^3LO}} &= \rho_{ccc}\, U_c^3 + \rho_{cc2}\, U_c^2 \left(U_{2+s} + U_r \right) \\
&+ \rho_{c22}\, U_c \left(U_{2+s}^2 + 2U_{2+s} U_r + \frac{3}{4} U_r^2 - 5 U_t^2 \mathbf{S^2} \right) \\
&+ \rho_{ck}\, U_c U_k + \rho_{k2}\, U_k \left(U_{2+s} + U_r/2 \right) \\
&+ \rho_{c3}\, U_c \left(U_3 + U_{3s} S^2 + \frac{1}{2} U_{rk} \right),
\end{aligned}
$$

where $U_{2+s} = U_2 + U_s \mathbf{S}^2$ and $\rho_{c22} = -m_R^2/4\pi^2$.

Can integrate RHS using NNLO values for U_i.

Only terms which run at NNLO are U_2 and U_3.

$$\int \text{const} = \ln \nu$$

$$\int \ln \nu = \frac{1}{2} \ln^2 \nu$$

$$\int \ln^2 \nu = \frac{1}{3} \ln^3 \nu$$

N^3LO series terminates after 3 terms

$$\alpha^6 \ln \alpha, \qquad \alpha^7 \ln^2 \alpha, \qquad \alpha^8 \ln^3 \alpha$$

Anomalous Dimensions

$$
\begin{aligned}
\rho_{ccc} &= -\frac{m_R^4}{64\pi^2}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)^2, & \rho_{c22} = -\frac{m_R^2}{4\pi^2}, \\
\rho_{cc2} &= -\frac{m_R^3}{8\pi^2}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right), & \rho_{c3} = \frac{2m_R}{\pi^2}, \\
\rho_{ck} &= \frac{m_R^2}{2\pi^2}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right), & \rho_{k2} = \frac{2m_R}{\pi^2}.
\end{aligned}
$$

Results for QED

$\alpha^8 \ln^3 \alpha$	Lamb	H	agree/new
		μ^+e^-, e^+e^-	new
	(no h.f.s.)		agree
$\alpha^4 \ln^3 \alpha$	(no $\Delta\Gamma/\Gamma$)		agree
$\alpha^7 \ln^2 \alpha$	Lamb	H, μ^+e^-, e^+e^-	agree
	h.f.s.	H, μ^+e^-, e^+e^-	agree
$\alpha^3 \ln^2 \alpha$	$\Delta\Gamma/\Gamma$	e^+e^- ortho and para	agree
$\alpha^6 \ln \alpha$	Lamb, h.f.s.	H, μ^+e^-, e^+e^-	agree
$\alpha^2 \ln \alpha$	$\Delta\Gamma/\Gamma$	e^+e^- ortho and para	agree

$\ln^3 \alpha$

$$\frac{1}{3}\,\gamma_0^2\,\rho_{c22}\,U_c^3(1)\,\ln^3\nu,$$

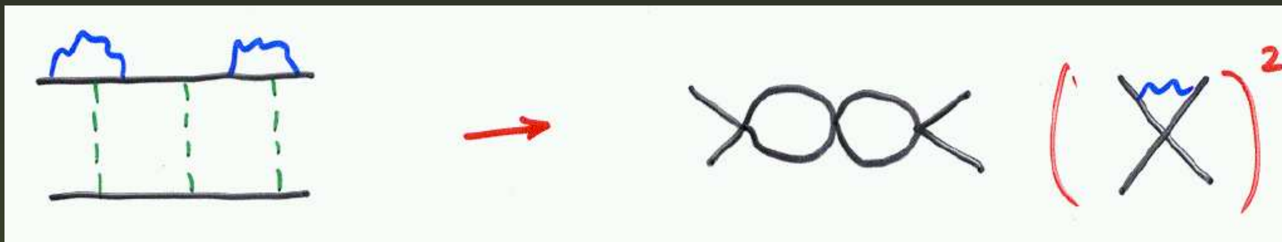
Lamb shift for the nS state (no HFS, Γ)

