Entropy and Instantons

München: November 16 2007

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universitätbonn

Micro States of Super Symmetric Black holes

- Strominger and Vafa 96: BH in 5d, MS from heterotic string on $K3 \times S^1$
- Beckenridge, Myers, Peet, Vafa 96: Spinning BH in 5d; MS from M-theory on $K3$ (or $T4$) $\times T^2$
- Katz, Vafa, AK 99: Spinning BH in 5d with $\frac{1}{2}$ SUSY, MS from M-theory on general Calabi-Yau M. Proposal: as it involved solving topological String theory on compact C-Y M.

- Review of BMPV and KKV proposal.

- Topological String theory:
  - holomorphicity,
  - boundary conditions,
  - modularity → solutions

    Huang, AK: `hep-th/0605195`, Aganagic, Bouchard, AK: `hep-th/0607100`, Huang, Quackenbush, AK: `arXiv:hep-th/0612125`, Grimm, Mariño, Weiss, AK: `arXiv:hep-th/0702187`

- Application of the results to the KKV proposal

  Huang, Mariño, Tavanfar, AK, `arXiv:0704.2440 [hep-th]`.

Macroscopic black hole description

Black hole has

Charge: $Q \in H_2(M, \mathbb{Z})$

Spin: $J \in SU(2)_L \in SO(4)$

With the effective action BMPV find a black hole solutions whose entropy ($\frac{1}{4}$ horizon area) is (BMPV96)

$$\boxed{S_0 = 2\pi \sqrt{\mathcal{Q}_g - J^2}}$$

Naively the graviphoton charge $\mathcal{Q}_g(moduli)|_{Horizon}$

could depend on values of vector moduli in the asymptotic region.

5D attractor phenomenon (damping term in attractor "flow" eq.) ensures

$$\boxed{\mathcal{Q}_g|_{Horizon} = \left(\frac{2}{9\kappa}\right)^{\frac{1}{3}} d}$$

For simplicity we set $\dim(H_2(M,\mathbb{Z})) = 1$ so that $Q = d\,(egree) \in \mathbb{Z}$. κ is triple intersection

$$\kappa := \int_M \omega_D^3 = D \cdot D \cdot D.$$

For $Hol(M) = SU(3)$ there are the following higher derivatives terms in LEA R_+^2, $R_+^2 F_+^{2g-2}$. The corrections to S_0 have been calculated

$$\boxed{S_1 = \frac{\pi}{24}(c_2 \cdot D)\left(\frac{6}{\kappa}\right)^{\frac{1}{3}} \sqrt{d^3 - J^2}\left(\frac{1}{d} + \frac{J^2}{3d^4}\right)}$$

Guica,L.Huang, Li, Strominger: `hep-th/0605195`, ...

$$\boxed{S_g \propto \chi \left(\frac{2d^3}{9\kappa}\right)^{\frac{1}{2}+\frac{g}{3}}}$$

Cardoso, de Wit, Mohaupt: `hep-th/9812083`, Gaiotto, Strominger, Yin, `hep-th/0503217`, ...

Lead to precise 5D spinning black hole problem. Identify microscopic states that reproduce for $Q = d \gg 1$ and $Q = d \gg J$

$$\boxed{S = b_0 d^{\frac{3}{2}} + b_1 d^{\frac{1}{2}} + b_2 d^{-\frac{1}{2}} + \dots}$$

$$\boxed{b_j(J, \kappa, c_2 \cdot D, \chi)}$$

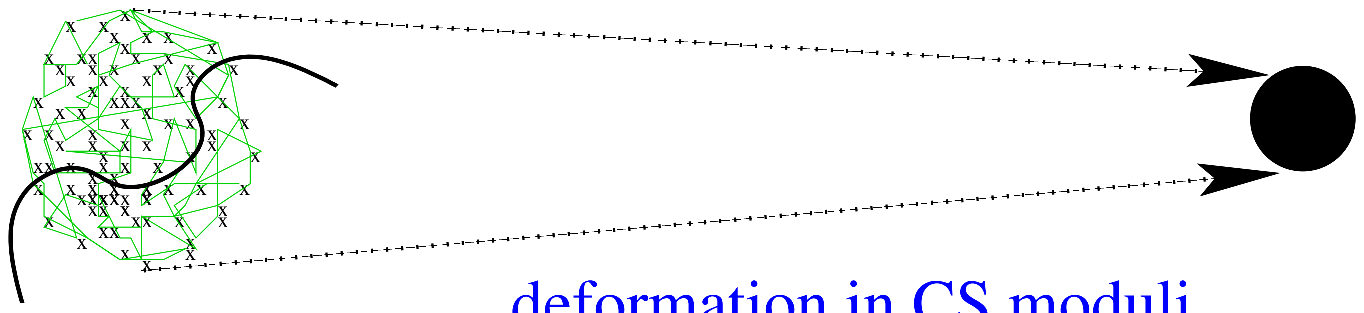
$$\boxed{\kappa := \int_M \omega_D^3, \quad c_2 \cdot D := \int_M \omega_D \wedge c_2, \quad \chi := \int_M c_3}$$

Precisely the data that according to R. Wald classify topological type of CY M.

