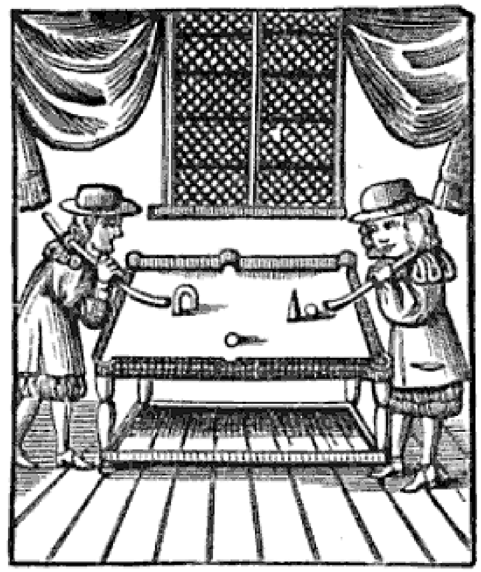
Scattering Amplitudes of Massive $\mathcal{N}=2$ Gauge Theories in Three Dimensions



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Abhishek Agarwal, Arthur Lipstein, DY, arXiv:1302.5288

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Outline

I. Invitation to amplitudeology

- Twistors
- BDS
- BCFW recursion relations

II. Mass-deformed $\mathcal{N} = 2$ amplitudes in $d = 3$

- Mass-deformed Chern-Simons theory (CSM)
- Yang-Mills-Chern-Simons theory (YMCS)
- Massive spinor-helicity in $d = 3$
- Trouble with YMCS external gauge fields
- On-shell SUSY algebras
- Four-point amplitudes: superamplitude for CSM
- Massive BCFW in $d = 3$

III. Conclusions and looking forward

Part I:
Amplitudeology

Amplitudeology

- Parke-Taylor formula: massive simplification of amplitudes in "spinor-helicity" variables; Witten's twistor string theory

- BCFW recursion: n-point amplitudes from $n-1$-point amplitudes

- Unitarity methods to construct loop-level amplitudes

- BCJ relations: duality between colour and kinematics

- KLT relations: gauge-theory amplitudes^2 = gravity amplitudes

- Grassmannian formulation

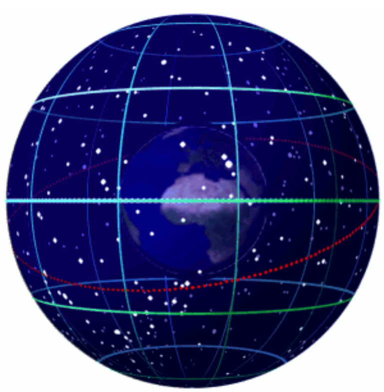
$\mathcal{N}=4$, ABJM	BDS formula
$\mathcal{N}=4$, ABJM	Dual superconformal symmetry, Yangian, null polygonal Wilson loops
$\mathcal{N}=4$, ABJM	Connections to spectral problem integrability

Penrose's twistors

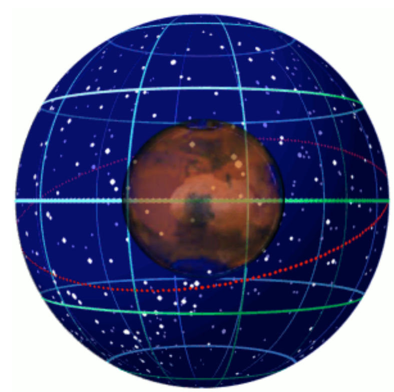
- Penrose's concept of twistors turns out to be an immensely powerful technique for describing massless amplitudes.

- Idea is to coordinatize space by the bundle of light-rays passing through a given point: i.e. by the local celestial sphere.

