Deep-inelastic structure functions at NNLO and beyond

Sven-Olaf Moch*

DESY Zeuthen

1. **Motivation**
2. **Method**
3. **First Results**
4. **Application to threshold resummation**
5. **Summary**

---

* in collaboration with **J.A.M. Vermaseren** and **A. Vogt**

Motivation

Structure of the proton

- Scaling violations of deep-inelastic structure functions F_2, F_3
  $\longrightarrow$ precision test of perturbative QCD

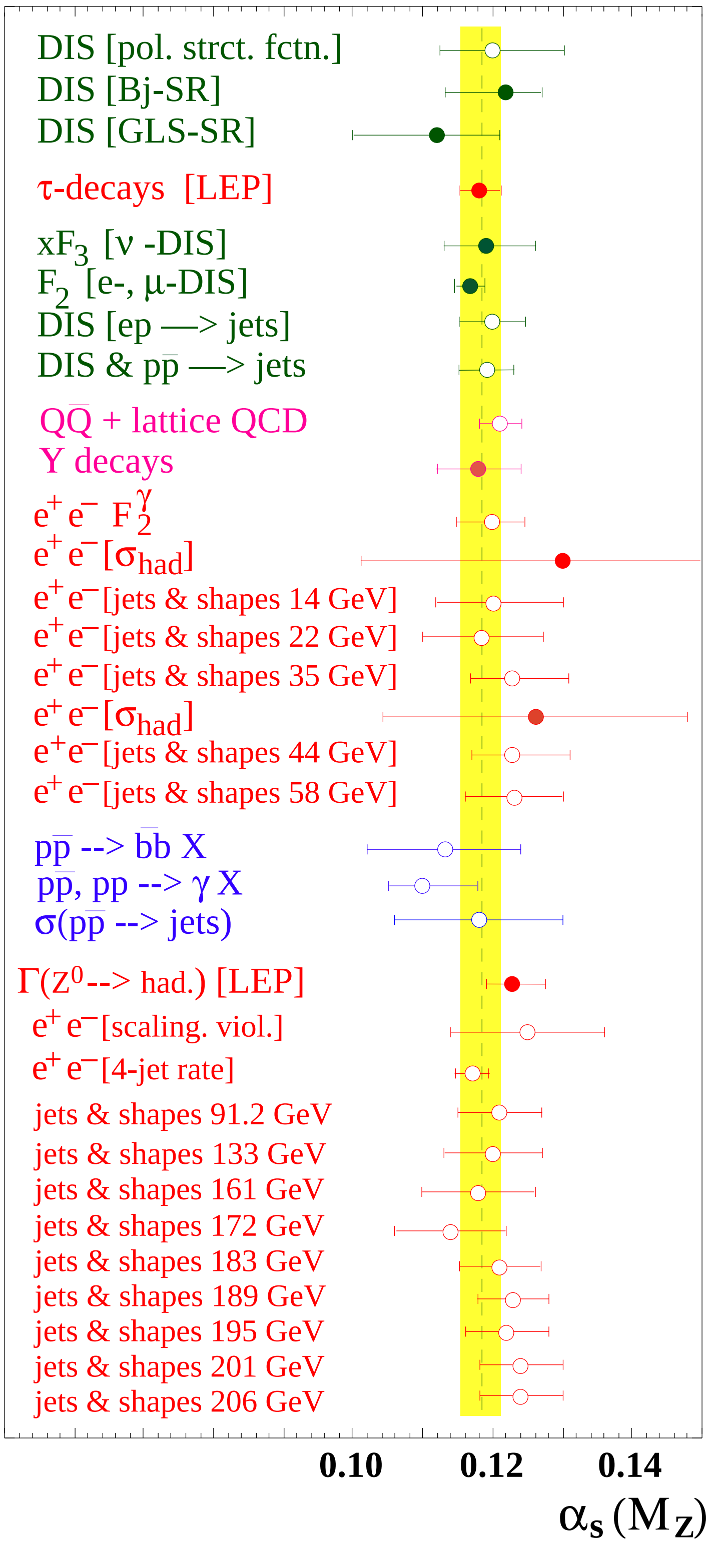
- Parton distributions
  - gluon distribution at small x, quark valence and sea distribution

  - important input for hard scattering reactions at hadron colliders
    $\longrightarrow$ precise parton flux at LHC needed for Higgs or SUSY searches

- **How well do we know (un)-polarized parton distributions ?**

α_s from DIS

- **How well do we know α_s ?**



Recent determinations of α_s

Bethke `hep-ex/0211012`

– **NLO** QCD analysis of HERA data for $F_2(x, Q^2)$
H1 coll. `hep-ph/0012053`

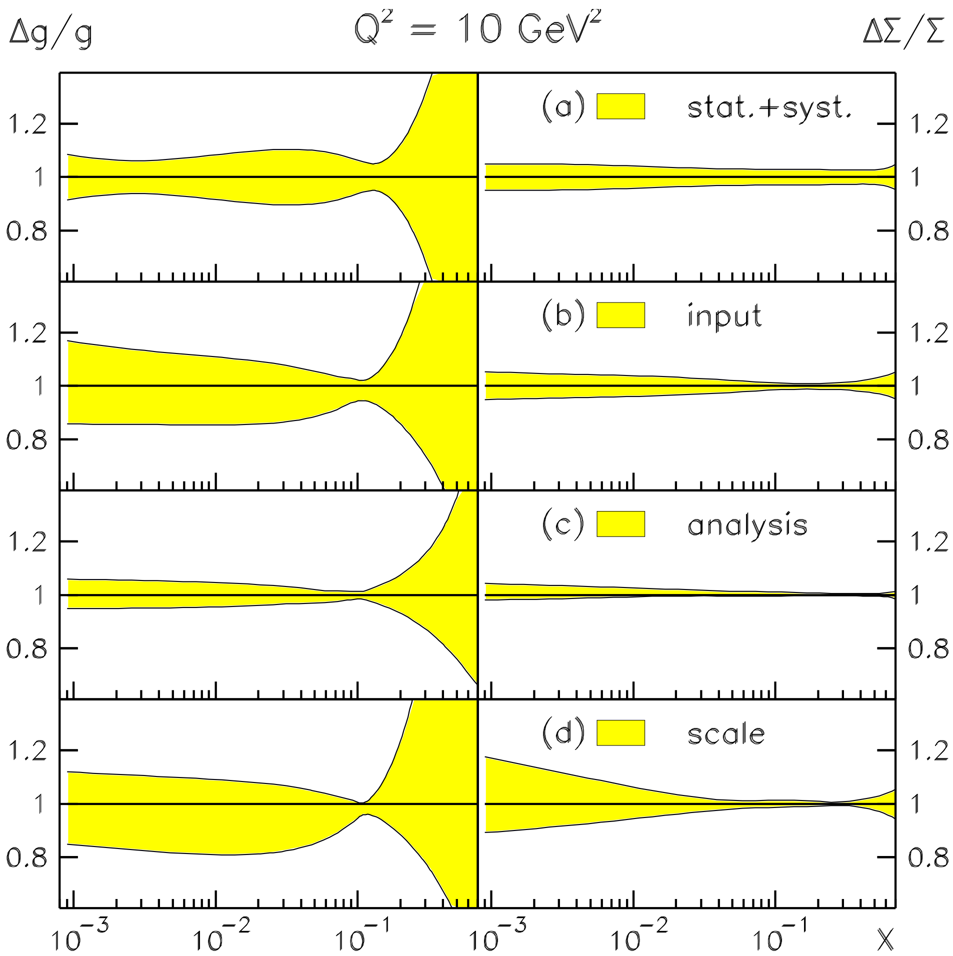
$$\alpha_s(M_Z^2) = 0.115 \pm 0.002(\mathrm{exp}) \pm 0.005(\mathrm{theo})$$