Microscopic state description

Entropy:

$$\boxed{S = \log(\#quantum\ states)}$$



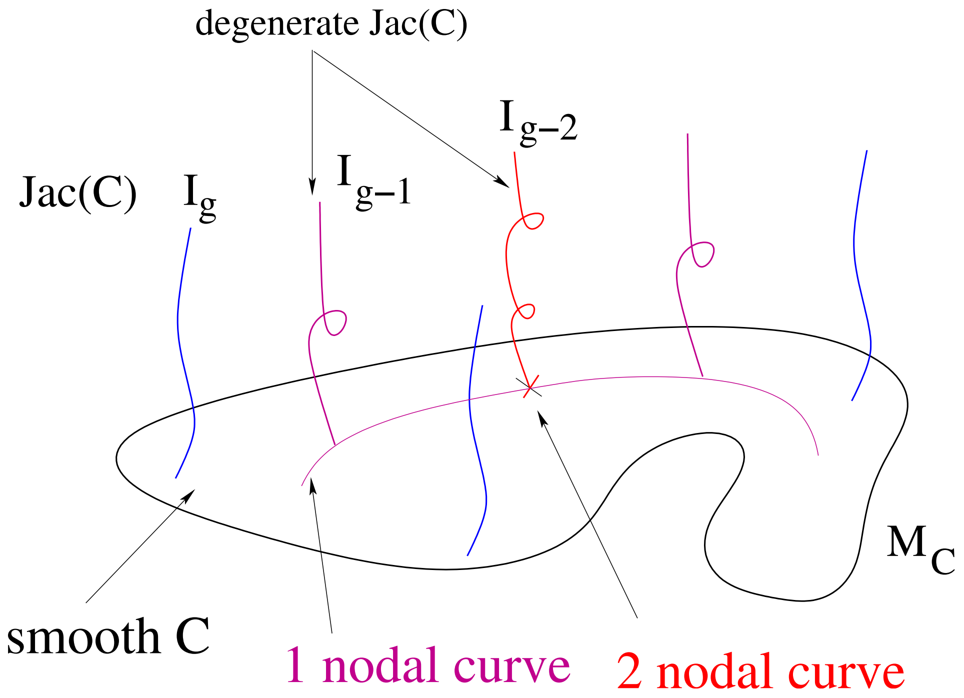
D2D0 brane system
N–> ∞
& open Strings

Black Hole

deformation in CS moduli
does not change the number
of relevant quantum states

Find the moduli space $\mathcal{M}_d$ of the solitonic object that creates the BH in 5d

- Soliton: M2 brane wrapped around the holomorphic curve $\mathcal{C}$ in the Q class in $H_2(M,\mathbb{Z})$ and $S^1_{11\,A}$.
- Decoupling: Problem depends on VM → decouples from IIA dilaton (HM) → $g_{II\,A}$ can set to any value → System: IIA $D2D0$ boundstate.
- Geometrization: System $D2D0$. If $\mathcal{C}_g$ is fixed genus g curve the $D2$ moduli are flat the $U(1)$-connections on $\mathcal{C}_g$ parametrising $Jac(\mathcal{C}_g)$. In addition $\mathcal{C}$ can be moved in M with deformation space $\mathcal{M}_{\mathcal{C}}$.



$$\boxed{\mathcal{M}_d = \{Jac(\mathcal{C}_g) \to \mathcal{M}_{C\ d}\}}$$

General microstates would be

$$\# BPS(d) = H^*(\mathcal{M}_d)$$

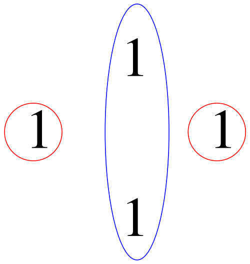
- Problem: $h^{p,q}(\mathcal{M}_d)$ varies with complex structure of M.

- Task: Relate (p,q) to (j_L, j_R) of 5dLG $SU(2)_L \times SU(2)_R$

- Solution: $\mathcal{M}_d$ has to Kählerforms (ω_L, ω_R) and therefore two Lefshetz actions $SU(2)_L \times SU(2)_R$ giving

$$H^*(\mathcal{M}_d) = \bigoplus_{J_L, J_R} H_{J_L, J_R}(\mathcal{M}_d)$$

SU(2) Lefshetz decomp. of T^2 cohomology



$= 2[0] + \left[\frac{1}{2}\right]$

- We are only interested in the $SU(2)_L \in SO(4)$ the spin content. Build CS-invariants n_d^g by tracing over the J_R.

$$\boxed{\sum_{J_R} \dim(H_{J_L,J_R}(\mathcal{M}_d))(-1)^{2J_R}(2J_R+1)\,[j_L] = \sum_{r=0^g} n_d^r I_r}$$

with $I_g = \left(2[0] + \left[\frac{1}{2}\right]\right)^g$. Since $\mathrm{Jac}(C_g) \sim (T^2)^g$ the hightest spin come from highest genus and is $n_d^g = e(\mathcal{M}_{\mathcal{C}})$ Gopakumar, Vafa 1998, ...

Microscopic entropy formula (KKV)

$$\boxed{\begin{aligned} S(d,2J) =&\ \log(\Omega(Q,J)) \\ \approx&\ b_0 d^{\frac{3}{2}} + b_1 d^{\frac{1}{2}} + b_2 d^{-\frac{1}{2}} + \ldots \end{aligned}}$$

for $d \gg 1$ and $d \gg J$ with

$$\boxed{\begin{aligned} \Omega(d,m) =&\ \textstyle\sum_{r=0}^{g} \binom{2r+2}{m+r+1} n_d^r \\ &\ b_j(J, \kappa, c_2 \cdot D, \chi)\ . \end{aligned}}$$

n_d^r Gopakumar-Vafa, r_d^r Gromov-Witten and N_d^r Donaldson-Thomas symplectic invariants

$$F_{GW} = \sum_{g,d} \lambda^{2g-2} e^{dt} r_\beta^g \ ,$$

where

$$r_\beta^g = \int_{\overline{\mathcal{M}}_{g(M,\beta)}} c^{vir}(g,\beta,M) \in \mathbb{Q}$$

are the G-W invariants.

$$e^{F_{GW}} = Z_{GV} = e^{\frac{c(t)}{\lambda^2}+l(t)} \exp\left(\sum_{g,d,m}^{\infty} \frac{n_d^g}{m} \left(2\sin\frac{m\lambda}{2}\right)^{2g-2} e^{mdt} \right)$$

$$Z^{\mathrm{hol}}_{\mathrm{GV}}(M, e^{i\lambda}, e^{t}) M(e^{i\lambda})^{\frac{\chi(M)}{2}} = Z^{\mathrm{hol}}_{\mathrm{DT}}(M, -e^{i\lambda}, e^{t})$$

with

$$Z^{\mathrm{hol}}_{\mathrm{DT}}(M, q_\lambda, q) = \sum_{d,k} N^k_d q^k_\lambda q^\beta$$

where $n^r_d \in \mathbb{Z}$ and $N^k_d \in \mathbb{Z}$.