- Imagine two observers at different places in the galaxy. Knowledge of their celestial spheres is enough to determine their locations:



Observer A



Observer B

Twistors

Homogeneous coordinates of CP^3:

$$Z_I = (Z_1,\, Z_2,\, Z_3,\, Z_4), \quad Z_I \sim \lambda Z_I, \quad \lambda \in \mathbb{C}.$$

For a given twistor Z_I, the incidence relation ($\Longrightarrow$ null condition)

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \sigma^\mu x_\mu \begin{pmatrix} Z_3 \\ Z_4 \end{pmatrix} \Longrightarrow \mathrm{Im}\,(Z_1 Z_3^* + Z_2 Z_4^*) = 0,$$

fixes $x^\mu = (0,\, \vec{x}_0) + k^\mu \tau$ with $k^2 = 0$, i.e. specifies a single light ray, going through a specific point in space.

Two (or more) twistors Z_I and Z'_I incident to the same point $\vec{x}_0$ specify two (or more) different light rays through that point, i.e. $(0,\, \vec{x}_0) + k^\mu \tau$ and $(0,\, \vec{x}_0) + k'^\mu \tau$.

For fixed $\vec{x}_0$, the incidence relation takes $CP^3 \to CP^1 \sim S^2$ which is nothing but the celestial sphere at $\vec{x}_0$.

Spinor-helicity variables

On-shell massless particle representations

$$p^{a\dot{a}} = p_\mu(\sigma^\mu)^{a\dot{a}} = \lambda^a\bar{\lambda}^{\dot{a}}, \quad \langle ij\rangle = \epsilon_{ab}\lambda_i^a\lambda_j^b, \quad [ij] = \epsilon_{\dot{a}\dot{b}}\bar{\lambda}_i^{\dot{a}}\bar{\lambda}_j^{\dot{b}},$$

with which the Parke-Taylor formula for MHV tree-level gluon scattering amplitudes is expressed:

$$\Big\langle \underbrace{-}_{1}, \dots, -, \underbrace{+}_{i}, -, \dots, -, \underbrace{+}_{j}, -, \dots, \underbrace{-}_{n} \Big\rangle \propto \delta^4\left(\sum_{i=1}^{n}\lambda_i^a\bar{\lambda}_j^{\dot{a}}\right)\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\dots\langle n1\rangle}.$$

Notice that expression is “holomorphic” i.e. does not depend on $\bar{\lambda}$. Fourier transform w.r.t. $\bar{\lambda}$ [Witten, 2003]

$$\int d^4x \int \prod_i \frac{d^2\bar{\lambda}_i}{(2\pi)^2}\exp\left(i\sum_i \mu_{i\dot{a}}\bar{\lambda}_i^{\dot{a}}\right)\exp\left(ix_{a\dot{a}}\sum_i\lambda_i^a\bar{\lambda}_i^{\dot{a}}\right)f(\{\lambda\})$$

$$= \int d^4x \prod_i \underbrace{\boxed{\delta^2\left(\mu_{i\dot{a}} + x_{a\dot{a}}\lambda_i^a\right)}}_{\text{INCIDENCE RELATION}} f(\{\lambda\}) \to \text{define twistor } Z_I = (\lambda^a,\ \mu_{\dot{a}}).$$

The particles (light rays) interact at a common point in space-time.

Colour ordering, BDS formula

In large-N gauge theories we have fields $\phi = \phi^a T^a$, where T^a is (for example) a $SU(N)$ generator. Colour ordering refers to (e.g. for 4-particle scattering)

$$\left\langle \phi^{a_1 \dagger}(p_1)\, \phi^{a_2 \dagger}(p_2)\, \phi^{a_3 \dagger}(p_3)\, \phi^{a_4 \dagger}(p_4) \right\rangle = \mathcal{M}\,(p_1, p_2, p_3, p_4)\, \mathrm{Tr}[T^{a_1} T^{a_2} T^{a_3} T^{a_4}] + \ldots$$

this restricts to the $(p_1 + p_2)^2$ and $(p_1 + p_4)^2$, i.e. adjacent, channels.