Future

– **NNLO** QCD analysis of HERA data for $F_2(x, Q^2)$ in 2006

$$\alpha_s(M_Z^2) = x \pm 0.001(\mathrm{exp}) \pm 0.001(\mathrm{theo})$$

PDF uncertainties at NLO



Botje '00

– Similar analysis Barone, Pascaud, Zomer '99

– Analysis of 1999 DIS data with errors

Botje '00

– Scale variation

$$Q/\sqrt{2} \leq \mu \leq \sqrt{2}Q$$

– Gluons (g) :

stat. $\oplus$ syst. $\simeq$ input $\simeq$ scale error

– Quarks (Σ) :

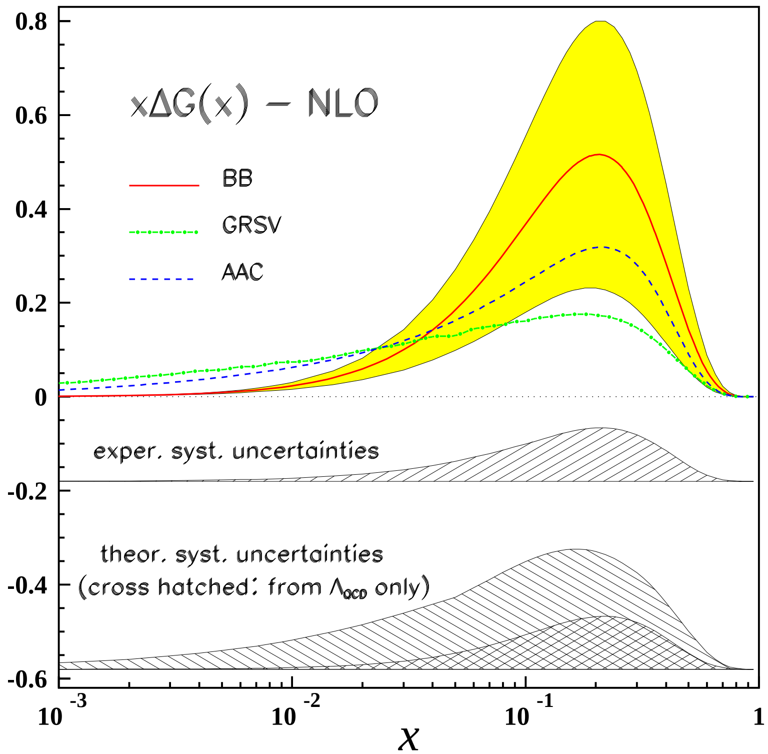
scale error already dominates

Upshot

– **NNLO** improvement of theory needed

$\longrightarrow$ three-loop splitting functions

Polarized deep-inelastic scattering



Blümlein, Böttcher '02

– Structure function g_1

– Analysis polarized DIS data with errors
Blümlein, Böttcher '02

– Polarized gluon distribution (ΔG) :
syst. $\simeq$ scale error from $\Lambda_{\rm QCD}$

– Theoretical uncertainties already dominate

Upshot

– Similar situation as for unpolarized scattering

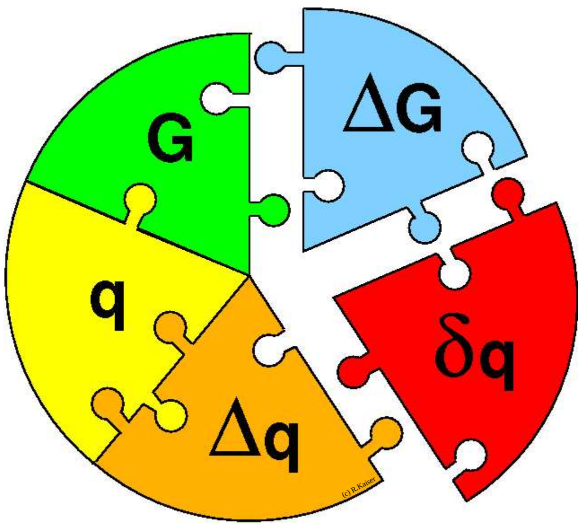
– **NNLO** improvement of theory needed
$\longrightarrow$ three-loop splitting functions
$\longrightarrow$ $g_1^{\rm c}$ for charm production at NLO

Twist-two parton distributions and the spin puzzle

– Twist-two deep-inelastic structure functions
  – Unpolarized scattering F_2, F_L valence and sea quarks, gluon distribution $q_{\rm val}, q_{\rm sea}, G$
    $\longrightarrow$ H1, ZEUS, ν-experiments

  – Longitudinally polarized scattering g_1, distributions $\Delta q, \Delta G$
    $\longrightarrow$ SLAC, HERMES, COMPASS

  – Transversely polarized scattering h_1, distribution δq $\longrightarrow$ HERMES, RHIC



Method

Optical theorem and the OPE

- Hadronic tensor $W_{\mu\nu}$

$$W_{\mu\nu} = e_{\mu\nu}\frac{1}{2x}F_L(x,Q^2) + d_{\mu\nu}\frac{1}{2x}F_2(x,Q^2) + \mathrm{i}\varepsilon_{\mu\nu\alpha\beta}\frac{p^\alpha q^\beta}{p\cdot q}F_3(x,Q^2)$$

- Optical theorem relates $W_{\mu\nu}$ to imaginary part of Compton amplitude $T_{\mu\nu}$

- Bjorken limit (x fixed, $Q^2 \to \infty$) allows for OPE of $T_{\mu\nu}$ for short distances $z^2 \simeq 0$

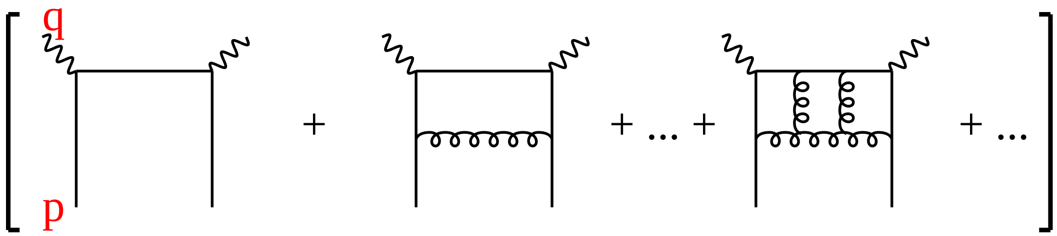
  Wilson '69 ; Brandt, Preparata '70 ; Zimmermann '70

$$\begin{aligned} T_{\mu\nu} &= \mathrm{i}\int d^4 \mathrm{z}\, \mathrm{e}^{\mathrm{i}q\cdot z}\langle \mathrm{P}|\, T\left(J_\mu^\dagger(\mathrm{z})J_\nu(0)\right)|\mathrm{P}\rangle = \\ &= \sum_{N,j}\left(\frac{1}{2x}\right)^N \left[e_{\mu\nu}C_{L,j}^N\left(\frac{Q^2}{\mu^2},\alpha_s\right) + d_{\mu\nu}C_{2,j}^N\left(\frac{Q^2}{\mu^2},\alpha_s\right) + \mathrm{i}\varepsilon_{\mu\nu\alpha\beta}\frac{p^\alpha q^\beta}{p\cdot q}C_{3,j}^N\left(\frac{Q^2}{\mu^2},\alpha_s\right)\right] A_{\mathrm{P},N}^j\left(\mu^2\right) \\ &\quad + \text{higher twist} \end{aligned}$$