GHRR shows that CY 3-folds with $c_1(M) = 0$ critical case for this invariants:

$$\boxed{\dim \overline{\mathcal{M}}_g(M, \beta) = c_1(M) \cdot \beta + (\dim(M) - 3)(1 - g) \geq 0}$$

Modular approach to topological string Theory:

Mirror symmetry → B-model. Parameters are henceforth complex structur def. in $\mathcal{M}_{CS}(W)$ of mirror W

➪ Failure of holomophicity *Holomorphic anomaly equations:*

$$\begin{aligned}\bar\partial_{\bar t_{\bar k}} F_g &= \int_{\overline{\mathcal{M}}(g)} \partial\bar\partial\lambda \\ &= \tfrac{1}{2}\bar C^{ij}_{\bar k}\left(D_iD_jF_{g-1}+\textstyle\sum_{r=1}^{g-1} D_iF_rD_jF_{g-r}\right)\ .\end{aligned}$$

May also be intepreted as *Wave function property of Z*

$$\Big[\frac{\partial}{\partial \bar{T}^I} + \frac{i}{8}\bar{C}_I{}^{JK}\frac{\partial^2}{\partial X^J \partial X^K}\Big] Z(X,T,\bar{T}) = 0$$

$$\Big[\frac{\partial}{\partial T^I} + \frac{i}{2}X^J C_{IJ}{}^K \frac{\partial}{\partial X^K} + \frac{i}{2} C_{IJK} X^J X^K - \frac{i}{4} C_{IJ}{}^J\Big] Z = 0\,.$$

Equations come from factorization of *higher genus world-sheets*, but leaves

- an *holomorphic ambiguity* (functions)
- *s-t modularity* → *modular ambiguity* (discrete data)
- eventually fixed by *gap conditions*.

Implementation of interplay between world-sheet and space-time arguments requires

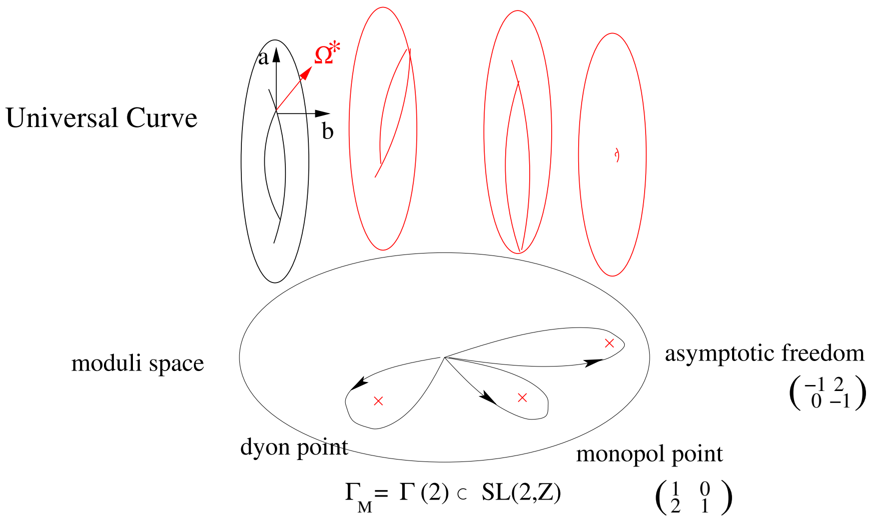
- an understanding of modular group Γ_M,
- control over the metaplectic transformation property of $Z(X,T,\bar{T})$ under Γ_M.

Easier the local case $\longrightarrow$ discussed next as example.

Coupling Seiberg-Witten gauge theory to gravity HK

Geometric engineering realizes e.g. N=2 SU(2) as double scaling limit of TST on $0(-2,-2) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$.

The mirror is an elliptic curve with $\Gamma(2) \in \mathrm{SL}(2,\mathbb{Z})$ monodromy.

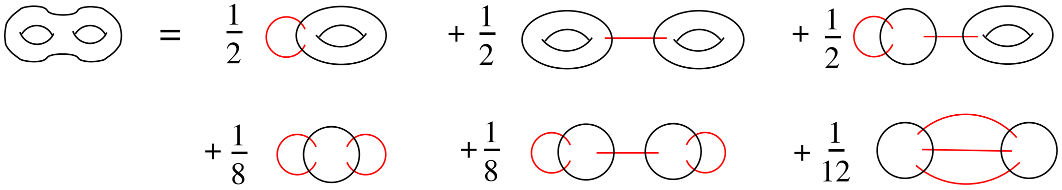


Modularity and WS degenerations:

➪ $F_g(\tau,\bar\tau)$ invariant under $\Gamma_M = \Gamma(2)$, e.g.

$$F_1 = -\log(\sqrt{\mathrm{Im}(\tau)}\eta\bar\eta)$$

➪ degenerations cap. by Feynmann rules:



➪ 'Propagator' transforms as form of weight 2 (derivative)

$$\rule[0.5ex]{3em}{0.4pt} = S = \frac{\partial}{\partial\tau} 2F_1 = \frac{1}{12}\left(E_2 - \frac{3}{\pi \mathrm{Im}\tau}\right) =: \hat{E}_2$$

➪ $F_g(\tau,\bar\tau) = \xi^{2g-2} \sum_{k=0}^{3(g-1)} \hat{E}_2^k(\tau,\bar\tau) c_k^{(g)}(\tau) =: \xi^{2g-2} f_g \;,x$

where $\xi = \frac{\theta_2^2}{1728\theta_3^4\theta_4^4} = \frac{1}{F^{(0)}_{aaa}}$ is of weight -3.