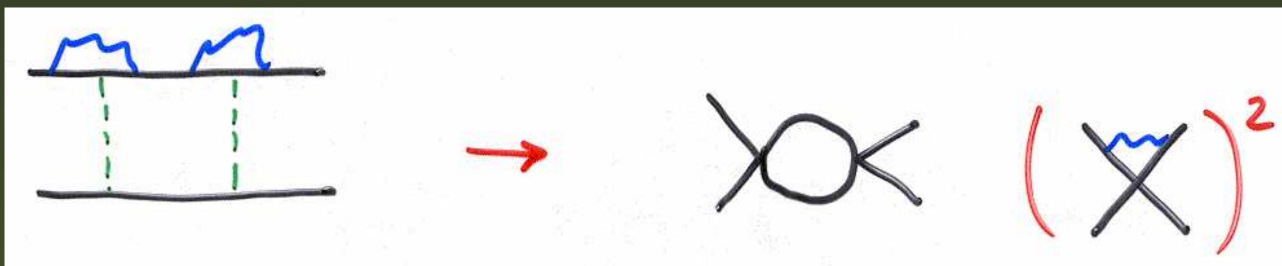
$$\Delta E = \frac{64 m_R^5 \alpha^8 Z^6}{27\pi^2 n^3}\left(\frac{1}{m_1^2} + \frac{Z}{4m_1 m_2} + \frac{Z^2}{m_2^2}\right)^2 \ln^3(Z\alpha)$$

$$= \frac{3 m_e \alpha^8 \ln^3\alpha}{8\pi^2 n^3} \quad \text{(positronium)}$$

(8 KHz for Hydrogen 2P–2S)

Karshenboim 1993	$a_{63} = -8/27$
Malampalli and Sapirstein PRL 1998	$a_{63} = -0.652$
Goidenko et al. PRL 1999	$a_{63} = -0.296$
Yerokhin hep-ph/0001327	$a_{63} = -0.652$





Now known: $\alpha^7 \ln \alpha$, $\alpha^8 \ln^2 \alpha$ ($m_p \to \infty$): first two terms of the next series and $\alpha^8 \ln \alpha$ ($m_p \to \infty$)

Could convert to anomalous dimensions and integrate.
Pachucki

Infinite Series

$$V(\nu = Z\alpha) = \exp\left[\frac{2\alpha}{3\pi}\left(\frac{1}{m_1} + \frac{Z}{m_2}\right)^2 \mathbf{k}^2 \ln Z\alpha\right]\frac{U_c(1)}{\mathbf{k}^2},$$

$$V(Z\alpha) = -\frac{Z\alpha}{r}\mathrm{Erf}\left[r\sqrt{\frac{3\pi}{8\alpha\ln[1/(Z\alpha)]}}\left(\frac{1}{m_1}+\frac{Z}{m_2}\right)^{-1}\right],$$

$\alpha^5 \ln Z\alpha(\alpha^3 \ln Z\alpha)^{n/2},\ n \geq 0.$

$$\nu \frac{dU_2}{d\nu} = \gamma_0 U_c + \rho_{c22} U_c U_2^2 .$$

$$U_2(\nu) = \frac{U_2(1) + \sqrt{\gamma_0/|\rho_{c22}|}\tanh\left[\sqrt{\gamma_0|\rho_{c22}|}\,U_c(1)\ln\nu\right]}{1 + \sqrt{|\rho_{c22}|/\gamma_0}U_2(1)\tanh\left[\sqrt{\gamma_0|\rho_{c22}|}\,U_c(1)\ln\nu\right]},$$

$\alpha^2 \ln\alpha(\alpha^3 \ln^2\alpha)^n.$

New Results: QED

1. Find a universal description of $\ln \alpha$ terms.
   A single RG equation gives the Lamb shift, hyperfine splitting and decay widths for Hydrogen, Muonium, and o,p-Positronium
2. Understand the structure of the series and why they terminate
   - LO series: $\alpha^5 \ln \alpha$
   - NLO series: $\alpha^6 \ln \alpha$, $\alpha^7 \ln^2 \alpha$, $\alpha^8 \ln^3 \alpha$
3. some ∞ series: $\alpha^2 \ln \alpha \left(\alpha^3 \ln^2 \alpha\right)^n$.
4. Computation of energy levels to order $\alpha^8 \ln^3 \alpha$

Experimental Situation

		Expt.(MHz)	Theory(MHz)	Agree?
H	Lamb	1057.845(9)	1057.833(6)	$< r_p^2 >$
			1057.814(6)	
	h.f.s	1420.405751767(9)	1420.399(2)	G_E, G_M
$\mu^+ e^-$	h.f.s	4463.302765(53)	4463.30267(27)	m_e/m_μ
$e^+ e^-$	Lamb	13012.4(1)	13012.41(8)	agree
	h.f.s	203389.10(74)	203391.69(16)	3σ
	$\Gamma_{\rm para}$	7990.9(1.7) μs^{-1}	7989.620(13) μs^{-1}	agree
	$\Gamma_{\rm ortho}$	7.0398(29) μs^{-1}	7.039968(10) μs^{-1}	6-9σ(?)
		7.0482(16) μs^{-1}		
		7.0514(14) μs^{-1}		

Potentials for $t\bar{t}$

(Hoang, Stewart, A.M.)

