Towards Tau Decays at $O(\alpha_s^4)$

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(with Baikov and Chetyrkin)

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I Introduction:

α_s and m_s from τ-decays

II Technicalities:

correlators and multi-loops

III Results:

fixed order $\Rightarrow$ FAC/PMS $\Rightarrow$ contour-improvement

IV Summary

I Introduction

α_s and m_s from τ-decays

one of the most precise results for α_s

$$\frac{\Gamma(\tau \rightarrow h_{s=0}\nu)}{\Gamma(\tau \rightarrow l\bar{\nu}\nu)} = |V_{ud}|^2 S_{\rm EW} R_\tau = 3.492 \pm 0.016$$

$$R_\tau = 3\,(1 + \delta_P + \underbrace{\delta_{\rm EW}}_{\text{small}} + \underbrace{\delta_{\rm NP}}_{0.003 \pm 0.003})$$

$\delta_P = 0.207 \pm 0.007$ (exp)

$$\delta_P = \frac{\alpha_s}{\pi} + 5.202\,(\frac{\alpha_s}{\pi})^2 + 26.37\,(\frac{\alpha_s}{\pi})^3 + ?$$

[if we would use the last calculated term to estimate the error on α_s, the result becomes irrelevant!]

$\Rightarrow$ standard strategy:

1. estimate higher order term (FAC,PMS)
2. contour-improved perturbation theory (CIPT), to include dominant (?) higher order terms

$\mathcal{O}(\alpha_s^3)$, no $\mathcal{O}(\alpha_s^4)$-terms

$$\begin{aligned}
\alpha_s^{\mathrm{FOPT}}(M_\tau) &= 0.345 \pm (0.025|0.037) \\
\alpha_s^{\mathrm{CIPT}}(M_\tau) &= 0.364 \pm (0.012|0.021) \\
\alpha_s^{\mathrm{FOPT}}(M_Z) &= 0.1209 \pm (0.0024|0.0037) \\
\alpha_s^{\mathrm{CIPT}}(M_Z) &= 0.1229 \pm (0.0011|0.0020)
\end{aligned}$$

questions:

- are FAC/PMS supported by higher order calculations
- does the difference between fixed order (FOPT) and CIPT decrease upon inclusion of α_s^4 ?

aim: evaluate α_s^4

$\Rightarrow$ absorptive part of 5-loop correlators interesting results for m_s

$$\frac{\Gamma(\tau\to h_{s=1}\nu)}{\Gamma(\tau\to l\bar{\nu}\nu)} = |V_{us}|^2 S_{\text{EW}} \left[R_\tau(m=0) + \frac{m_s^2}{m_\tau^2}\,\text{terms}\right]$$

and spin 1 vs spin 0 hadronic final states separately

[and moments of spectral function: ...]

typical result [ALEPH; Chetyrkin et al.; Chen et al.; Maltman]

$m_s(1GeV) = 149 \pm 30$ MeV

problems with perturbative series $(a_s \equiv \alpha_s/\pi)$

$$\delta^{00}_{us,2} = -8\frac{m_s^2}{M_\tau^2}\Big(1 + 5.33\,a_s + 46.0\,a_s^2 + 284\,a_s^3$$

$$+ 0.75\,a_s^3 k_2^{[2]3} + a_s^4(723. + 0.25\,d_2^{[1]4} + 9.84\,k_2^{[2]3} + 0.75\,k_2^{[2]4})\Big)$$

$$= -8\frac{m_s^2}{M_\tau^2}(3.2 \pm 0.6)$$

II Technicalities: correlators and multiloops

$$R_\tau = 6i\pi \int_{|s|=M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left[\Pi^{[2]}(s) - \frac{2}{M_\tau^2}\Pi^{[1]}(s)\right]$$

where

$$i \int dx\, e^{iqx} \langle T[\, j_\mu(x)(j_\nu)^\dagger(0)\,]\rangle = g_{\mu\nu}\Pi^{[1]}(q^2) + q_\mu q_\nu \Pi^{[2]}(q^2)$$

massless case: $\Pi^{[1]} = -q^2\Pi^{[2]}$

$m_s \neq 0$: expand in $\frac{m_s^2}{q^2}$

$\frac{m^2}{q^2}$ of (V-A) correlator $\widehat{=}$ $\frac{m^2}{q^2}$ expansion of vector correlator
and scalar correlator [spin 0 contribution]

determination of α_s

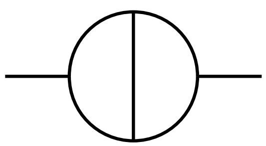
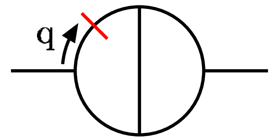
consider $m_s = 0$:

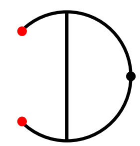
α_s^4 requires absorptive part of 5-loop correlator

$\widehat{=}$ divergent part $(1/\varepsilon)$ of 5-loop correlator

A finite part of 4-loop $\Rightarrow$ div. part of 5-loop

systematic, automatized algorithm (Chetyrkin)

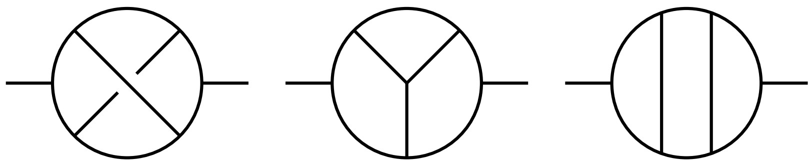
div  $\widehat{=} \int dq^2$ 

requires 

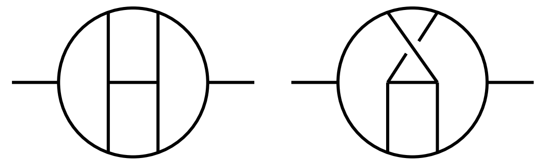
B finite part of 4-loop massless propagators difficult!

compare 3- and 4-loop calculation

3 topologies without insertions



11 topologies without insertion



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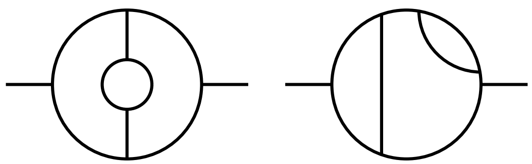
14 topologies with+without insertions



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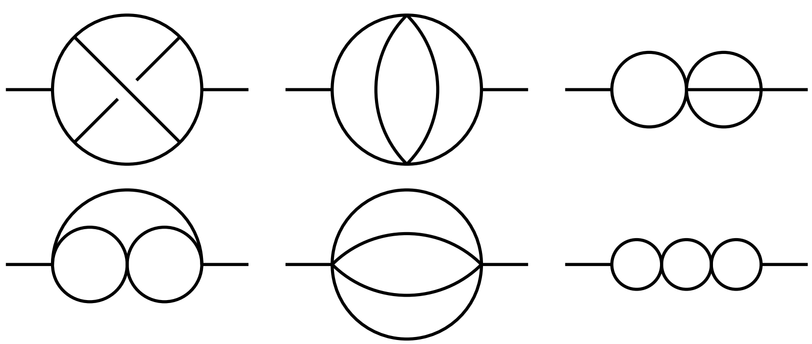
∼150 topologies with+without insertions

. . .

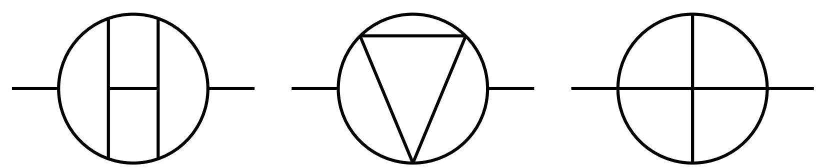
reduction to master integrals: MINCER

6 master integrals



reduction to master integrals ???

28 master integrals



. . .

MINCER: 3-loop [Larin, Tkatchov, Vermaseren]

recursion relations based on integration by parts identities!

reduction algorithm and program constructed “manually” for 14 topologies.

4-loop:

more complicated identities

$\sim$ 150 topologies . . .

straightforward generalization of MINCER difficult

$\Rightarrow$ fully automatized construction of program; new concept?