➪ Invariance means mathematically

$$\boxed{f_g \in \hat{\mathcal{M}}_{6(g-1)}(\hat{E}_2, \Delta, h)}$$

the *ring* of *almost holomorphic functions* of $\Gamma(2)$ of weight $6(g-1)$ *finitely generated* by

$$\boxed{(\hat{E}_2, h = \theta_2^4 + 2\theta_4^4, \Delta = \theta_3^4\theta_4^4)\ .}$$

Direct integration:

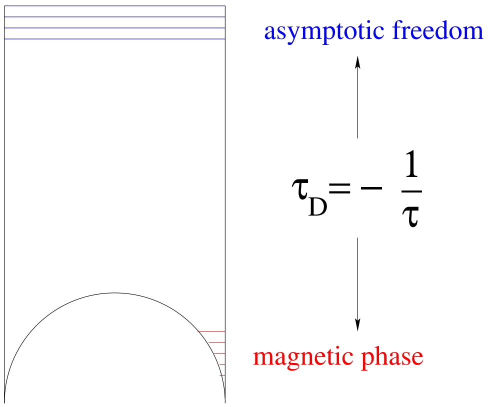
The only antiholomorphic dependence is in the $S \propto \hat{E}_2$: $\frac{\partial}{\partial\bar{\tau}} \longrightarrow \frac{\partial}{\hat{E}_2}$:

$$\begin{aligned} \frac{1}{24^2}\frac{\mathrm{d}}{\mathrm{d}\hat{\mathrm{E}}_2} f_g &= d_\xi^2 f_{g-1} + \tfrac{1}{3}\frac{(\partial_\tau \xi)}{\xi} d_\xi f_{g-1} + \textstyle\sum_{r=1}^{g-1} d_\xi f_r d_\xi f_{g-r}, \\ \text{with} \quad d_\xi f_k &= \partial_\tau f_k + \tfrac{k}{3}\frac{(\partial_\tau \xi)}{\xi} f_k \quad \text{Serre} \quad \text{operator} \end{aligned}$$

- ➪ Only the degree 0 part in $\hat{E}_2$ remains undetermined. Ambiguity is a holomorphic modular form $c_0^{(g)}(\tau) \in \mathcal{M}_{6(g-1)}(\Delta, h)$.

- ➪ $\dim\left(\mathcal{M}_{6(g-1)}(h,\Delta)\right) = \left[\frac{3g}{2}\right]$ number of required boundary conditions

Global properties:



$$\mathbb{F}(\Gamma(2))$$

$$F_g^D(\tau_D, \bar{\tau}_D) = F_g(-\frac{1}{\tau_D}, -\frac{1}{\bar{\tau}_D})$$

- ST-instanton expansion

$$\mathcal{F}_g(\tau(a)) = lim_{\bar{\tau}\to\infty} F_g(\tau, \bar{\tau})$$

- Strong-coupling expansion

$$\mathcal{F}_g^D(\tau_D(a_D)) = lim_{\bar{\tau}_D\to\infty} F_g^D(\tau_D, \bar{\tau}_D)$$

Can be seen as metaplectic transformation on $\Psi = Z$

The strong coupling gap :

$$\mathcal{F}_g^D = \frac{B_{2g}}{2g(2g-2)a_D^{2g-2}} + \ldots + k_1^{(g)} a_D + \mathcal{O}(a_D^2)$$

$$\uparrow$$

$$2g-2 \text{ independent vanishing conditions}$$

$$\boxed{2g-2 > \left[\frac{3g}{2}\right]}$$

➪ theory completly solved

- Other methods: W-S Instanton localisation or vertex, S-T Inst. localisation Nekrasov, Flume, Poghossian, Nakajima, Okounkov, Yoshioka,. . .

- are perturbatively defined near $\frac{1}{a} = 0$. First coefficients check and confirm the gap.

Why the Gap ?

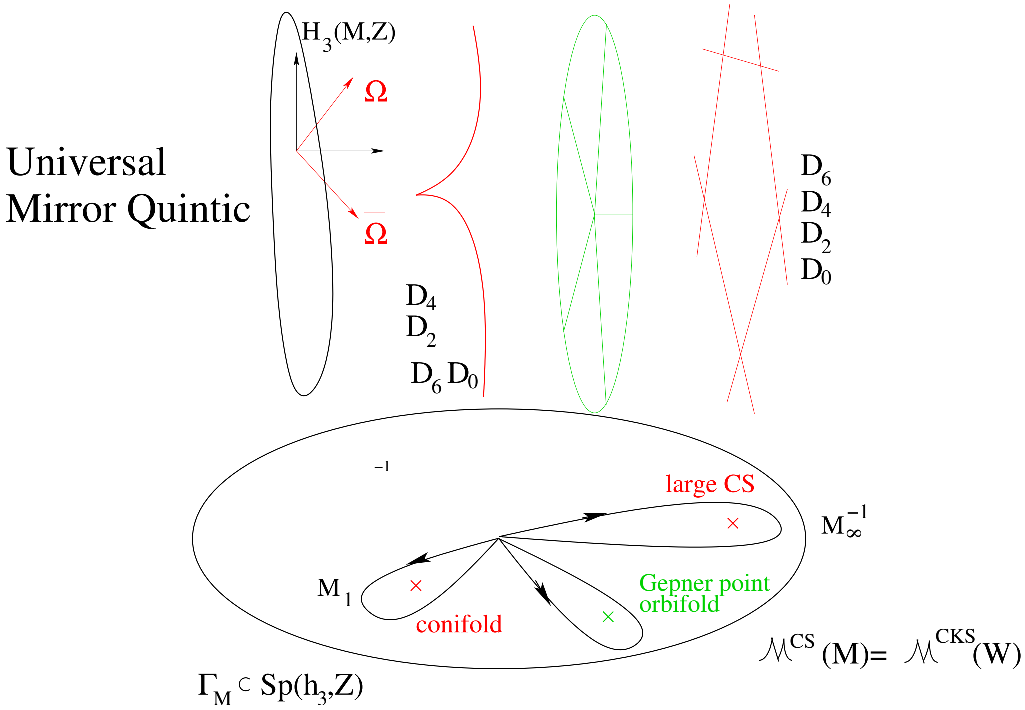
- Dijkgraaf & Vafa: SW is described by a matrix model: Typical in MM is a pole $\frac{1}{s^{2g-2}}$ from the measure followed by a regular perturbative expansion.