	$\mathcal{V}_r^{(s)}$	$\mathcal{V}_2^{(s)}$	$\mathcal{V}_s^{(s)}$	$\mathcal{V}_\Lambda^{(s)}$	$\mathcal{V}_t^{(s)}$
$\nu = 1$	-1.81	0	0.60	0.15	2.71
$\nu = v$	-1.39	0.61	0.53	0.16	3.11

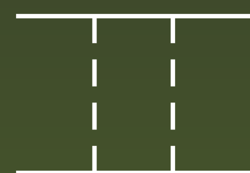
$$\mathcal{V}_2^{(1)}(\nu) = \frac{4\pi C_1}{\beta_0}\left[\alpha_s(m) - \alpha_s(m\nu)\right]\ln\left(\frac{m\nu}{m}\right) - \frac{16\pi C_1}{3\beta_0}\,\alpha_s(m)\ln\left[\frac{\alpha_s(m\nu)}{\alpha_s(m\nu^2)}\right]$$

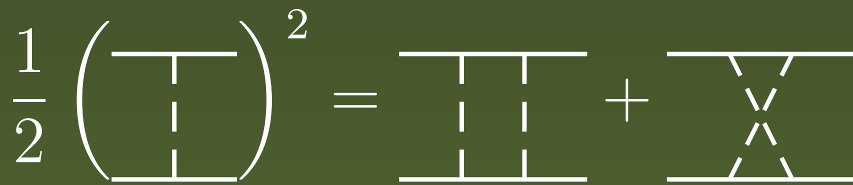
Error in calculation found by Pineda: Has to do with operator which has zero matching coefficient, but is induced by running. (Hoang, Stewart)

Soft indices are labels. Do the ultrasoft RG, and then do the soft integrals as sums over labels later. (Hoang, Stewart)

Deal with soft theory pinch singularities using the exponentiation theorem. (Gatheral, Frenkel, Taylor)