C Baikov: recursion relations can be solved “mechanically” in the limit of large dimension d:

consider amplitude f:

$f($topology, power of prop, $d)$
$= \sum_{\alpha=\text{masters}} C^{(\alpha)}($topology, power of prop, $d) \star f^{(\alpha)}(d)$

$f^{(\alpha)}$: 28 masters, analytically or numerically solvable

$C^{(\alpha)}$: rational function $\frac{P^n(d)}{Q^m(d)}$, to be calculated

expand $C^{(\alpha)}$:

$C^{(\alpha)} = \sum_k c_k^{(\alpha)}($topology, power of prop$)(1/d)^k + \dots$

sufficiently many terms $c_k^{(\alpha)} \Rightarrow C^{(\alpha)}$

$m,\ n$ depend on power of propagators!

evaluation of $c_k^{(\alpha)}$:

handling of polynomials of 9 variables of degree k

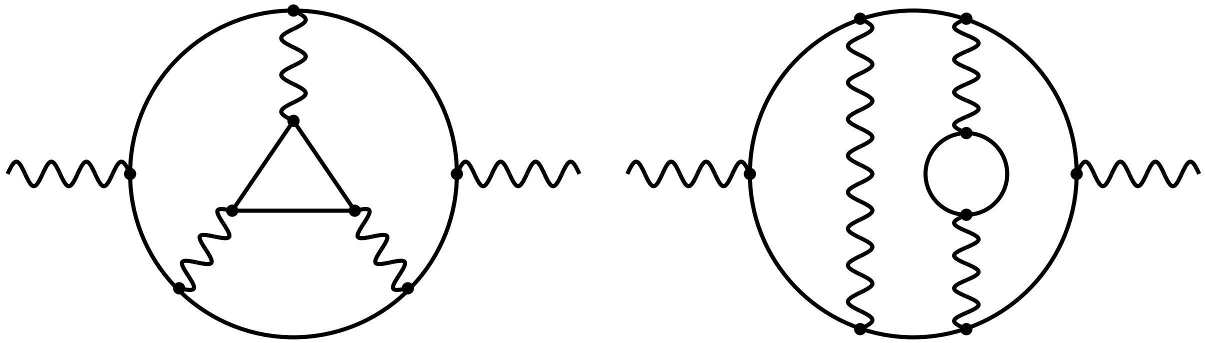
$\frac{(9+k)!}{9!\,k!}$ terms $\quad k = 40 \;\Rightarrow\; 2 \cdot 10^9$ terms

$\quad k = 24 \;\Rightarrow\; 4 \cdot 10^7$ terms $\quad$ (4 GB disk $\longrightarrow$ 40 GB)

weeks of runtime

addition information on structure of $P^n(d)$, $Q^m(d)$ may lead to drastic reduction of hardware requirements

status: four-loop n_f-terms done



$\Rightarrow$ leading and subleading n_f terms for $R_{e^+e^-}$, R_τ, m^2/s-terms:

$\alpha_s^4 n_f^3$ (renormalon chain)

$\alpha_s^4 n_f^2$ new

III Results

R_τ, $m = 0$

fixed order:

consider $D_0^{[1]}(Q^2) \equiv -\frac{3}{4} Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2} \Pi_0^{[1]}$

(Adler function, μ independent)

$$D_0^{[1]}(q^2) = 1 + a_s + a_s^2\,(-0.1153\, n_f + 1.986) \\ + a_s^3 \left(0.08621\, n_f^2 - 4.216\, n_f + 18.24\right) \\ + a_s^4 \left(-0.01009\, n_f^3 + 1.875\, n_f^2 + d_{0,1}^{[1]4}\, n_f + d_{0,0}^{[1]4}\right)$$

relation to FAC/PMS

n_f	$d_3^{\rm exact}$	$d_3^{\rm FAC}$	$d_3^{\rm PMS}$	$d_4^{\rm FAC/PMS}$	$d_5^{\rm FAC}$
3	6.371	5.604	6.39	27 ± 16	145 ± 100
4	2.758	4.671	5.26	8 ± 28	40 ± 160
5	-0.68	3.762	4.16	-8 ± 44	-3 ± 230

$O(\alpha_s^3)$ from $O(\alpha_s^2)$:

$18.24 - 4.216\, n_f + 0.08621\, n_f^2$ exact

$8.54 - 1.013\, n_f + 0.0116\, n_f^2$ FAC

$9.93 - 1.23\, n_f + 0.0125\, n_f^2$ PAC

- n_f^0 and n_f^1 contributions of comparable size
- sign and order of magnitude correct

  (better agreement for large corrections: m_s^2-terms!)
- total prediction significantly better than individual coefficients

  deviation of subleading term $\Rightarrow$ conservative error

$O(\alpha_s^4)$: based on (Kataev, Starshenko)

$$d_0^{[1]4} = 127.58 - 44.211\, n_f + 3.6439\, n_f^2 - 0.0100928\, n_f^3$$

use new input: $\alpha_s^4 n_f^2$-term

$$d_0^{[1]4}(\mathrm{FAC/PMS}, n_f = 3,4) = \boxed{105.7 - 31.8\, n_f} + 1.875\, n_f^2 - 0.01009\, n_f^3$$

$$d_0^{[1]4}(\mathrm{FAC/PMS}, n_f = 4,5) = \boxed{107.7 - 32.3\, n_f} + 1.875\, n_f^2 - 0.01009\, n_f^3$$

$$d_0^{[1]4}(\mathrm{FAC/PMS}, n_f = 3,5) = \boxed{106.4 - 32.0\, n_f} + 1.875\, n_f^2 - 0.01009\, n_f^3$$

$$\Rightarrow \quad \begin{array}{l} d_0^{[1]4}|_{n_f=3} = 27 \pm 16 \\ d_0^{[1]5}|_{n_f=3} = 145 \pm 10 \end{array}$$

Implication for α_s

with $\alpha_s^4 \rightarrow 0$

$$
\begin{aligned}
\alpha_s^{\rm FOPT}(M_\tau) &= 0.345 \pm (0.025|0.037) \\
\alpha_s^{\rm CIPT}(M_\tau) &= 0.364 \pm (0.012|0.021) \\
\alpha_s^{\rm FOPT}(M_Z) &= 0.1209 \pm (0.0024|0.0037) \\
\alpha_s^{\rm CIPT}(M_Z) &= 0.1229 \pm (0.0011|0.0020)
\end{aligned}
$$

with α_s^4 and α_s^5

Method	$\alpha_s(M_\tau)$	$\Delta\delta_P^{\rm exp}$	$\Delta\mu$	$\Delta d_0^{[1]4}$	$\Delta d_0^{[1]5}$
FOPT	$0.330 \pm 0.006 \pm 0.02$	0.006	0.019	0.0045	0.0011
CIPT	$0.354 \pm 0.009 \pm 0.006$	0.009	0.0036	0.0042	0.0019

$$\Rightarrow \quad \alpha_s^{\mathrm{FOPT}}(M_Z) = 0.1192 \pm 0.0007 \pm 0.002$$
$$\alpha_s^{\mathrm{CIPT}}(M_Z) = 0.1219 \pm 0.001 \pm 0.0006$$

uncertainty is reduced; difference between FOPT and CIPT remains!

[this difference is reduced for a fictitious heavy lepton of 3 GeV]

m_s^2-terms

expansion of $D^{[1]}$

$$D_2^{[1]} = m_s^2 \Big(1 + 1.667\, a_s + a_s^2 \{ -0.5736\, n_f + 12.57 \} + a_s^3 \Big\{ 0.1646\, n_f^2 - 14.31\, n_f + 149. \Big\} + a_s^4 \Big\{ -0.01563\, n_f^3 + 6.067\, n_f^2 + d_{2,1}^{[1]4}\, n_f + d_{2,0}^{[1]4} \Big\} \Big)$$

expansion of $\Pi^{[2]}$

$$\Pi_2^{[2]} = -4 m_s^2 \Big(1 + 2.333\, a_s + a_s^2 \{ -1.309\, n_f + 23.51 \} + a_s^3 \Big\{ 0.4647\, n_f^2 - 32.08\, n_f + k_{2,0}^{[2]3} \Big\} \Big)$$