- String LEEA explanation: $F(\lambda, t)$ graviphoton couplings given by Schwinger-Loop calculation Antoniadis, Gava, Narain, Taylor, Gopakumar, Vafa. For one HM at conifold Strominger t_D mass of HM

$$F(\lambda, t_D) = \int_\epsilon^\infty \frac{\mathrm{d}s}{s} \frac{e^{-st_D}}{4\sin^2(s\lambda/2)} = \sum_{g=2}^{\infty} \left(\frac{\lambda}{t_D}\right)^{2g-2} \frac{(-1)^{g-1} B_{2g}}{2g(2g-2)} .$$

Compact Calabi-Yau HKQ

$$W=\sum_{i=1}^{5}x_i^5-j_q^{\frac{1}{5}}\prod_{i=1}^{5}x_i=0\in\mathbb{P}^4,$$



$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 5 & -3 & 1 & -1 \\ -8 & -5 & 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_\infty^{-1} = \begin{pmatrix} -4 & 3 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ 5 & -3 & 1 & -1 \\ 8 & -5 & 0 & 1 \end{pmatrix}.$$

generate a discrete subgroup of $\Gamma_M = \mathrm{Sp}(4,\mathbb{Z})$ acting on $H^3(W,\mathbb{Z})$ on periods $\Pi(z) = \int_\Gamma \Omega(z)$ fullfilling

$$[\theta^4 - 5 j_q^{-1} \prod_{i=1}^{4} (\theta + i)]\, \Pi(z) = 0, \qquad \theta := -j_q \frac{\mathrm{d}}{\mathrm{dj_q}}\,.$$

Properties of Γ_M, even if of finite index unknown, but we can build modular objects using the periods and special geometry.

E.g. from the mirror map an analog of j-function, $q = \exp(\int_C \omega) = \exp(\Pi_1(j_q)/\Pi_0(j_q))$

$$\begin{aligned} j_q &= \tfrac{1}{q} + 770 + 421375\, q + 274007500\, q^2 + 236982309375\, q^3 + \ldots \\ (j_e &= \tfrac{1}{q} + 744 + 196884\, q + 21493760\, q^2 + 864299970\, q^3 + \ldots) \end{aligned}$$

Regularity at the Gepner point is maintained if we introduce $P_g = \xi^{g-1} F_g$, where $\xi = \frac{j_q}{1-j_q} = j_q X$. From the *gap behaviour* of the F_g at the conifold $j_q = 1$ and from regularity at the large CS, we conclude that the *holomorphic* and *modular ambiguity* in P_g is given by

$$\boxed{c_0^{(g)} = \sum_{i=0}^{3g-3} a_i X^i}$$

The generators of the ring of almost holomorphic modular (tensor) forms of Γ_M for Calabi-Yau are not known, but Yau, Yamaguchi `hep-th/0406078`, showed following BCOV,KKV that the P_g can be written as polynomials in 3 an-holomophic and one holomorphic generator

$$\begin{array}{rl} A_p & := \frac{(j\partial_j)^p G_{j,\bar{j}}}{G_{j\bar{j}}}, \quad B_p := \frac{(j\partial_j)^p e^{-K}}{e^{-K}}, \quad p = 1, \ldots \\ C: & = C_{jjj} j^3, \quad X = \frac{1}{1-j} \end{array}$$

➪ Special geometry & Picard-Fuchs eq. truncate to A_1, B_1, B_2, B_3, X.

- ➪ One combination does not appear in $P_g = C^{g-1}F_g$. $B_1 = u$, $A_1 = v_1 - 1 - 2u$, $B_2 = v_2 + uv_1$, $B_3 = v_3 - uv_2 + uv_1X - c_1uX$

- ➪ The P_g are degree $3g-3$ weighted inhomogeneous polynomials in v_1, v_2, v_3, X,

- ➪ hol. anom. eq.

$$(\partial_{v_1} - u\partial_{v_2} - u(u+X)\partial_{v_3})\, P_g = -\frac{1}{2}\left(P^{(2)}_{g-1} + \sum_{r=1}^{g-1} P^{(1)}_r P^{(1)}_{g-r}\right)$$

Boundary conditions:

- ➪ Gap at the conifold $j = 1$

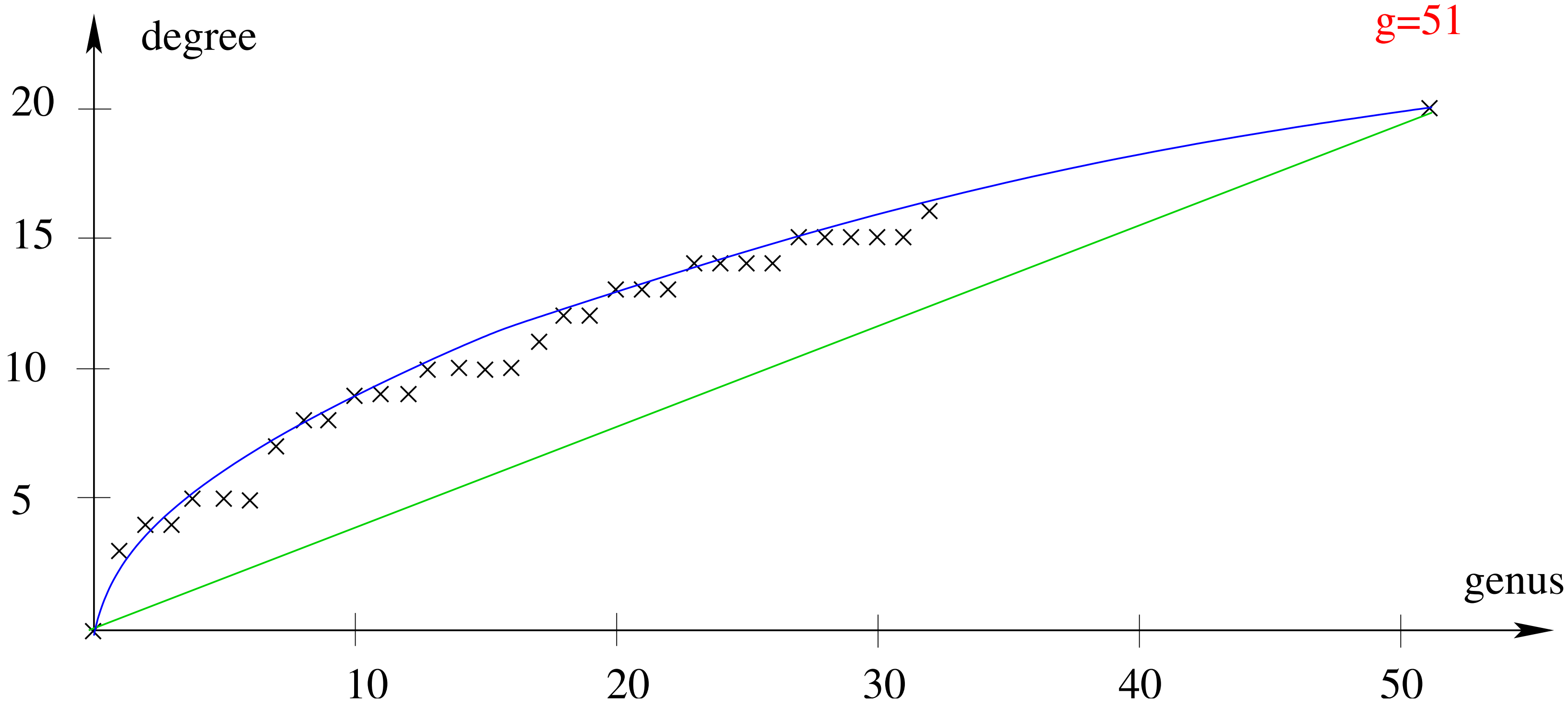
$$\mathcal{F}_g^D = \frac{B_{2g}}{2g(2g-2)t_D^{2g-2}} + k_g^1 + \mathcal{O}(t_D)$$

provides $2g - 2$ conditions.

- ➪ Regularity at Gepner point $j = 0$ provides $\left[\frac{3(g-1)}{5}\right]$ conditions $\rightarrow$ $\left[\frac{2(g-1)}{5}\right]$ unkowns.

- ➪ Castelnouvo's bound for GV invariants at large radius. From aqjunction formula in $\mathbb{P}^4$ ones find there are no

genus g curves for $d \leq \sqrt{g}$



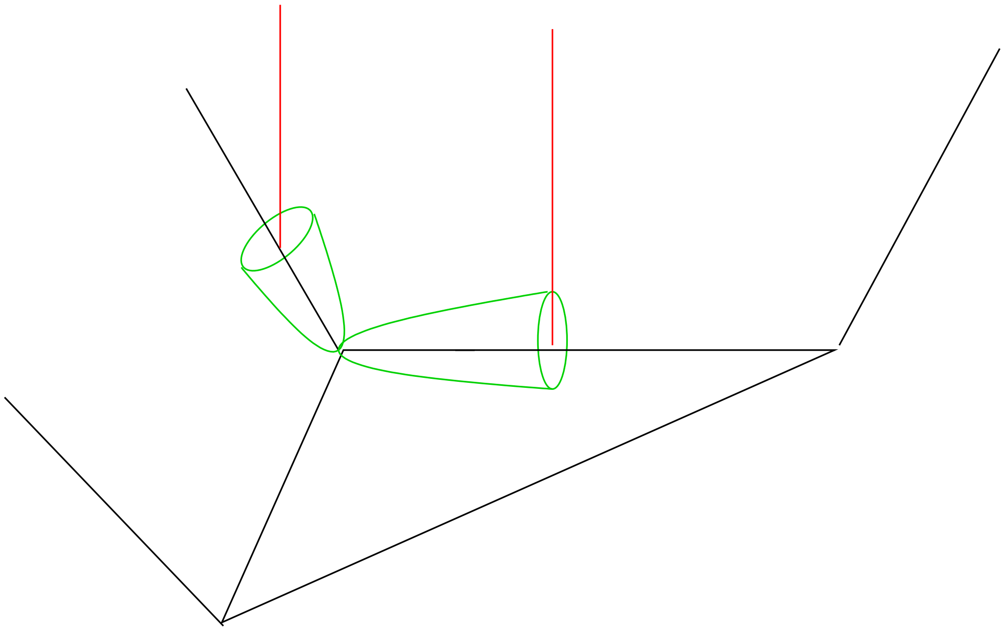
genus	degree=18
0	144519433563613558831955702896560953425168536
1	491072999366775380563679351560645501635639768
2	826174252151264912119312534610591771196950790
3	866926806132431852753964702674971915498281822
4	615435297199681525899637421881792737142210818
5	306990865721034647278623907242165669760227036
6	109595627988957833331561270319881002336580306
7	28194037369451582477359532618813777554049181
8	5218039400008253051676616144507889426439522
9	688420182008315508949294448691625391986722
10	63643238054805218781380099115461663133366
11	4014173958414661941560901089814730394394
12	166042973567223836846220100958626775040
13	4251016225583560366557404369102516880
14	61866623134961248577174813332459314
15	451921104578426954609500841974284
16	1376282769657332936819380514604
17	1186440856873180536456549027
18	2671678502308714457564208
19	-59940727111744696730418
20	1071660810859451933436
21	-13279442359884883893
22	101088966935254518
23	-372702765685392
24	338860808028
25	23305068
26	-120186
27	-5220
28	-90
29	0

Application to open string amplitudes

Bouchard, Mariño, Pasquetti, AK hep-th/0790.1453, Bouchard, Mariño, Pasquetti, Weiss, AK to appear