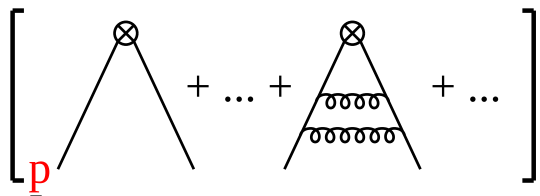
- Coefficient functions $C_{2,i}^N$ in Mellin space
- Matrix elements $A_{\mathrm{P},N}^i$ of operators O^i of leading twist

The parton picture

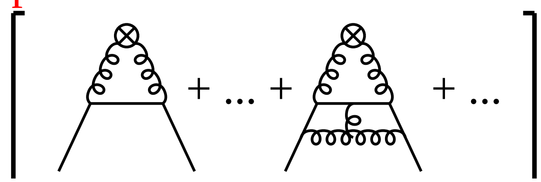
– Apply OPE to parton Green's functions, e.g. external quarks



$$= \sum_N \left(\frac{1}{2x}\right)^N \Bigg\{ \left(C^N_{i,\mathrm{q}} Z^{\mathrm{qq}} + C^N_{i,\mathrm{g}} Z^{\mathrm{gq}}\right)$$



$$+ \left(C^N_{i,\mathrm{q}} Z^{\mathrm{qg}} + C^N_{i,\mathrm{g}} Z^{\mathrm{gg}}\right) \quad \Bigg\}$$



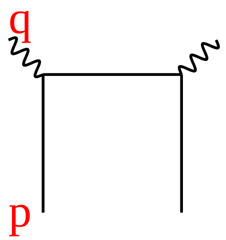
– Apply $\mathcal{P}_N$ to project the N-th moment ; Gorishnii, Larin, Tkachev '83 ; Gorishnii, Larin '87

$$\mathcal{P}_N \equiv \left[\frac{q^{\{\mu_1} \cdots q^{\mu_N\}}}{N!} \frac{\partial^N}{\partial p^{\mu_1} \cdots \partial p^{\mu_N}} \right] \Bigg|_{p=0} \qquad \frac{1}{2x} = -\frac{p \cdot q}{q \cdot q}$$

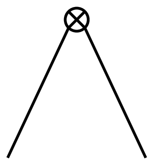
Upshot

– Projection with $\mathcal{P}_N \longrightarrow$ only tree level operator matrix elements survive

$\mathcal{P}_N \Big[$



$+ \ldots \Big] = \left(C^N_{i,\mathrm{q}} Z^{\mathrm{qq}} + C^N_{i,\mathrm{g}} Z^{\mathrm{gq}} \right)$



– **Anomalous dimensions** $\gamma(\alpha_s, N)$ from scale dependence of renormalized operators

$$O^{\mathrm{bare}} = Z\, O^{\mathrm{ren}} \qquad\qquad \frac{d}{d\ln\mu^2} O^{\mathrm{ren}} = -\gamma\, O^{\mathrm{ren}}$$

Structure functions in Mellin space

– Parameters of OPE are directly related to Mellin moments of F_2, F_3 and F_L

$$\int_0^1 dx\, x^{N-2} F_2(x, Q^2) \;=\; \sum_{i=\mathrm{ns,q,g}} C^N_{2,i}\left(\frac{Q^2}{\mu^2}, \alpha_s\right) A^i_{\mathrm{P},N}(\mu^2)$$

Gluonic currents

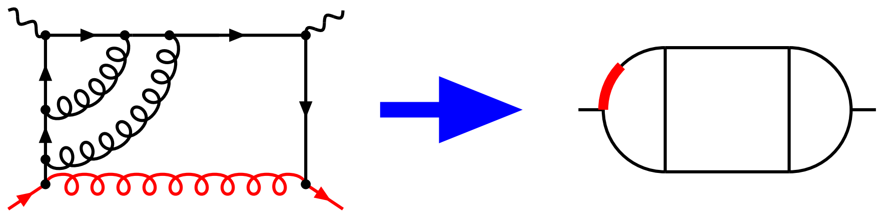
Larin, Nogueira, van Ritbergen, Vermaseren '97

– Problem : photons only couple to quarks
$\longrightarrow$ γ_{gq}, γ_{gg} only to $(l-1)$-loop in l-calculation

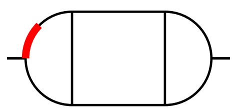
– Solution : add scalar particle ϕ which couples only to gluons
  – Feynman rules from adding term $\phi F^a_{\mu\nu} F_a^{\mu\nu}$ to QCD Lagrangian
    Kluberg-Stern, Zuber '75 ; Collins, Duncan, Joglekar '77
  – different current product in Compton amplitude T
  – same parton opertor matrix elements and **same OPE**

Integrals and how we break them to little pieces

- Reduction scheme for given diagram

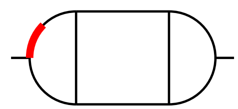


- Scalar diagram with external momenta p and q



$$= \int d^D l_1 d^D l_2 d^D l_3 \, \frac{1}{(p-l_1)^2} \frac{1}{l_1^2 \ldots l_8^2}$$

  - N-th moment $\longrightarrow$ coefficient c_N

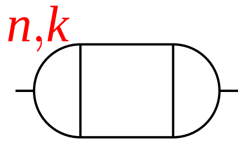


$$= \frac{(2\, p \cdot q)^N}{(q^2)^{N+\alpha}} \, c_N$$

  - Taylor expansion

$$\frac{1}{(p-l_1)^2} = \sum_i \frac{(2p \cdot l_1)^i}{(l_1^2)^{i+1}} \quad \longrightarrow \quad \frac{(2p \cdot l_1)^N}{(l_1^2)^N}$$

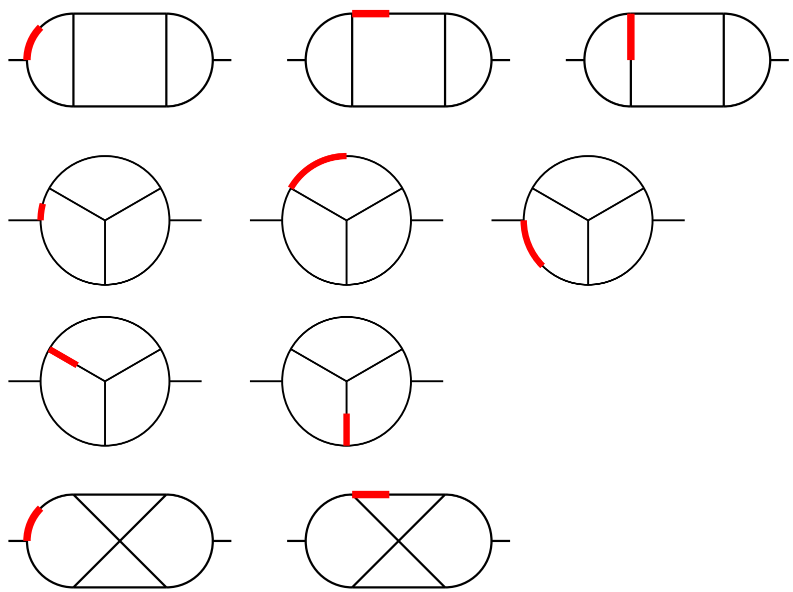
- Two-point functions with symbolic powers



$$= \int d^D l_1 d^D l_2 d^D l_3 \frac{(2p\cdot l_1)^k}{(l_1^2)^n} \frac{1}{l_2^2 \ldots l_8^2}$$