In $\mathcal{N} = 4$, $d = 4$ SYM, the MHV amplitudes have a conjectured all-orders form [Bern, Dixon, Smirnov, 2005]

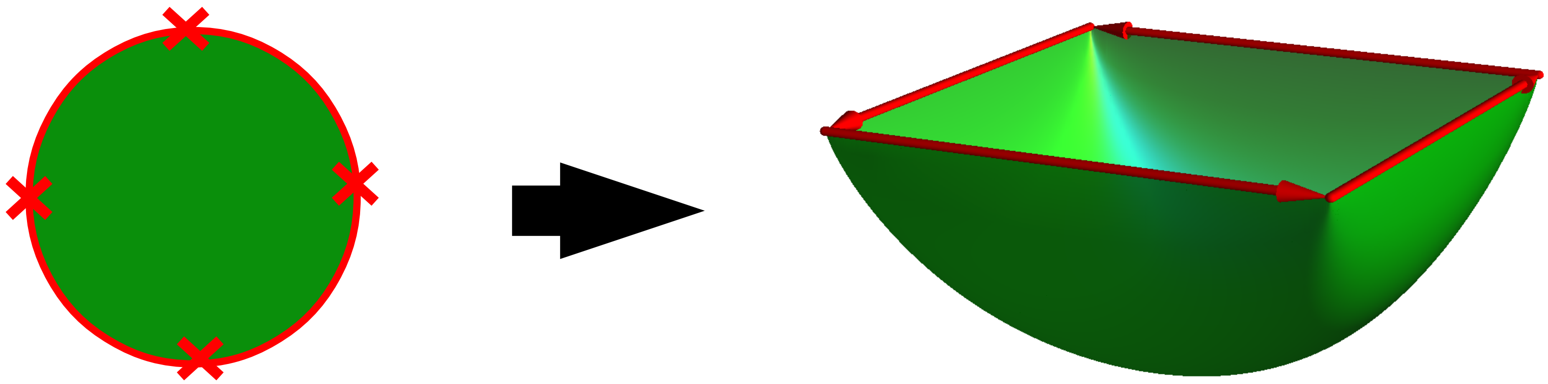
$$\log \frac{\mathcal{M}_{\mathrm{MHV}}}{\mathcal{M}^{\mathrm{tree}}_{\mathrm{MHV}}} = -\sum_{i=1}^{n} \left[\frac{1}{8\epsilon^2} f^{(-2)}\left(\frac{\lambda \mu_{IR}^{2\epsilon}}{(-s_{i,i+1})^{\epsilon}} \right) + \frac{1}{4\epsilon} g^{(-1)}\left(\frac{\lambda \mu_{IR}^{2\epsilon}}{(-s_{i,i+1})^{\epsilon}} \right) \right] + f(\lambda) \frac{R}{4} + \text{finite}.$$

where $f^{(-n)}(\lambda)$ in the n-th logarithmic integral of the cusp anomalous dimension $f(\lambda)$.

IR divergences have been regulated by going above four dimensions, i.e. $d = 4 - 2\epsilon$ with $\epsilon < 0$.

Dual superconformal symmetry and Wilson loops

Alday & Maldacena taught us that at strong coupling, the dual of the amplitude is the dual of a null-polygonal Wilson loop: i.e. a string worldsheet:

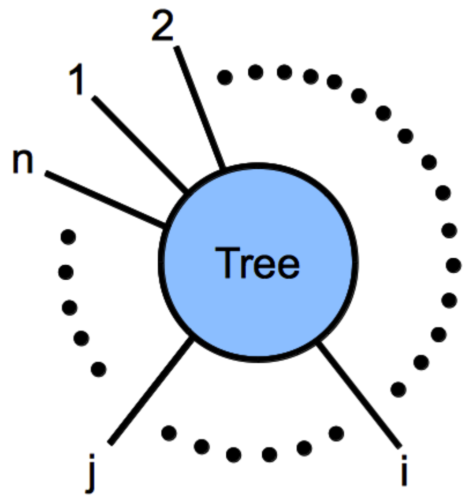


$$\frac{\mathcal{M}_{\mathrm{MHV}}}{\mathcal{M}^{\mathrm{tree}}_{\mathrm{MHV}}} = \left\langle \frac{1}{N}\,\mathrm{Tr}\,P\exp\oint_C d\tau\, i\dot{x}^\mu A_\mu \right\rangle = \exp\left(-\frac{\sqrt{\lambda}}{2\pi}(\text{Area of Min. Surf.})\right)$$

