# Worldsheet Instantons, Torsion Curves, and Non-perturbative Superpotentials

Workshop on Recent Developments in
String Effective Actions and D-instantons
MPI Munich

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based on the series of articles with V. Braun, M. Kreuzer, and B. Ovrut

- Worldsheet Instantons, Torsion Curves, and Non-perturbative Superpotentials hep-th/0703134
- Worldsheet Instantons and Torsion Curves, Part A: Direct Computation hep-th/0703182
- Worldsheet Instantons and Torsion Curves, Part B: Mirror Symmetry. 0704.0449 [hep-th]

# Outline

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- Crucial ingredient: $B$-field ($A = \int_C \omega = \text{Area} + iB$).

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- Schoen Calabi-Yau / $\mathbb{Z}_3 \times \mathbb{Z}_3$ — Braun, Kreuzer, Ovrut, E.S.

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- Start with the $E_8 \times E_8$ heterotic string, embed an SU(4) gauge instanton in the $E_8$, breaking it to the commutant Spin(10) and add a $\mathbb{Z}_3 \times \mathbb{Z}_3$ Wilson line

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- Can construct a bundle $E$ on $X$ which yields the Standard Model matter. Braun, He, Ovrut, Pantev

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$C$ rational, isolated.

$$W_{tot} = \sum_{\beta\in H_2(X,\mathbb{Z})}\sum_{i=1}^{n_\beta} W(C_i)$$

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where $n_\beta$ is the number of holomorphic curves $C_i \in \beta$ in the cohomology class $\beta$.

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- Question 2: Is there an $X$ with curves $C_i$ such that $n_\beta = 1$ ?

# Outline

# Schoen's Calabi-Yau manifold

- $dP_9$ surface (=rational elliptic surface): Take two cubics in $\mathbb{P}^2$. Generically, they intersect in 9 points. Blow these 9 points up, i.e. glue in $\mathbb{P}^1 \Rightarrow B$.

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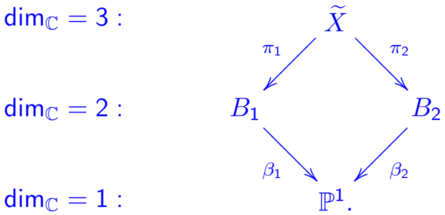

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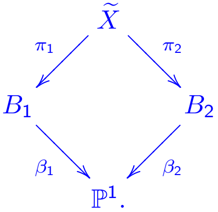


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$$\begin{array}{lccc} \dim_{\mathbb{C}} = 3 : & & \widetilde{X} & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ \dim_{\mathbb{C}} = 2 : & B_1 & & B_2 \\ & \searrow_{\beta_1} & & \swarrow_{\beta_2} \\ \dim_{\mathbb{C}} = 1 : & & \mathbb{P}^1 . & \end{array}$$

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- Other realization: $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of $T^6$

# $\mathbb{Z}_3 \times \mathbb{Z}_3$ quotient of Schoen's Calabi-Yau

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- The defining equations allow for three complex parameters $\lambda_1$, $\lambda_2$, $\lambda_3$ respecting the free $\mathbb{Z}_3 \times \mathbb{Z}_3$ action on $\widetilde{X}$: $(\zeta = e^{2\pi i/3})$

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- $\widetilde{X}$ is self-mirror! Crucial for the computation of $n_\beta$.

# Free quotients in toric geometry

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\begin{array}{ccccc}
\widetilde{X} \subset \mathbb{P}_{\Delta^*} & \Delta^* \subset N & \widetilde{X}^* \subset \mathbb{P}_{\nabla^*} & \nabla^* \subset M & (19,19) \\
\downarrow \langle g_1 \rangle & \downarrow \langle g_1 \rangle & \downarrow \langle g_1^* \rangle & \downarrow \langle g_1^* \rangle & \\
\overline{X} \subset \mathbb{P}_{\bar{\Delta}^*} & \bar{\Delta}^* \subset \bar{N} & \overline{X}^* \subset \mathbb{P}_{\bar{\nabla}^*} & \bar{\nabla}^* \subset \bar{M} & (7,7) \\
\downarrow \langle g_2 \rangle & & \downarrow \langle g_2^* \rangle & & \\
X & & X^* & & (3,3)
\end{array}
$$

# Free quotients in toric geometry

- Discrete quotients by $G$ of toric varieties respecting the torus action are always described by lattice refinements $N \to \bar{N}$, such that $N/\bar{N} = G$.
- On compact toric varieties the action can never be free, on complete intersections it can

$$
\begin{array}{ccccccccc}
\widetilde{X} \subset \mathbb{P}_{\Delta^*} & & \Delta^* \subset N & & \widetilde{X}^* \subset \mathbb{P}_{\nabla^*} & & \nabla^* \subset M & & (19,19) \\
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X & & & & X^* & & & & (3,3)
\end{array}
$$

- Only $\widetilde{X}, \overline{X}, \widetilde{X}^*, \overline{X}^*$ are complete intersections, $X, X^*$ are not.