Basic building blocks

- 10 topologies for basic building blocks
  - ladder
  - benz
  - nonplanar



General strategy of mapping

- Mapping of complicated topologies to simpler topologies
  - hierachy : non-planar $\longrightarrow$ benz $\longrightarrow$ ladder
  - composite building blocks $\longrightarrow$ basic building blocks

- Integration by parts, scaling identies, form factor relations (Passarino–Veltman), ...

  ’t Hooft, Veltman ‘72 ; Chetyrkin, Tkachov ‘81 ; Larin ‘81 ; S.M., Vermaseren ‘99

$$\int d^D l_1 d^D l_2 d^D l_3 \frac{\partial}{\partial l_i^\mu}\left[l_j^\mu f(l_1,\ldots l_n)\right] = 0 \quad , \qquad q^\mu \frac{\partial}{\partial q^\mu}, \qquad p^\mu \frac{\partial}{\partial q^\mu}, \qquad p^\mu \frac{\partial}{\partial p^\mu}$$

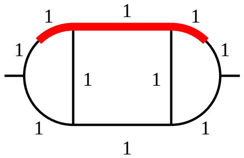
$$\frac{\partial}{\partial q^\mu}\int d^D l_1 d^D l_2 d^D l_3 \left[l_j^\mu f(l_1,\ldots l_n)\right] = \frac{\partial}{\partial q^\mu}\left[q^\mu I^{(q)} + p^\mu I^{(p)}\right]$$

Recursion relations

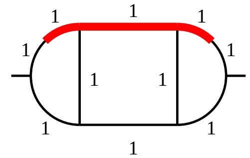
- Recursion relations $\longrightarrow$ difference equations

$$a_0(N)F(N) - \ldots - a_n(N)F(N-n) - G(N) = 0$$

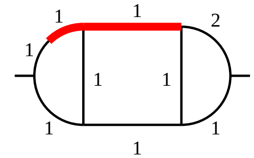
– Example : single-step difference equation in N



$$= -\frac{N+3+3\varepsilon}{N+2}\,\frac{2\,p\cdot q}{q^2}$$



$$+\ \frac{2}{N+2}$$



– Formal solution of single-step difference equations

$$\mathbf{F}(\mathbf{N}) \quad = \quad \frac{\prod_{j=1}^{N} a_1(j)}{\prod_{j=1}^{N} a_0(j)}\mathbf{F}(\mathbf{0}) + \sum_{\mathbf{i=1}}^{\mathbf{N}} \frac{\prod_{j=i+1}^{N} a_1(j)}{\prod_{j=i}^{N} a_0(j)}\mathbf{G}(\mathbf{i})$$

– Implementation in computer algebra system FORM Vermaseren

```
id lafun(n1?,l1?,0,0,l2?,k2?,n3?,l3?,0,<n4?,0,0>,...,<n8?,0,0>,k9?) =
      theta_(N-k2)*sign_(N)* fac_(n1+n3-k2)*invfac_(N+n1+n3-k2)*
      Gamma(-7+N+3*ep+n1+n3+...+n8-k2-k9+l1+l2+l3+1)*
      InvGamma(-7+3*ep+n1+n3+...+n8-k2-k9+l1+l2+l3+1)*
    LA(n1+l1,l2,n3+l3,n4,n5,n6,n7,n8,0,k2,0,0,0,0,0,0,k9)
- sum1(j1,1,N)* theta_(N-j1)*sign_(N-j1)* fac_(j1+n1+n3-k2-1)*invfac_(N+n1+n3-k2)*
      Gamma(-7+N+3*ep+n1+n3+...+n8-k2-k9+l1+l2+l3+1)*
      InvGamma(-7+j1+3*ep+n1+n3+...+n8-k2-k9+l1+l2+l3+1)* (
      -n1*lafun(1+n1,-1+l1,0,0,l2,k2,n3,l3,0,<n4,0,0>,...,<n8,0,0>,k9)
      -n3*lafun(n1,l1,0,0,l2,k2,1+n3,-1+l3,0,<n4,0,0>,...,<n8,0,0>,k9) );
```

Mathematics of harmonic sums

- Harmonic sums $S_j(N)$ Gonzalez-Arroyo, Lopez, Ynduráin ‘79 ; Vermaseren ‘98 ; Blümlein, Kurth ‘98
  - Recursive definition $S_{m_1,\ldots,m_k}(N) = \sum_{i=1}^{N} \frac{1}{i^{m_1}} S_{m_2,\ldots,m_k}(i)$
- Algebra of multiplication $S_j(N)S_k(N) \longrightarrow S_{\{j,k\}}(N)$
- Harmonic sums originate from :
  - expansion of Γ-functions in powers of ε
  - solutions of recursion relations
  - sums of type : $\sum_{i=1}^{N} (-1)^i \binom{N}{i} \frac{1}{i^3} = -S_{1,1,1}(N)$
- Multiple polylogarithms in x-space

  Goncharov ‘98 ; Borwein, Bradley, Broadhurst, Lisonek ‘99 ; Remiddi, Vermaseren ‘99

Recursion relations and algebra of harmonic sums $\longrightarrow$ Breakthrough in technology

First results

Anomalous dimensions in Mellin space

– One-loop : Gross, Wilczek ’73

$$\gamma_{\rm ns}^{(0)}(N) \quad = \quad C_F\Big(2(\mathbf{N}_- + \mathbf{N}_+)S_1 - 3\Big)$$

– Two-loop : Floratos, Ross, Sachrajda ‘79 ; Gonzalez-Arroyo, Lopez, Ynduráin ‘79

$$\gamma_{\rm ns}^{(1)}(N) \;=\; 4C_AC_F\left(2\,\mathbf{N}_+S_3 - \frac{17}{24} - 2S_{-3} - \frac{28}{3}S_1 + (\mathbf{N}_- + \mathbf{N}_+)\left[\frac{151}{18}S_1 + 2S_{1,-2} - \frac{11}{6}S_2\right]\right)$$
$$+\,4C_Fn_f\left(\frac{1}{12} + \frac{4}{3}S_1 - (\mathbf{N}_- + \mathbf{N}_+)\left[\frac{11}{9}S_1 - \frac{1}{3}S_2\right]\right) + 4{C_F}^2\left(4S_{-3} + 2S_1 + 2S_2 - \frac{3}{8}\right.$$
$$\left. +\,\mathbf{N}_-\Big[S_2 + 2S_3\Big] - (\mathbf{N}_- + \mathbf{N}_+)\Big[S_1 + 4S_{1,-2} + 2S_{1,2} + 2S_{2,1} + S_3\Big]\right)$$