Moreover, the duality holds also at *weak* coupling [Brandhuber, Heslop, Travaglini, 2007]. Reason: under T-duality $p_i \leftrightarrow x_{i+1} - x_i$, and AdS is mapped to *itself*. Amplitude is dual to high energy scattering on an IR brane à la Gross & Mende, T-duality maps it to the null-polygon in the UV, i.e. on the boundary.

The picture which has emerged is that there is a full dual $PSU(2,2|4)$ symmetry and a Yangian symmetry relating the two [Drummond, Henn, Plefka, 2009].

Recursion relations

$\mathcal{A}(z) \quad =$ 

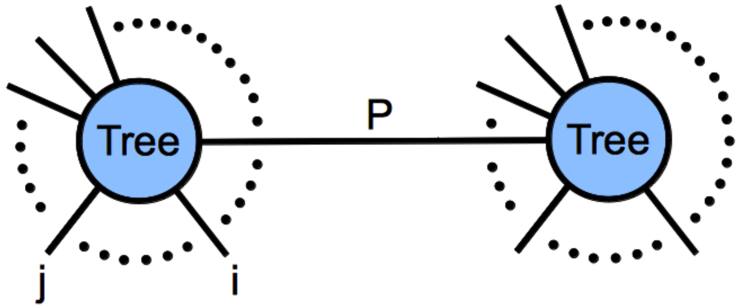
$$p_i \to \tilde{p}_i = p_i + z\,q,$$
$$p_j \to \tilde{p}_j = p_j - z\,q,$$
$$\tilde{p}_i^2 = 0 = \tilde{p}_j^2,$$
$$q^2 = q \cdot p_i = q \cdot p_j = 0.$$

[Britto, Cachazo, Feng, Witten, 2005]

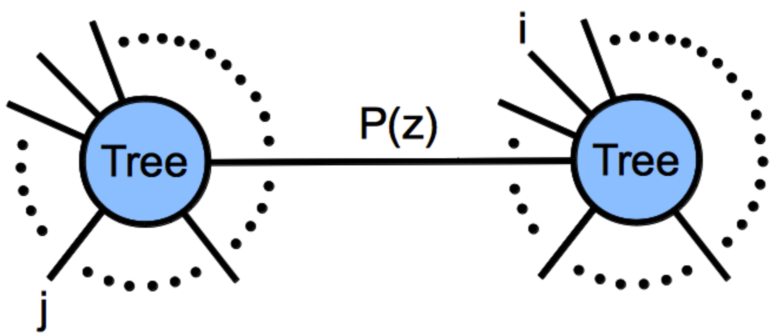
- $\mathcal{A}(z) = \frac{F(z)}{G(z)}$ i.e. amplitude is rational.
- Poles in z are simple.
- $\lim_{z\to\infty} \mathcal{A}(z) = 0.$

$$\Longrightarrow \quad \mathcal{A}(z) = \sum_p \mathrm{Res}[\mathcal{A}(z),\, z_p]/(z - z_p)$$