$$\int dk^0 \frac{1}{k^0 + i\epsilon} \frac{1}{k^0 - i\epsilon}$$



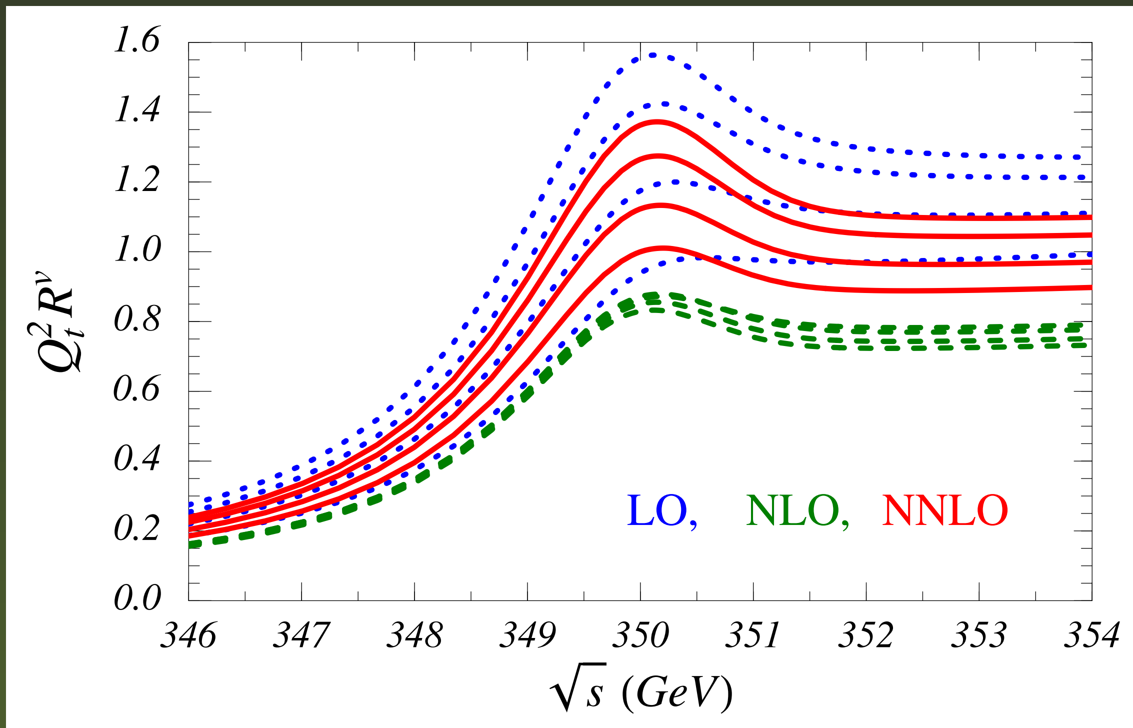


$$C_F \left(C_F - \frac{1}{2} C_A \right) - C_F^2$$

Cross sections

Hoang, A.M., Stewart, & Teubner

1S Mass - no running



$\nu =$

0.1

0.125

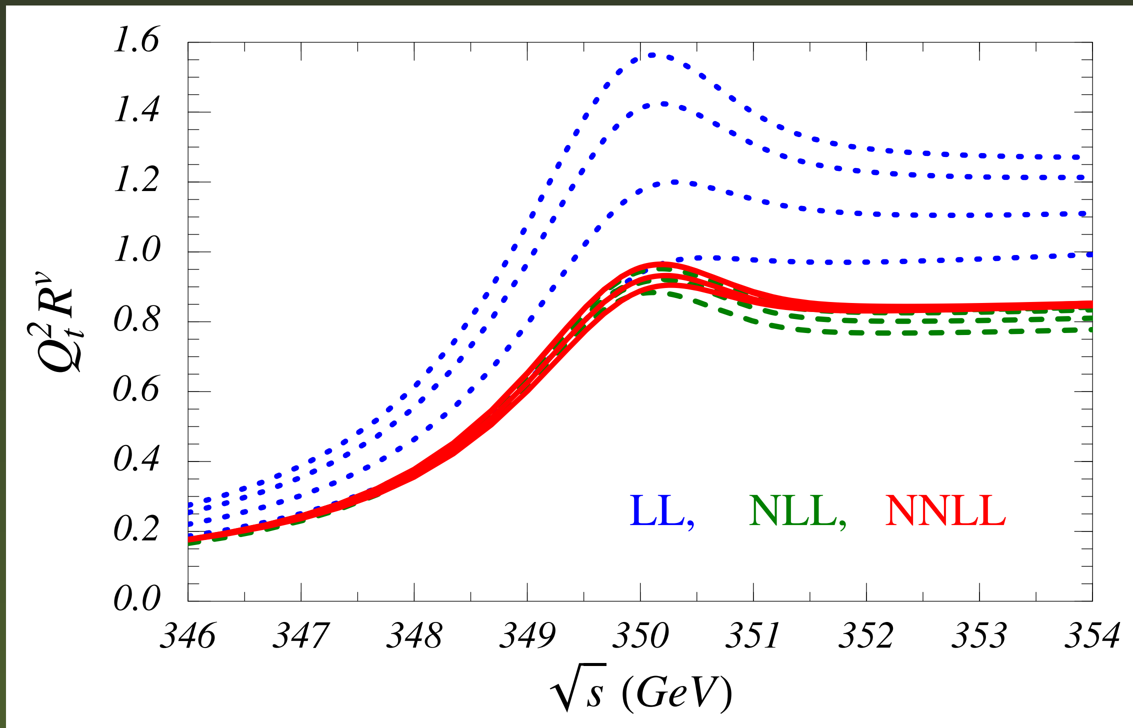
0.2

0.4

Cross sections

Hoang, A.M., Stewart, & Teubner

1S Mass - with running



$\nu =$

0.1

0.125

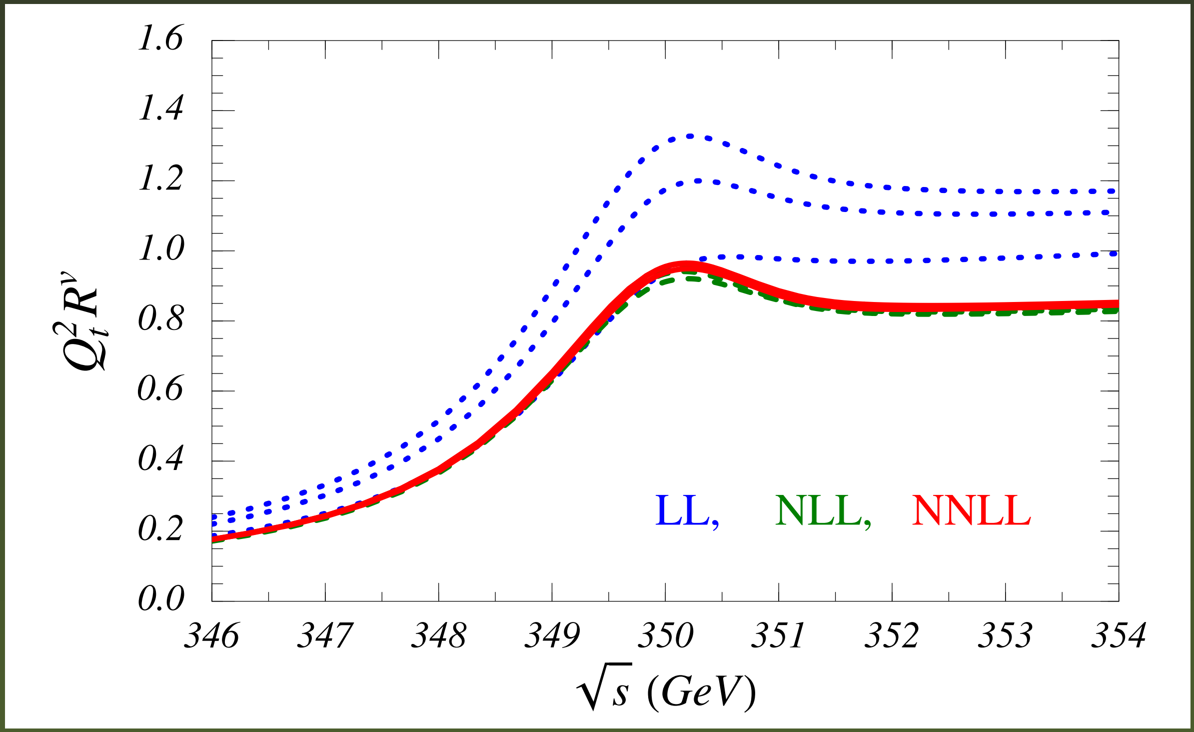
0.2

0.4

Cross sections