– Compact notation : $\mathbf{N}_\pm\, f(N) \;=\; f(N\pm 1)\ ,\qquad \mathbf{N}_{\pm\mathbf{i}}\, f(N) \;=\; f(N\pm i)$

– Three-loop : fermionic contributions to nonsinglet anomalous dimension

$$\begin{aligned}
\gamma_{\rm ns}^{(2)}(N) \; = \; & 16 C_A C_F n_f \Bigg(\frac{3}{2}\zeta_3 - \frac{5}{4} + \frac{10}{9}S_{-3} - \frac{10}{9}S_3 + \frac{4}{3}S_{1,-2} - \frac{2}{3}S_{-4} + 2S_{1,1} - \frac{25}{9}S_2 + \frac{257}{27}S_1 \\
& - \frac{2}{3}S_{-3,1} - \mathbf{N}_+ \left[S_{2,1} - \frac{2}{3}S_{3,1} - \frac{2}{3}S_4 \right] + (1-\mathbf{N}_+) \left[\frac{23}{18}S_3 - S_2 \right] - (\mathbf{N}_- + \mathbf{N}_+) \Bigg[S_{1,1} + \frac{1237}{216}S_1 \\
& + \frac{11}{18}S_3 - \frac{317}{108}S_2 + \frac{16}{9}S_{1,-2} - \frac{2}{3}S_{1,-2,1} - \frac{1}{3}S_{1,-3} - \frac{1}{2}S_{1,3} - \frac{1}{2}S_{2,1} - \frac{1}{3}S_{2,-2} + S_1\zeta_3 + \frac{1}{2}S_{3,1} \Bigg] \Bigg) \\
& + 16 C_F n_f^2 \left(\frac{17}{144} - \frac{13}{27}S_1 + \frac{2}{9}S_2 + (\mathbf{N}_- + \mathbf{N}_+) \left[\frac{2}{9}S_1 - \frac{11}{54}S_2 + \frac{1}{18}S_3 \right] \right) + 16 C_F^2 n_f \Bigg(\frac{23}{16} - \frac{3}{2}\zeta_3 \\
& + \frac{4}{3}S_{-3,1} - \frac{59}{36}S_2 + \frac{4}{3}S_{-4} - \frac{20}{9}S_{-3} + \frac{20}{9}S_1 - \frac{8}{3}S_{1,-2} - \frac{8}{3}S_{1,1} - \frac{4}{3}S_{1,2} + \mathbf{N}_+ \Bigg[\frac{25}{9}S_3 - \frac{4}{3}S_{3,1} \\
& - \frac{1}{3}S_4 \Bigg] + (1-\mathbf{N}_+) \left[\frac{67}{36}S_2 - \frac{4}{3}S_{2,1} + \frac{4}{3}S_3 \right] + (\mathbf{N}_- + \mathbf{N}_+) \Bigg[S_1\zeta_3 - \frac{325}{144}S_1 - \frac{2}{3}S_{1,-3} + \frac{32}{9}S_{1,-2} \\
& - \frac{4}{3}S_{1,-2,1} + \frac{4}{3}S_{1,1} + \frac{16}{9}S_{1,2} - \frac{4}{3}S_{1,3} + \frac{11}{18}S_2 - \frac{2}{3}S_{2,-2} + \frac{10}{9}S_{2,1} + \frac{1}{2}S_4 - \frac{2}{3}S_{2,2} - \frac{8}{9}S_3 \Bigg] \Bigg)
\end{aligned}$$

Splitting functions in x-space

$$
\begin{aligned}
P^{(2)}_{\mathrm{ns}}(x) \;=\; & 16 C_A C_F n_f \Big(\mathbf{p_{qq}(x)} \Big[\frac{5}{9}\zeta_2 - \frac{209}{216} - \frac{3}{2}\zeta_3 + \frac{1}{3}\mathrm{Li}_3(x) - \frac{167}{108}\ln(x) + \frac{1}{3}\ln(x)\zeta_2 - \frac{1}{4}\ln^2(x)\ln(1-x) \\
& - \frac{7}{12}\ln^2(x) - \frac{1}{18}\ln^3(x) - \frac{1}{2}\ln(x)\mathrm{Li}_2(x) \Big] + p_{qq}(-x) \Big[\frac{1}{2}\zeta_3 - \frac{5}{9}\zeta_2 - \frac{2}{3}\ln(1+x)\zeta_2 + \frac{1}{6}\ln(x)\zeta_2 - \frac{10}{9}\ln(x)\ln(1+x) \\
& + \frac{5}{18}\ln^2(x) - \frac{1}{6}\ln^2(x)\ln(1+x) + \frac{1}{18}\ln^3(x) - \frac{10}{9}\mathrm{Li}_2(-x) - \frac{1}{3}\mathrm{Li}_3(-x) - \frac{1}{3}\mathrm{Li}_3(x) + \frac{2}{3}\mathrm{H}_{-1,0,1}(x) \Big] \\
& + (1+x)\Big[\frac{1}{6}\zeta_2 + \frac{1}{2}\ln(x) - \frac{1}{2}\mathrm{Li}_2(x) - \frac{2}{3}\mathrm{Li}_2(-x) - \frac{2}{3}\ln(x)\ln(1+x) + \frac{1}{24}\ln^2(x) \Big] + (1-x)\Big[\frac{1}{3}\zeta_2 - \frac{257}{54} \\
& + \ln(1-x) - \frac{17}{9}\ln(x) - \frac{1}{24}\ln^2(x) \Big] + \delta(1-x)\Big[\frac{5}{4} - \frac{167}{54}\zeta_2 + \frac{1}{20}{\zeta_2}^2 + \frac{25}{18}\zeta_3 \Big] \Big) + 16 C_F {n_f}^2 \Big(\mathbf{p_{qq}(x)} \Big[-\frac{1}{54} \\
& + \frac{5}{54}\ln(x) + \frac{1}{36}\ln^2(x) \Big] + (1-x)\Big[\frac{13}{54} + \frac{1}{9}\ln(x) \Big] - \delta(1-x)\Big[\frac{17}{144} - \frac{5}{27}\zeta_2 + \frac{1}{9}\zeta_3 \Big] \Big) + 16 {C_F}^2 n_f \Big(\mathbf{p_{qq}(x)} \Big[\frac{5}{3}\zeta_3 - \frac{55}{48} \\
& - \frac{2}{3}\mathrm{Li}_3(x) + \frac{5}{24}\ln(x) + \frac{1}{3}\ln(x)\zeta_2 + \frac{10}{9}\ln(x)\ln(1-x) + \frac{1}{4}\ln^2(x) + \frac{2}{3}\ln^2(x)\ln(1-x) + \frac{2}{3}\ln(x)\mathrm{Li}_2(x) - \frac{1}{18}\ln^3(x) \Big] \\
& + p_{qq}(-x) \Big[\frac{10}{9}\zeta_2 - \zeta_3 + \frac{4}{3}\ln(1+x)\zeta_2 - \frac{1}{3}\ln(x)\zeta_2 - \frac{5}{9}\ln^2(x) + \frac{20}{9}\ln(x)\ln(1+x) - \frac{1}{9}\ln^3(x) + \frac{1}{3}\ln^2(x)\ln(1+x) \\
& + \frac{20}{9}\mathrm{Li}_2(-x) + \frac{2}{3}\mathrm{Li}_3(-x) + \frac{2}{3}\mathrm{Li}_3(x) - \frac{4}{3}\mathrm{H}_{-1,0,1}(x) \Big] + (1+x)\Big[\frac{7}{36}\ln^2(x) - \frac{67}{72}\ln(x) + \frac{4}{3}\ln(x)\ln(1+x) \\
& + \frac{1}{12}\ln^3(x) + \frac{2}{3}\mathrm{Li}_2(x) + \frac{4}{3}\mathrm{Li}_2(-x) \Big] + (1-x)\Big[\frac{1}{9}\ln(x) - \frac{10}{9} - \frac{4}{3}\ln(1-x) + \frac{2}{3}\ln(x)\ln(1-x) - \frac{1}{3}\ln^2(x) \Big] \\
& - \delta(1-x)\Big[\frac{23}{16} - \frac{5}{12}\zeta_2 - \frac{29}{30}{\zeta_2}^2 + \frac{17}{6}\zeta_3 \Big] \Big)
\end{aligned}
$$