Braun, Kreuzer, Ovrut, E. S.

# Fundamental group

- The only fixed points in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$ of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ action lie outside $\widetilde{X}$, hence the action is free on $\widetilde{X}$

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$$H_1(X) = \frac{\pi_1(X)}{[\pi_1(X), \pi_1(X)]} \Rightarrow H_1(X) = \mathbb{Z}_3 \oplus \mathbb{Z}_3.$$

# Curves of degree $(3, 0, 0)$

$$x_0^3 + x_1^3 + x_2^3 = 0 = x_0 x_1 x_2, \quad y_0^3 + y_1^3 + y_2^3 = 0 = y_0 y_1 y_2.$$

Two cubics intersect in 9 points $\Rightarrow 9 \cdot 9 = 81$ genus-0 curves

$$\widetilde{C}_{ij} : \mathbb{P}^1 \to \widetilde{X},\ [t_0 : t_1] \mapsto \Big( [x_0^{(i)} : x_1^{(i)} : x_2^{(i)}], [t_0 : t_1], [y_0^{(j)} : y_1^{(j)} : y_2^{(j)}] \Big)$$

on $\widetilde{X}$, each wrapping the $\mathbb{P}^1$. The $\mathbb{Z}_3 \times \mathbb{Z}_3$ group action identifies them in 9-tuples and, hence, they define 9 genus-0 curves

$$\{s_0, \ldots, s_8\} = \Big\{ \widetilde{C}_{ij} / (\mathbb{Z}_3 \times \mathbb{Z}_3) \Big\}$$

on the quotient threefold $X$.

# Quotient homology

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- Non-trivial torsion classes represented by $s_i - s_0$, $i = 1, \ldots, 8$.
- In general, the coinvariant homology is only one ingredient in the Cartan-Leray spectral sequence. Here, this is sufficient $\Rightarrow$

$$H_2\big(X, \mathbb{Z}\big) = H_2\big(\widetilde{X}, \mathbb{Z}\big)_{\mathbb{Z}_3\times\mathbb{Z}_3} = \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3.$$

In summary:

$$h^{p,q}(X) = \begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 3 & & 0 & \\ 1 & & 3 & & 3 & & 1, \\ & 0 & & 3 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix} \qquad H_i(X,\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=6 \\ 0 & i=5 \\ \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 & i=4 \\ \mathbb{Z}^8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 & i=3 \\ \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 & i=2 \\ \mathbb{Z}_3 \oplus \mathbb{Z}_3 & i=1 \\ \mathbb{Z} & i=0. \end{cases}$$

# Outline

- A-model computation: Analytic answer for the prepotential by counting the curves directly, but only to first order in $t_1$
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- Straightforward to implement the second $\mathbb{Z}_3$ quotient to obtain $\mathcal{F}_B^{(0)}(X^*, z)$. Braun, Kreuzer, Ovrut, E.S.

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- However, this only requires that $H \in H^3(X,\mathbb{Z})$ vanishes in $H^3(X,\mathbb{R})$.
- $B$ need not be globally defined $\Rightarrow \int_\beta B$ ill-defined.
- View the instanton contribution to the path integral $e^{iS}$ as an abstract map $e^{iS} : H_2(X,\mathbb{Z}) \to \mathbb{C}^*$. Aspinwall, Morrison

# Instanton numbers

- Define this map by the image of the generators:

$$H_2(X,\mathbb{Z})_{\text{free}} = \mathbb{Z}^3 : \qquad e^{iS}(s_0) = p = e^{2\pi i t^1},\ q = e^{2\pi i t^2},\ r = e^{2\pi$$

$$H_2(X,\mathbb{Z})_{\text{tors}} = \mathbb{Z}_3 \oplus \mathbb{Z}_3 : \quad b_1, b_2 \qquad b_1^3 = 1, b_2^3 = 1$$

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- The instanton factor is then

$$e^{iS}(\beta) = p^{n_1} q^{n_2} r^{n_3} b_1^{m_1} b_2^{m_2}.$$

for any curve class

$$\beta = (n_1, n_2, n_3, m_1, m_2) \ \in \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 = H_2(X,\mathbb{Z})$$