Recursion relations cont'd



$$P = \sum_L p = -\sum_R p = \text{no } z \text{ dep.}$$



$$P = \sum_L p = -\sum_R p = \text{depends on } z$$

$$\mathcal{A}(z) = \sum_{\text{splittings}} \frac{\mathcal{A}_L(z_p)\mathcal{A}_R(z_p)}{P^2(z)} \Longrightarrow \mathcal{A} = \mathcal{A}(0) = \sum_{\text{splittings}} \frac{\mathcal{A}_L(z_p)\mathcal{A}_R(z_p)}{P^2},$$

$$\mathcal{A}_L(z_p) = \mathcal{A}\big(\ldots, p_i(z_p), \ldots, P(z_p)\big), \quad \mathcal{A}_R(z_p) = \mathcal{A}\big(\ldots, p_j(z_p), \ldots, -P(z_p)\big),$$

$$P^2(z_p) = 0.$$

A few slides motivating amplitudes in three-dimensions

Three-dimensional theories: ABJM

- Propagating degrees of freedom are scalars and fermions. Results have not been interpreted in terms of "helicity".

- $\mathcal{A}_4^{\text{tree}} = \delta^3(P)\delta^6(Q)/\sqrt{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$, six-partilcle result also known [Agarwal, Beisert, McLoughlin, 2008], [Bargheer, Loebbert, Meneghelli, 2010].

- BCFW and dual super-conformal invariance [Gang, Huang, Koh, Lee, Lipstein, 2011].

- Extensions to loop-level performed [Chen, Huang, 2011] [Bianchi, Leoni$^{(2)}$, Mauri, Penati, Santambrogio, 2011] [Caron-Huot, Huang, 2013].

- Yangian constructed [Bargheer, Loebbert, Meneghelli, 2010].

- Grassmanian proposed [Lee, 2010].

- Light-like Wilson loop seems to match amplitudes [Henn, Plefka, Wiegandt, 2010], [Bianchi, Leoni, Mauri, Penati, Santambrogio, 2011].

Three-dimensional theories: $\mathcal{N} = 8$ SYM and ABJM

- Strong coupling IR fixed point of $\mathcal{N} = 8$ SYM is believed to be ABJM.
- Can be seen using M2-to-D2 Higgsing of ABJM.
- On-shell supersymmetry algebras of the two theories (and analogues with less SUSY) may be mapped to each other [Agarwal, DY (2012)].
- One-loop MHV vanish, one-loop non-MHV are finite [Lipstein, Mason (2012)]. – ABJM 1-loop amps either vanish or are finite.
- $\mathcal{N} = 8$ amps have dual conformal covariance [Lipstein, Mason (2012)], ABJM amps have dual conformal invariance.
- 4-pt. 2-loop amplitudes agree in the Regge limit between the two theories [Bianchi, Leoni (2013)].

Mass-deformed three-dimensional theories: $\mathcal{N} \geq 4$ Chern-Simons-Matter amplitudes

[Agarwal, Beisert, McLoughlin (2008)]

- Amplitudes computed at the tree and one-loop level.
- Exploited $SU(2|2)$ algebra to relate amplitudes to one another – same contraints at play in $\mathcal{N} = 4$ SYM spin chains!

Now we will look at massive Chern-Simons-Matter theory with $\mathcal{N} = 2$, and also at another way of introducing mass: Yang-Mills-Chern-Simons theory.

Part II:
Mass-deformed $\mathcal{N} = 2$ amplitudes in $d = 3$

$\mathcal{N} = 2$ massive Chern-Simons-matter theory

$$
\begin{aligned}
S_{CSM} = \kappa \int \epsilon^{\mu\nu\rho} \, \mathrm{Tr}(A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho) \\
- 2 \int \mathrm{Tr}\, |D_\mu \Phi|^2 + 2i \int \mathrm{Tr}\, \bar{\Psi}(D_\mu \gamma^\mu \Psi + m\Psi) \\
- \frac{2}{\kappa^2} \int \mathrm{Tr}\left(|[\Phi, [\Phi^\dagger, \Phi]] + e^2 \Phi|^2 \right) + \frac{2i}{\kappa} \int \mathrm{Tr}([\Phi^\dagger, \Phi][\bar{\Psi}, \Psi] + 2[\bar{\Psi}, \Phi][\Phi^\dagger, \Psi])
\end{aligned}
$$

- Gauge field is non-dynamical: external states are Φ's and Ψ's.
- Mass is set by e: this quantity does not run, $m = e^2/\kappa$.
- $\kappa = k/(4\pi)$, k is CS level.
- Couplings in potential include Φ^6, Φ^4, and $\Phi^2\Psi^2$.