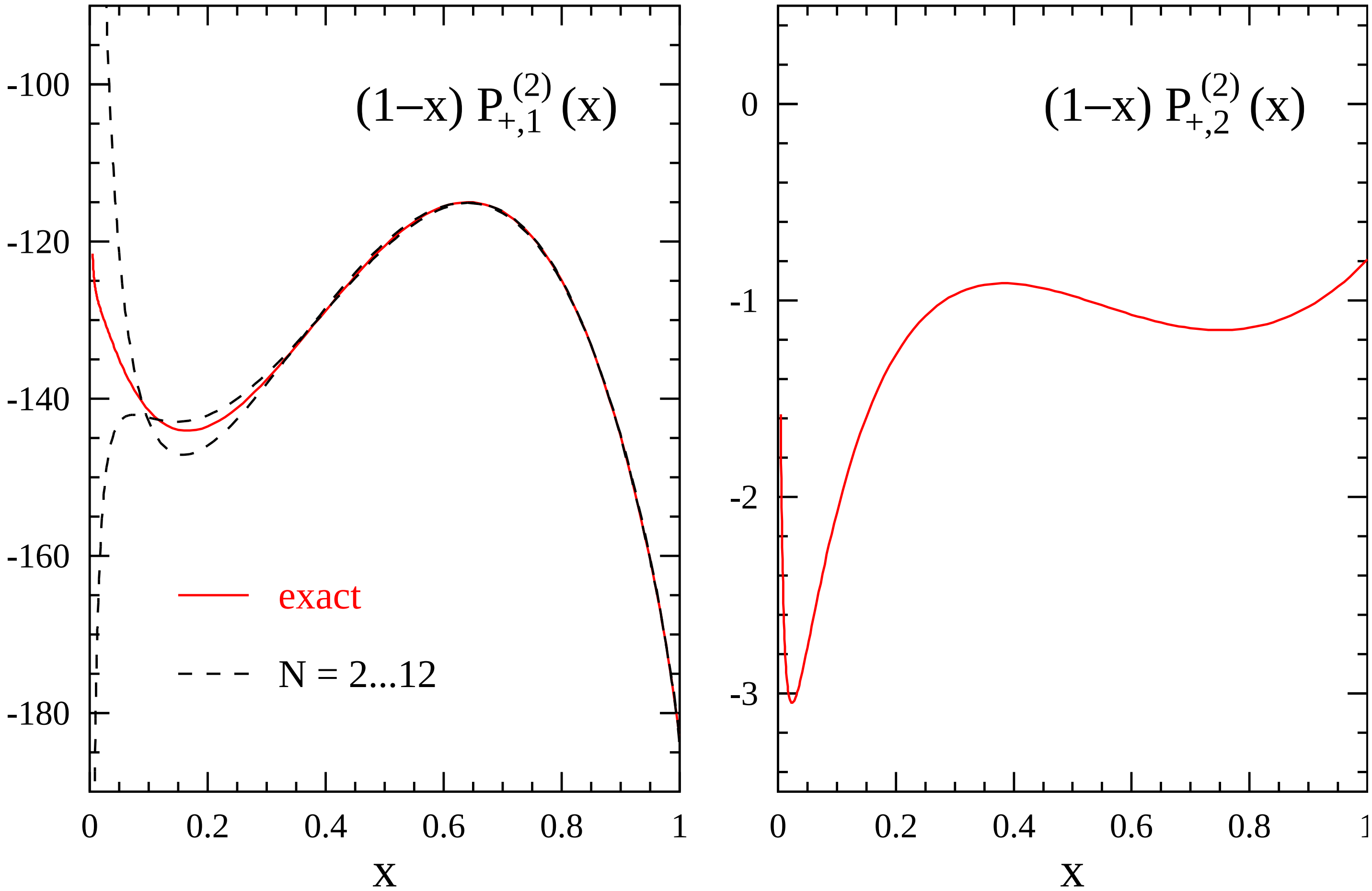
Easy-to-use parametrization

- Combine exact limits for $x \longrightarrow 0$ and $x \longrightarrow 1$ with smooth fit for intermediate x
- Notation : end-point logarithms $L_0 = \ln(x)$, $L_1 = \ln(1-x)$, +-distributions $\mathcal{D}_i = \left[\frac{\ln^i(1-x)}{1-x}\right]_+$

$$
\begin{aligned}
P^{(2)}_{\rm ns}(x) \quad \cong \quad & n_f \Big(-183.187\,\mathcal{D}_0 - 173.927\,\delta(1-x) - 5120/81\,L_1 - 197.0 + 381.1\,x + 72.94\,x^2 \\
& \quad + 44.79\,x^3 - 1.497\,xL_0^3 - 56.66\,L_0L_1 - 152.6\,L_0 - 2608/81\,L_0^2 - 64/27\,L_0^3\Big) \\
+ \quad & n_f^2\,\Big(-\mathcal{D}_0 - (51/16 + 3\,\zeta_3 - 5\,\zeta_2)\,\delta(1-x) + x\,(1-x)^{-1}L_0\,(3/2\,L_0 + 5) + 1 \\
& \quad + (1-x)\,(6 + 11/2\,L_0 + 3/4\,L_0^2)\Big)\;64/81
\end{aligned}
$$

Comparison with estimates from fixed moments

van Neerven, Vogt ‘00



Three-loop coefficient functions

– OPE and optical theorem
$\longrightarrow$ obtain simultaneously anomalous dimension $\gamma_{\rm ns}(N)$ and coefficient function $C^N_{2,\rm ns}$ or $C^N_{L,\rm ns}$

– Coefficient function $C^N_{2,\rm ns}$ at three loops
$\longrightarrow$ numerically the most relevant part of **NNNLO** corrections

– **Longitudinal** coefficient function $C^N_{L,\rm ns}$ at three loops
$\longrightarrow$ needed for **NNLO** analysis of $R = \sigma_L/\sigma_T$