FAC/PMS give reasonable estimates

for all α_s^3-terms and for $\alpha_s^4 n_f^2$-term

n_f	d_3^{exact}	d_3^{FAC}	d_3^{PMS}	$d_4^{\text{FAC/PMS}}$
3	107.5	89.86	91.17	1200 ± 400
4	94.37	79.13	80.11	950 ± 300
5	81.54	68.66	69.33	750 ± 200

n_f	k_3^{FAC}	k_3^{PMS}	k_4^{PMS}
3	199.1	200 ± 60	2200 ± 1500
4	171.2	170 ± 50	1800 ± 1100
5	144.7	145 ± 40	1400 ± 900

result (fixed order)

$$
\begin{aligned}
\delta^{00}_{us,2} &= -8\frac{m_s^2}{M_\tau^2}(1. + 5.33\,a_s + 46.0\,a_s^2 + 284\,a_s^3 \\
&+ 0.75\,a_s^3 k_2^{[2]3} + a_s^4(723. + 0.25\,d_2^{[1]4} + 9.84\,k_2^{[2]3} + 0.75\,k_2^{[2]4})) \\
&= -8\frac{m_s^2}{M_\tau^2}(3.2 \pm 0.6)
\end{aligned}
$$

result (contour improved)

$$
\begin{aligned}
\tilde{\delta}^{00}_{us,2} &= -8\frac{m_s^2}{M_\tau^2}(1.44 + 3.65\,a_s + 30.9\,a_s^2 + 72.2\,a_s^3 + 1.18\,a_s^3 k_2^{[2],3} \\
&\qquad + a_s^4(0.678\,d_2^{[1]4} + 1.06\,k_2^{[2]4})) \\
&= -8\frac{m_s^2}{M_\tau^2}(1.44 + 0.389 + 0.349 + 0.371 \pm 0.09 + 0.403) \\
&= -8\frac{m_s^2}{M_\tau^2}(2.95 \pm 0.4)
\end{aligned}
$$

spin 1 and spin 0 separately [moment (0,1)]

spin 1

$$\delta^{(1)01}_{us,2} = -5\frac{m_s^2}{M_\tau^2}(1.+4.83a+35.7\,a_s^2+276.\,a_s^3+a_s^4(1350+d_2^{[1]4})))$$

$$= -5\frac{m_s^2}{M_\tau^2}(1.+0.514+0.404+0.331+0.326)$$

$$= -5\frac{m_s^2}{M_\tau^2}(2.58\pm 0.33)$$

$$\tilde{\delta}^{(1)01}_{us,2} = -5\frac{m_s^2}{M_\tau^2}(1.37+2.55a+16.1\,a_s^2+135\,a_s^3+0.895\,a_s^4 d_2^{[1]4})$$

$$= -5\frac{m_s^2}{M_\tau^2}(1.37+0.271+0.182+0.163+0.137)$$

$$= -5\frac{m_s^2}{M_\tau^2}(2.12\pm 0.14)$$

spin 0

$$\delta^{(0)01}_{us,2} = \frac{3}{2}\frac{m_s^2}{M_\tau^2}(1. + 9.33a + 110\,a_s^2 + 1323\,a_s^3 + a_s^4(12200 + d_2^{[1]4} + 17.5k_2^{[2]3})$$

$$= \frac{m_s^2}{M_\tau^2}(1. + 0.992 + 1.24 + 1.59 + 2.16)$$

$$= \frac{3}{2}\frac{m_s^2}{M_\tau^2}(7.0 \pm 2)$$

$$\tilde{\delta}^{(0)01}_{us,2} = \frac{3}{2}\frac{m_s^2}{M_\tau^2}(3.19 + 11.2a + 126.\,a_s^2 + 289.\,a_s^3 + 6.63\,a_s^3 k_2^{[2]3} + a_s^4(2.71 d_2^{[1]4} + 7.76 k_2^{[2]4}))$$

$$= \frac{3}{2}\frac{m_s^2}{M_\tau^2}(3.19 + 1.19 + 1.42 + 1.94 + 2.6 \pm 1.31)$$

$$= \frac{3}{2}\frac{m_s^2}{M_\tau^2}(10.3 \pm 2.6)$$

rapidly increasing coefficients!

IV Summary

- α_s^4-terms for $R_{e^+e^-}$ and R_τ are important for improved determination of α_s
- in principle they can be calculated
- subleading n_f terms are available
- reasonable agreement with previous estimates

  $\Rightarrow$ improved value for α_s
- m_s: poor convergence of perturbative series