Hoang, A.M., Stewart, & Teubner

1S Mass - with running



$\nu =$

0.15

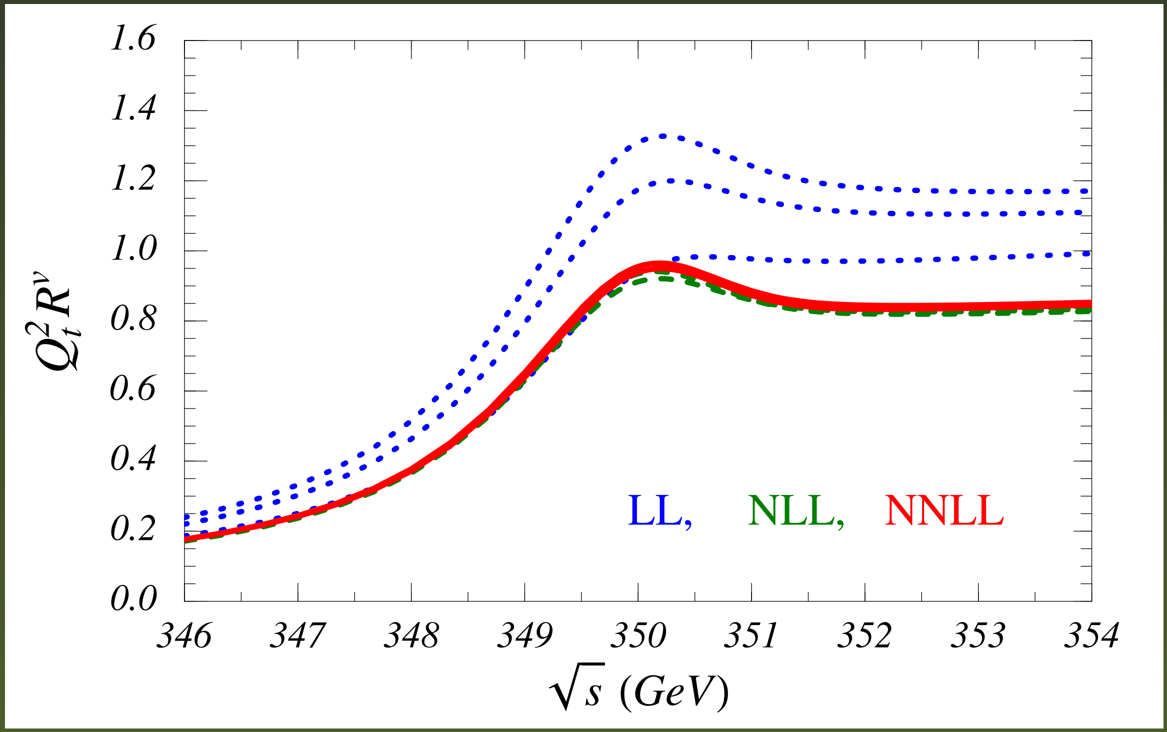
0.2

0.4

Cross sections

Hoang, A.M., Stewart, & Teubner

1S Mass - with running



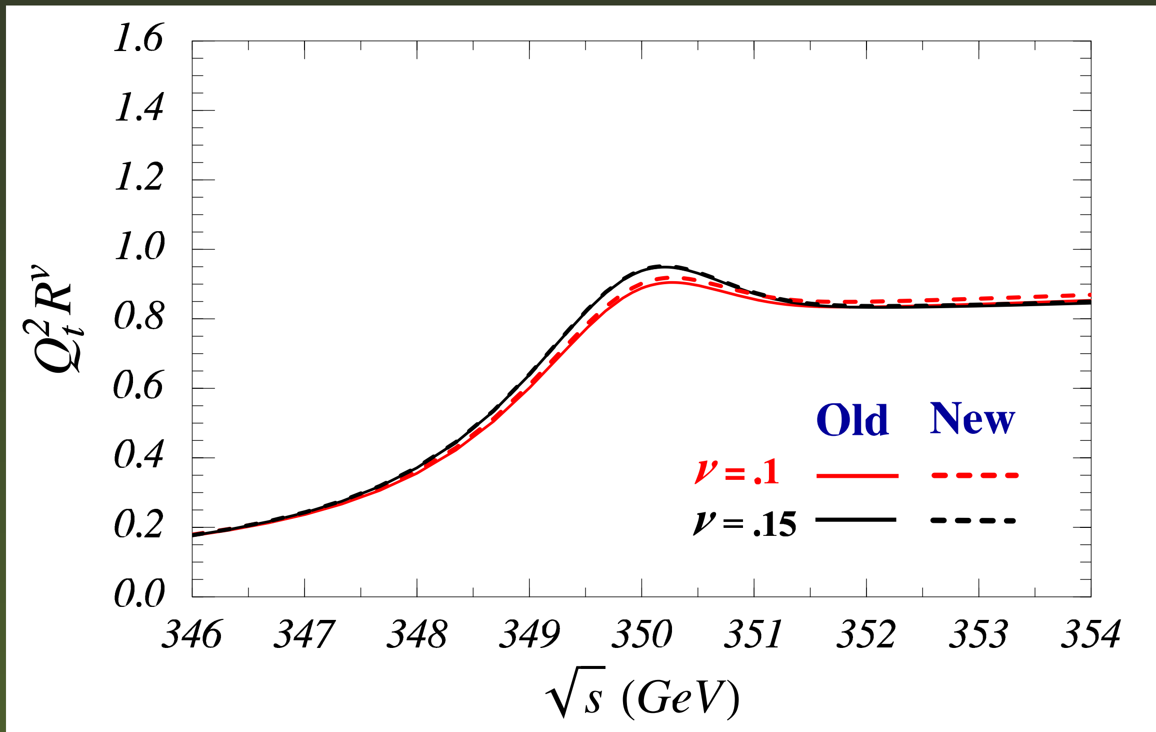
$\nu =$
0.15
0.2
0.4

- convergent expansion
- small scale uncertainties (conservative $\pm 3\%$)

Cross sections

Hoang, A.M., Stewart, & Teubner

1S Mass - with running



$\nu =$

0.1

0.15

0.2

0.4

Conclusions

1. Systematic way to separate scales in non-relativistic bound states
2. All large logs summed using the velocity RG
3. Universal description of QED logs.
   Checks: $\alpha^5 \ln \alpha$, $\alpha^6 \ln \alpha$, $\alpha^7 \ln^2 \alpha$, $\alpha^8 \ln^3 \alpha$
4. QCD: can distinguish $\alpha(mv)$ and $\alpha(mv^2)$. Both can appear in the same equation
5. Leads to an RG improved analysis of $\bar{t}t$ and Υ with much smaller uncertainties
6. VRG: apply to other problems with correlated scales