– Easy-to-use parametrizations

$$
\begin{aligned}
c^{(3)}_{L,\rm ns}(x) \quad &\cong \quad n_f \Big(1024/81\, L_1^3 - 112.4\, L_1^2 + 340.3\, L_1 + 409 - 210\, x - 762.6\, x^2 - 1792/81\, x L_0^3 \\
&\qquad + L_0 L_1 \,(969.2 + 304.8\, L_0 - 288.2\, L_1) + 200.8\, L_0 + 64/3\, L_0^2 + 0.046\, \delta(1-x)\Big) \\
&+ \quad n_f^2 \Big(3x\, L_1^2 + (6 - 25\, x)\, L_1 - 19 + (317/6 - 12\, \zeta_2)\, x - 6x\, L_0 L_1 + 6x\, \mathrm{Li}_2(x) \\
&\qquad + 9x\, L_0^2 - (6 - 50\, x)\, L_0\Big)\, 64/81
\end{aligned}
$$

$$
\begin{aligned}
c^{(3)}_{2,\mathrm{ns}}(x) \quad \cong \quad & n_f \Big(640/81\, \mathcal{D}_4 - 6592/81\, \mathcal{D}_3 + 220.573\, \mathcal{D}_2 + 294.906\, \mathcal{D}_1 - 729.359\, \mathcal{D}_0 \\
& \quad + 2572.597\, \delta(1-x) - 640/81\, L_1^4 + 167.2\, L_1^3 - 315.3\, L_1^2 + 4742\, L_1 \\
& \quad + 762.1 + 7020\, x + 989.4\, x^2 + L_0 L_1 \,(326.6 + 65.93\, L_0 + 1923\, L_1) \\
& \quad + 260.1\, L_0 + 186.5\, L_0^2 + 12224/243\, L_0^3 + 728/243\, L_0^4 \Big) \\
+ \quad & n_f^2 \Big(64/81\, \mathcal{D}_3 - 464/81\, \mathcal{D}_2 + 7.67505\, \mathcal{D}_1 + 1.00830\, \mathcal{D}_0 - 103.2655\, \delta(1-x) \\
& \quad - 64/81\, L_1^3 + 15.46\, L_1^2 - 51.71\, L_1 + 59.00\, x + 70.66\, x^2 + L_0 L_1 \,(-80.05 \\
& \quad - 10.49\, L_0 + 41.67\, L_1) - 8.050\, L_0 - 1984/243\, L_0^2 - 368/243\, L_0^3 \Big)
\end{aligned}
$$

Application to threshold resummation

– Coefficient function $C_{2,\mathrm{ns}}$
$\longrightarrow$ large double logarithmic corrections for $N \to \infty$ cf. $x \to 1$

$$\alpha_s^l \left[\frac{\ln^{2l-1}(1-x)}{1-x} \right]_+ \qquad \longleftrightarrow \qquad \alpha_s^l \ln^{2l}(N)$$

– Large double logarithms originate from soft and collinear regions in Feynman diagrams

– Use **factorization** properties in soft/collinear limit $\longrightarrow$ **resummation** of large logarithms
Collins, Soper '81 ; Sterman '87 ; Catani, Trentadue '89 ; Magnea, Sterman '90 ; Catani, Webber '98 ; etc. ...

$$C_{2,\mathrm{ns}}(\alpha_s, N) = g(Q^2) \exp[G_{\mathrm{DIS}}(\alpha_s, N)] =$$

$$(1 + \alpha_s g_{01} + \alpha_s^2 g_{02} + \ldots) \exp\left[\ln(N)\, g_1(\alpha_s \ln(N)) + g_2(\alpha_s \ln(N)) + \alpha_s g_3(\alpha_s \ln(N)) + \ldots\right]$$

Physical picture

- Functions g_l depend on coefficients $A_{i\leq l}$, $B_{i\leq l-1}$ and $D^{\rm DIS}_{i\leq l-1}$

- **Factorization** in soft and collinear limit
  - soft gluons collinear to initial state partons $\longrightarrow$ universal A_i
  - soft gluons collinear to final state partons $\longrightarrow$ universal B_i
  - soft gluons at large angles $\longrightarrow$ process-dependent $D^{\rm DIS}_i$

Resummation of the next-to-next-to-leading logarithm

- Resumming to NNLL accuracy requires $g_3 \longrightarrow$ new coefficients A_3, B_2 and $D^{\rm DIS}_2$
  - $g_1 \longrightarrow A_1$
  - $g_2 \longrightarrow A_1, A_2, B_1, D^{\rm DIS}_1$
  - $g_3 \longrightarrow A_1, A_2, A_3, B_1, B_2, D^{\rm DIS}_1, D^{\rm DIS}_2$

Matching

- Fermionic contributions to A_3 from $\gamma_{\rm ns}^{(2)}$
  - independent check Berger '02

$$A_3\Big|_{n_f} = C_A C_F n_f \left[-\frac{836}{27} + \frac{160}{9}\zeta_2 - \frac{112}{3}\zeta_3 \right] + C_F^2 n_f \left[-\frac{110}{3} + 32\,\zeta_3 \right] + C_F {n_f}^2 \left[-\frac{16}{27} \right]$$

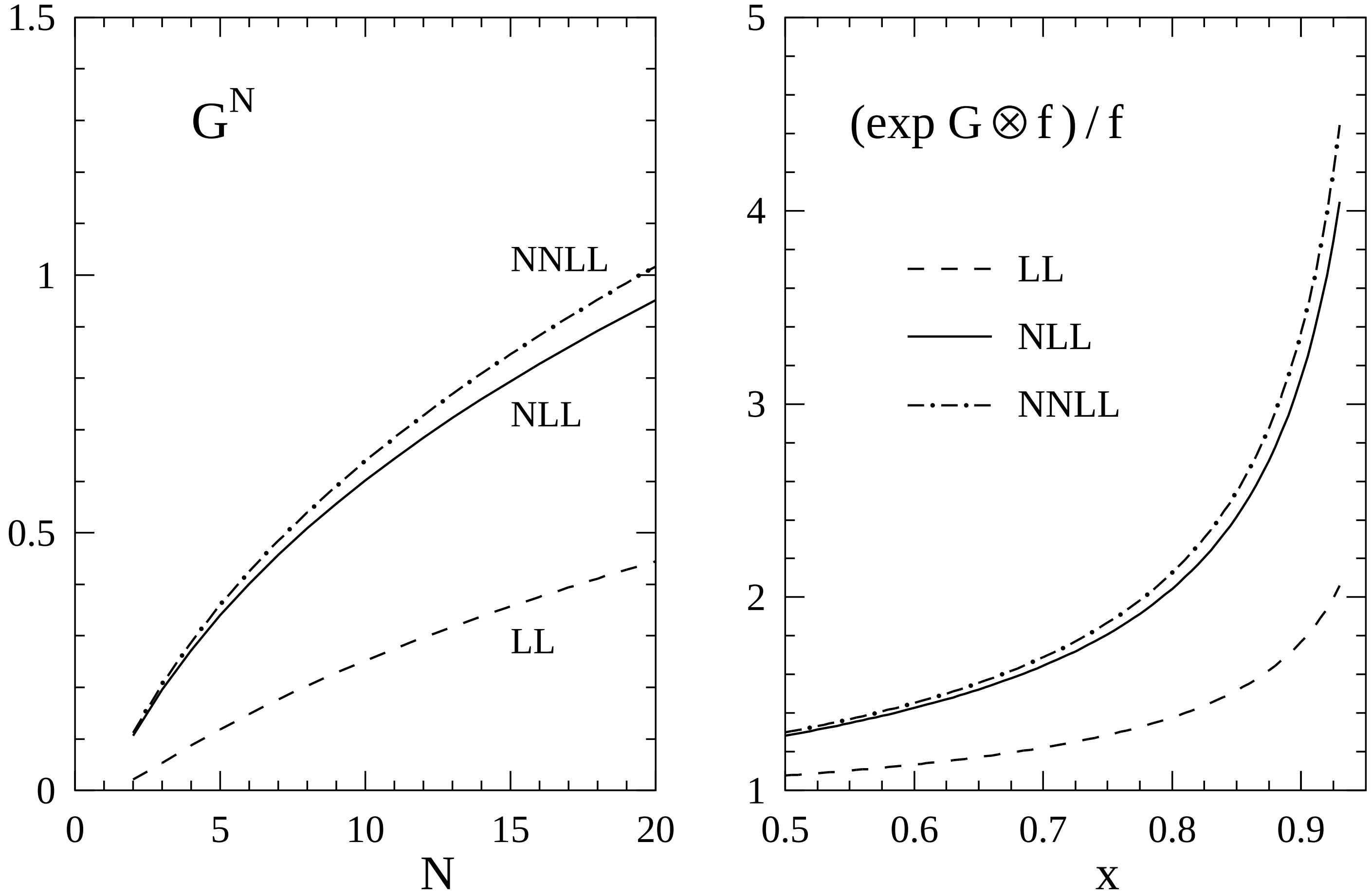
- Determine $B_2, D_2^{\rm DIS}$ with help of three-loop coefficient function
  - $B_2 + D_2^{\rm DIS}$ previously known
  - $c_{2,\rm ns}^{(3)}$ involves $\beta_0 (B_2 + 2\,D_2^{\rm DIS})$