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- Hence

$$\mathcal{F}^{(0)}(X,t) = \sum_{\beta \in H_2(X,\mathbb{Z})} n_\beta \; \mathrm{Li}_3\left(p^{n_1} q^{n_2} r^{n_3} b_1^{m_1} b_2^{m_2}\right).$$

# Instanton numbers and modular forms

Closed form for the prepotential:

$$\mathcal{F}^{(0)}(X,p,q,r,b_1,b_2)\Big|_{p^n} = \frac{p^n}{8^{n-1}} \left( \sum_{i,j\in\mathbb{Z}_3} b_1^i b_2^j \right) \Big( P(q)^4 P(r)^4 \Big)^n M_{2n-2}(q,r)$$

if $n$ is not a multiple of 3 and, slightly weaker, that

$$\mathcal{F}^{(0)}(X,p,q,r,1,1)\Big|_{p^n} = \frac{9p^n}{8^{n-1}} \Big( P(q)^4 P(r)^4 \Big)^n M_{2n-2}(q,r)$$

if $n$ is a multiple of 3. Here, $P(q)$ is the usual generating function of partitions

$$P(q) = \sum_{i=0}^{\infty} p(i) q^i = \frac{q^{\frac{1}{24}}}{\eta(\frac{1}{2\pi i}\ln q)}.$$

The $M_{2n-2}$ are modular forms of weight $(2n-2, 2n-2)$ for $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ and can be written in terms of polynomials in the Eisenstein series $E_2(q)$, $E_4(q)$, $E_6(q)$ and $E_2(r)$, $E_4(r)$, $E_6(r)$, starting with

$$M_{-2}(q,r) = 0$$

$$M_0(q,r) = 1 \qquad \text{(also obtained analytically from A-model computation)}$$

$$M_2(q,r) = E_2(q)E_2(r)$$

$$M_4(q,r) = \tfrac{13}{108}E_4(q)E_4(r) + \tfrac{1}{4}\Big(E_4(q)E_2(r)^2 + E_2(q)^2E_4(r)\Big) + \tfrac{7}{4}E_2(q)^2E_2(r)^2$$

$$\begin{aligned} M_6(q,r) = {} & \tfrac{1}{27}E_6(q)E_6(r) + \tfrac{13}{54}\Big(E_6(q)E_4(r)E_2(r) + E_4(q)E_2(q)E_6(r)\Big) \\ & + \tfrac{1}{6}\Big(E_6(q)E_2(r)^3 + E_2(q)^3E_6(r)\Big) + \tfrac{79}{108}E_4(q)E_2(q)E_4(r)E_2(r) \\ & + \tfrac{5}{4}\Big(E_2(q)^3E_4(r)E_2(r) + E_4(q)E_2(q)E_2(r)^3\Big) + \tfrac{47}{12}E_2(q)^3E_2(r)^3 \end{aligned}$$

# Outline

Answer to Question 1:

$n^X_{(1,n_2,n_3,0,0)}$

| $n_2 \backslash n_3$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 4 | 14 | 40 | 105 |
| 1 | 4 | 16 | 56 | 160 | |
| 2 | 14 | 56 | 196 | | |
| 3 | 40 | 160 | | | |
| 4 | 105 | | | | |

$n^X_{(1,n_2,n_3,m_1,m_2)}$, $(m_1, m_2) \neq (0,0)$

| $n_2 \backslash n_3$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
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$n^X_{(2,n_2,n_3,0,0)}$

| $n_2 \backslash n_3$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | −2 | −28 | −192 |
| 1 | −2 | 32 | 440 | |
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$n^X_{(3,n_2,n_3,0,0)}$

| $n_2 \backslash n_3$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | **3** | **36** |
| 1 | **3** | **108** | |
| 2 | **36** | | |

$n^X_{(3,n_2,n_3,m_1,m_2)}$, $(m_1, m_2) \neq (0,0)$

| $n_2 \backslash n_3$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | **0** | **27** |
| 1 | **0** | **81** | |
| 2 | **27** | | |

Table: Instanton numbers $n^X_{(n_1,n_2,n_3,m_1,m_2)}$ computed by mirror symmetry.

Answer to Question 2: Expanding $\mathcal{F}^{(0)}(X, p, q, r, b_1, b_2)$ we find

$$n_{(1,0,0,m_1,m_2)} = 1, \qquad \forall\, m_1, m_2 \in \mathbb{Z}_3.$$

Furthermore, these curves have normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Hence, there are indeed 9 smooth rigid rational curves $C$ which are alone in their homology class.

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$$\frac{\text{Pfaff}\left(\bar{\partial}_{E(-1)}(C)\right)}{\left(\det \bar{\partial}_{\mathcal{O}(-1)}(C)\right)^2}$$

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- Applications to E1-instantons in type I theories