$\mathcal{N} = 2$ Yang-Mills-Chern-Simons theory

Chern-Simons theory as a mass-term in $d = 3$ [Deser, Jackiw, Templeton (1982)]:

$$S_{YM} = \frac{\text{Tr}}{e^2} \int \Big[-\frac{1}{2} F_{\mu\nu} F^{\mu\nu} - D_\mu \Phi D^\mu \Phi + F^2 + i\bar{\Psi}_I \gamma^\mu D_\mu \Psi_I + \epsilon_{AB} \bar{\Psi}_A [\Phi, \Psi_B] \Big],$$

$$S_{CS} = \frac{m}{e^2} \text{Tr} \int \Big[\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \frac{2i}{3} \epsilon^{\mu\nu\rho} A_\mu A_\nu A_\rho + i\bar{\Psi}_I \Psi_I + 2F\Phi \Big]$$

Magic Arithmetic:

$$\begin{array}{r} S_{YM} \\ + S_{CS} \\ \hline S_{YMCS} \end{array} \quad \Longrightarrow \quad \begin{array}{r} \text{massless} \\ + \text{non-dynamical} \\ \hline \text{massive} \end{array}$$

- Auxilliary field F gives mass term for Φ.
- Fermion mass-term present in S_{CS}.
- Gauge field kinetic term is $A^\mu \left(\partial^2 \eta_{\mu\nu} - \partial_\mu \partial_\nu - m\epsilon_{\mu\nu\rho} \partial^\rho \right) A^\nu \implies$

$$\Delta_{\mu\nu}(p) = \frac{1}{p^2(p^2 + m^2)} \left(p^2 \eta_{\mu\nu} - p_\mu p_\nu + im\epsilon_{\mu\nu\rho} p^\rho \right).$$

Massive spinor-helicity in $d = 3$

Recall: $p^{\alpha\dot{\alpha}} = \lambda^{\alpha}\bar{\lambda}^{\dot{\alpha}}$ for massless spinors in $d = 4$. The reason for this is that the Lorentz group (up to signature) is $SO(4) \sim SU(2) \times SU(2)$, hence we have α and $\dot{\alpha}$.

- Massive momentum in $d = 3$ also has 3 d.o.f.

- Lorentz group is $SO(3) \sim SU(2)$, thus distinction between α and $\dot{\alpha}$ dissappears; extra momentum-component becomes a three-dimensional mass

$$p^{\alpha\beta} = \lambda^{\alpha}\bar{\lambda}^{\beta} - im\epsilon^{\alpha\beta}.$$

- $p^2 = -m^2$, $\langle\lambda\bar{\lambda}\rangle = \epsilon_{\beta\alpha}\lambda^{\alpha}\bar{\lambda}^{\beta} = -2im$.

- Square-bracket from four dimensions is replaced by a barred notation: $\langle ij\rangle$, $\langle\bar{i}j\rangle$, $\langle i\bar{j}\rangle$, and $\langle\bar{i}\bar{j}\rangle$.

Trouble with external gauge fields in YMCS

In YMCS, the electric field does not commute with itself:

$$[E^i(x), E^j(x')] \sim \epsilon^{ij}\delta^2(x - x')$$

A usual mode expansion like

$$A^a_\mu(x) = \int \frac{d^2p}{(2\pi)^2}\frac{1}{\sqrt{2p^0}}\left(\epsilon_\mu(p)a_1^{a\dagger}(p)e^{ip\cdot x} + \epsilon^*_\mu(p)a_1^a(p)e^{-ip\cdot x}\right)$$

does not fit the bill! [Haller, Lim-Lombridas, (1994)].