Local $O(-3) \longrightarrow IP^2$ geometry after the moment map

with HL special lagrangian branes and open string instantons



Annulus: Geometrically Bergmann Kernel

$W_0(p,q) = \partial_p \partial_q E(p,q)$, where $E(p,q)$ is the prime form.

$$W_0(p,q) = \left(-\frac{1}{2(p-q)^2} - \frac{G(t,\bar{t}) - f(p,q,z)}{4\sqrt{\sigma(p)\sigma(q)}} \right) \mathrm{dpdq} \ ,$$

$$G(\bar{t},t) = z_t \frac{S^{tt}(\bar{t},t)}{z^2} z_t$$

with

$$W_0(p) = \frac{z_t}{\sigma(p)} = \partial_t \int_{p^*}^{p} \lambda = \int_{p^*}^{p} \frac{\mathrm{dx}}{y}$$

One point closed string insertion of the disk instanton superpotential, which corresponds to the Abelian integral.

E.g. at the Gepner or Landau-Ginzburg point in flat coordinates (P, Q, σ):

$$W_0(P,Q) = \frac{PQ\left(3Q+3P^3Q+2P^2Q^2+3P\left(1+Q^3\right)\right)}{6} - \frac{PQ\left(9PQ+8Q^2+16P^3Q^2+8P^2\left(1+2Q^3\right)\right)\sigma}{12} + \frac{PQ\left(1+15P^2Q+15PQ^2+14Q^3+2P^3\left(7+26Q^3\right)\right)\sigma^2}{18} + \ldots$$

Generally

$$W_g(\underline{p}_h) = \frac{1}{\Delta^{2g-2+h}\prod_{i=1}^{h}\sqrt{\sigma(p_i)}} \sum_{i=0}^{3g-3+2h} G^i f_{g,i}(z, \underline{p}_h) \ ,$$

where

$$f_{g,0}(z,\underline{p}_h)=\frac{Q(z,\underline{p}_h)}{\left(\prod_{i=1}^{h}\sigma(p_i)\right)^{3g-2+h}}$$

are rational functions and

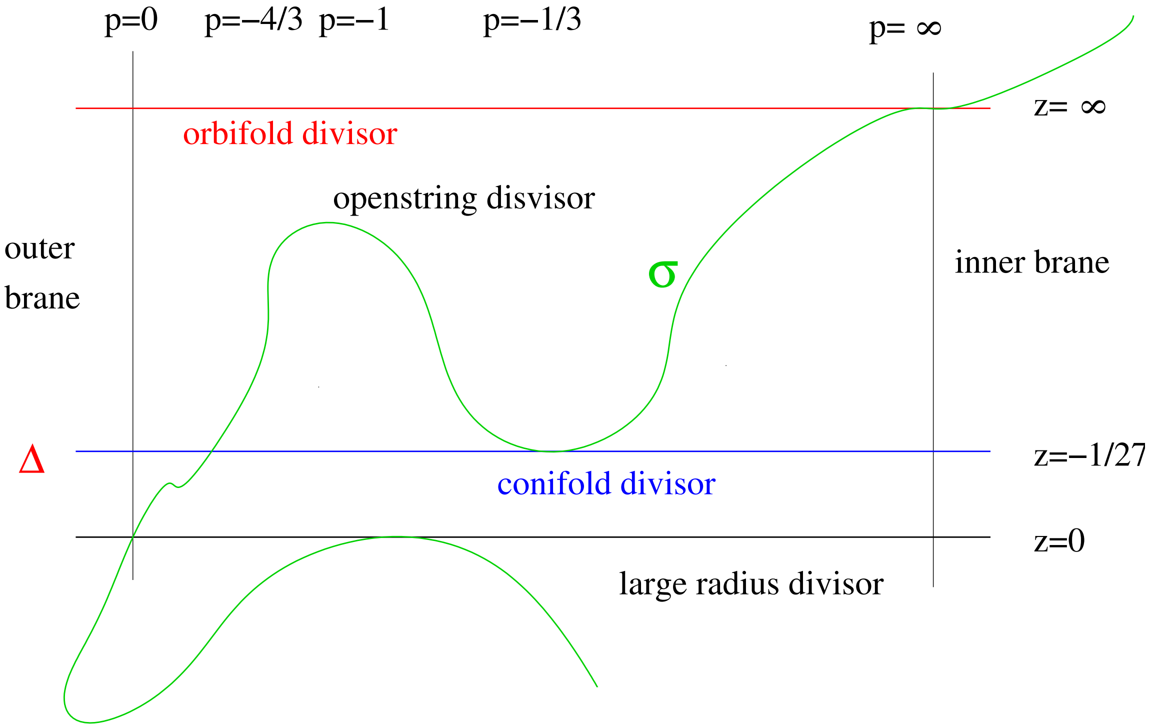
$$\sigma(q)=q^2(1+q)^2-4qz, \qquad \Delta=(1+27z),$$

are the open and closed string discriminants.

E.g. one-loop one hole or three hole at the G/LG point

$$W_1(p)=\tfrac{1}{24}(P^3\ \left(5+33\,P^3+170\,P^6+805\,P^9\right)-$$
$$\tfrac{1}{72}(P\ \left(-1+40\,P^3+462\,P^6+3400\,P^9+20930\,P^{12}\right)\ \sigma)+$$

$$W_0(Q,P,R) = \frac{1}{24}(P\,Q\,R\,(3\,Q\,(8\,P^2+9\,P\,R+8\,R^2)+ 8\,(2+3\,P^2\,R+3\,P\,R^2)+8\,Q^2\,(3\,P+3\,R+4\,P^2\,R^2)))- \frac{1}{18}(P\,Q\,R\,(15\,P+15\,R+56\,P^2\,R^2+7\,Q^2\,(8\,P^2+9\,P\,R+ 8\,R^2)+3\,Q\,\left(5+21\,P^2\,R+21\,P\,R^2\right)))\,\sigma+\ldots$$