$$B_2 = C_F^2 \left[-\frac{3}{2} - 24\,\zeta_3 + 12\,\zeta_2 \right] + C_F C_A \left[-\frac{3155}{54} + 40\,\zeta_3 + \frac{44}{3}\zeta_2 \right] + C_F n_f \left[\frac{247}{27} - \frac{8}{3}\zeta_2 \right]$$

$$D_2^{\rm DIS} = 0$$

- Remarkable result $D_1^{\rm DIS} = D_2^{\rm DIS} = 0$
  - $D_i^{\rm DIS} = 0$ to all orders from renormalization group arguments Forte, Ridolfi '03

Numerical analysis



– **LL, NLL, NNLL resummed** exponent $G^N(Q^2)$ for $\mu = Q$, $n_f = 4$ and $\alpha_s = 0.2$

– $G^N(Q^2)$ convoluted with typical input shape $xf = x^{1/2}(1-x)^3$

Upshot

- Large perturbative corrections to structure functions for $x \longrightarrow 1$

- Use **NNLO + NNLL resummed** perturbative QCD

- Investigate implications for higher twist
  - common phenomenological ansatz

$$F_2^{\mathrm{DATA}} = F_2^{\mathrm{QCD}} \left(1 + \frac{ht(x)}{Q^2} \right)$$

  - ansatz mixes $1/Q^2$-corrections with leading twist perturbative QCD corrections

Summary

What do we want ?

- **NNLO** analysis of deep-inelastic structure functions $F_2, F_3 \longrightarrow$ **high precision**
  - match experimental accuracy in final HERA data
  - get parton luminosity at LHC precisely

What do we learn ?

- Mellin moments and nested sums $\longrightarrow$ **powerful technology**
  - apply innovative and efficient method to solve multi-loop integrals
  - different process : $e^+e^- \rightarrow$ 3 jets at **NNLO** (n_f-terms) S.M., Uwer, Weinzierl '02
- **Key aspects** of technology
  - solve loop integrals via difference equations $\longrightarrow$ analytical solutions
  - use algebraic properties of harmonic sums $\longrightarrow$ algorithmic solution of nested sums
  - **check efficienctly** $\longrightarrow$ compute fixed values of N

What do we get (in the end) ?

- Calculation of deep-inelastic structure functions F_2, F_3 and F_L at **NNLO** (and beyond)
  - nonsinglet n_f-terms done S.M., Vermaseren, Vogt ‘02
  - complete calculation under way

- Application of formalism
  - Photon structure function at order $O(\alpha\alpha_s^2)$ and further distributions

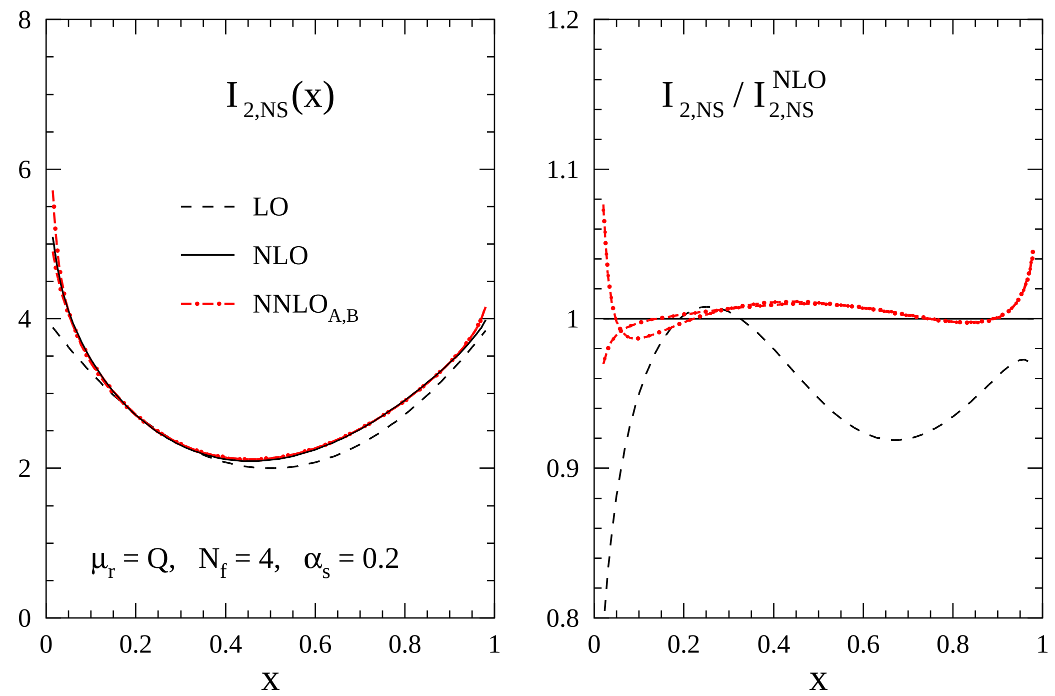
- New level of **precision phenomenology** in deep-inelastic scattering

The photon structure function

S.M, Vermaseren, Vogt '01

– Mellin moments $N=2,\ldots,12$ of photon structure function F_2^γ calculated at order $\alpha\,\alpha_s^2$

– Comparison of inhomogeneous evolution kernel at NLO and NNLO for non-singlet $1/x\,F_{2,NS}^\gamma$
  – approximate x-space expressions



– Present information on photon-parton splitting functions sufficient for $x \gtrsim 0.05$ at order $\alpha\,\alpha_s^2$