Application of the results to the KKV proposal

To check againts marcoscopic result in limit $d \gg 1$ and $d \gg m$ for $J = 0$ define

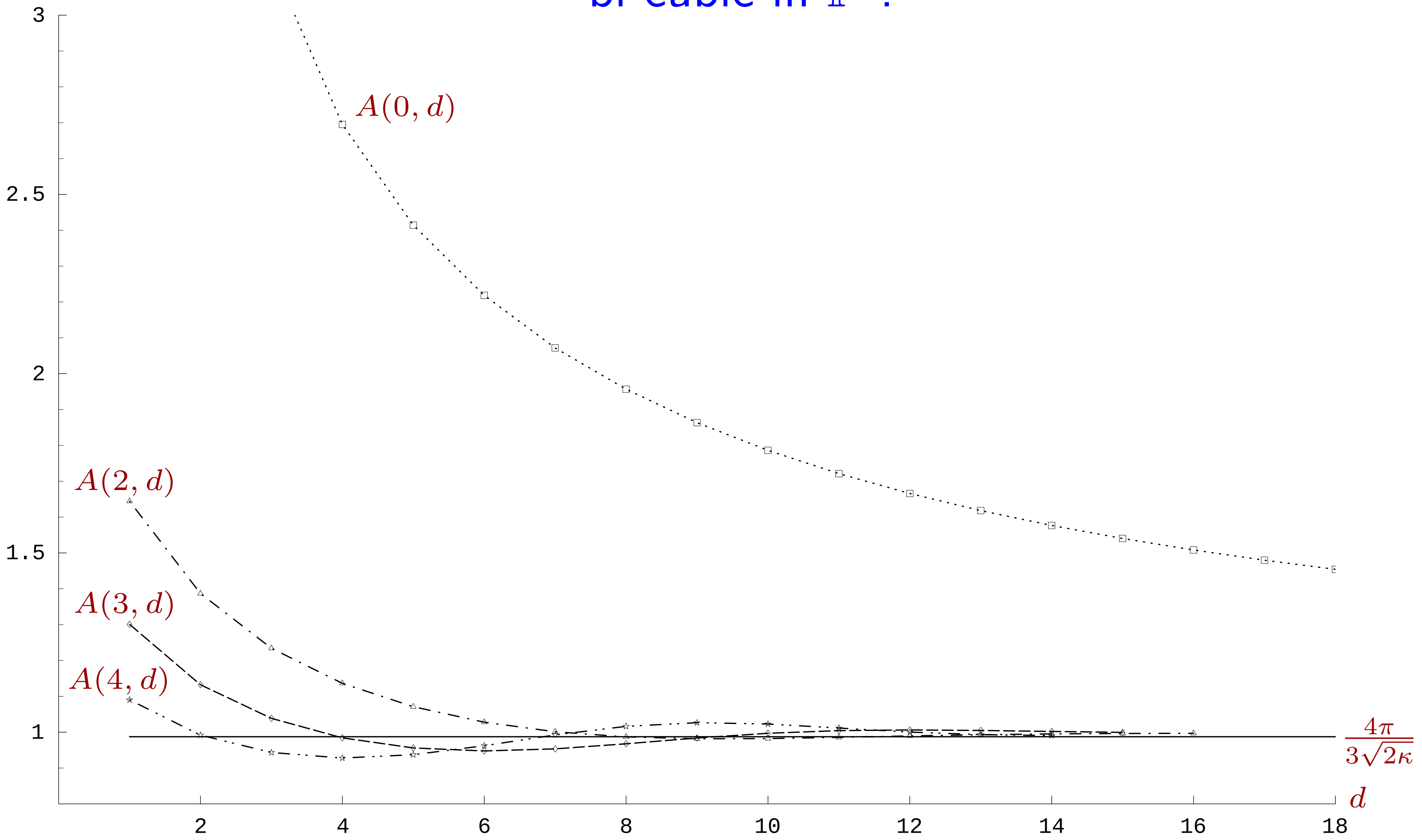
$$f(d) = \frac{\log(\Omega(d,0))}{d^{\frac{3}{2}}} \to \frac{4\pi}{3\sqrt{2\kappa}}$$

and as

$$A(m,d)$$

the m'th Richardson transform of $f(d)$, eliminating subleading term in $\frac{1}{d}$. THen we get the following plot

bi-cubic in $\mathbb{P}^5$:



Calabi-Yau	d_{max}	$A(d_{max}-3,3)$	$b_0=\frac{4\pi}{3\sqrt{2\kappa}}$	error
$X_5(1^5)$	14	1.35306	1.32461	2.15 %
$X_6(1^4,2)$	10	1.75559	1.71007	2.66 %
$X_8(1^4,4)$	7	2.11454	2.0944	0.96 %
$X_{10}(1^3,2,5)$	5	2.99211	2.96192	1.02 %
$X_{3,3}(1^6)$	17	1.00204	0.987307	1.49 %
$X_{4,2}(1^6)$	15	1.07031	1.0472	2.21 %
$X_{3,2,2}(1^7)$	10	0.821169	0.855033	-3.96 %
$X_{2,2,2,2}(1^8)$	13	0.722466	0.74048	-2.43 %
$X_{4,3}(1^5,2)$	11	1.21626	1.2092	0.58 %
$X_{6,2}(1^5,3)$	11	1.52785	1.48096	3.17 %
$X_{4,4}(1^4,2^2)$	7	1.42401	1.48096	-3.85 %
$X_{6,4}(1^3,2^2,3)$	5	2.06899	2.0944	-1.21 %
$X_{6,6}(1^2,2^2,3^2)$	4	2.95082	2.96192	-0.37 %

Table 2: Comparing the extrapolated value of b_0 with the macroscopic prediction.

- For $J = m = 1$ the agreement is in 10% range. Microscopic prediction lower.

- The R^2 and the F^2R^2 contributions is confirmed in micro state counting

- Indictions that the Dennef, Moore scaling $k \sim 2$