We learned this the hard way:

- YMCS amplitudes with external gauge fields computed using a standard mode expansion do not respect the SUSY algebra!

Two different theories, two different SUSY algebras

- CSM theory: $\Phi = \Phi_1 + i\Phi_2$, $\Psi = \Psi_1 + i\Psi_2$, $SO(2)$ R-symmetry

$$\{Q^{\beta J}, Q^{\alpha I}\} = \frac{1}{2}\left(P^{\alpha\beta}\delta^{IJ} + m\epsilon^{\beta\alpha}\epsilon^{JI}R\right)$$

- YMCS theory: Real scalar $\Phi \sim \Phi_2$, gauge field d.o.f. $A \sim \Phi_1$: no R-symmetry although Ψ_1 and Ψ_2 do enjoy $SO(2)$

$$\{Q^{\beta J}, Q^{\alpha I}\} = \frac{1}{2}P^{\alpha\beta}\delta^{IJ}$$

On-shell SUSY algebras

Solutions to the massive Dirac equation ($\not{p}\,\bar{\lambda} = im\bar{\lambda},\ \not{p}\,\lambda = -im\lambda$):

$$\bar{\lambda}(p) = \frac{-1}{\sqrt{p_0 - p_1}} \begin{pmatrix} p_2 + im \\ p_1 - p_0 \end{pmatrix}, \quad \lambda(p) = \frac{1}{\sqrt{p_0 - p_1}} \begin{pmatrix} p_2 - im \\ p_1 - p_0 \end{pmatrix}.$$

CSM:

$$Q_I|\Phi_1\rangle = -\frac{1}{2}\bar{\lambda}|\Psi_I\rangle,$$
$$Q_I|\Phi_2\rangle = -\frac{1}{2}\bar{\lambda}\epsilon^{IJ}|\Psi_J\rangle,$$
$$Q_I|\Psi_J\rangle = \frac{1}{2}\delta_{IJ}\lambda|\Phi_1\rangle + \frac{1}{2}\epsilon^{IJ}\lambda|\Phi_2\rangle.$$

YMCS:

$$Q_I|A\rangle = \frac{1}{2}\lambda|\Psi_I\rangle,$$
$$Q_I|\Phi\rangle = -\frac{1}{2}\bar{\lambda}\epsilon^{IJ}|\Psi_J\rangle,$$
$$Q_I|\Psi_J\rangle = -\frac{1}{2}\delta_{IJ}\bar{\lambda}|A\rangle + \frac{1}{2}\epsilon^{IJ}\lambda|\Phi\rangle.$$

CSM theory has $SO(2)$ R-symmetry: $a_\pm \equiv (\Phi_1 \pm i\Phi_2)/\sqrt{2}$, $\chi_\pm = (\Psi_1 \pm i\Psi_2)/\sqrt{2}$

$$Q_+|a_+\rangle = -\frac{1}{\sqrt{2}}\bar{\lambda}|\chi_+\rangle, \ \ Q_+|\chi_-\rangle = \frac{1}{\sqrt{2}}\lambda|a_-\rangle,$$
$$Q_-|a_-\rangle = -\frac{1}{\sqrt{2}}\bar{\lambda}|\chi_-\rangle, \ \ Q_-|\chi_+\rangle = \frac{1}{\sqrt{2}}\lambda|a_+\rangle,$$
$$Q_-|a_+\rangle = Q_+|\chi_+\rangle = Q_+|a_-\rangle = Q_-|\chi_-\rangle = 0.$$

CSM four-point amplitudes

- SUSY algebra is a powerful constraint:

$$0 = Q_-\langle \chi_+ a_+ a_- a_- \rangle = \lambda_1 \langle a_+ a_+ a_- a_- \rangle + \bar{\lambda}_3 \langle \chi_+ a_+ \chi_- a_- \rangle + \bar{\lambda}_4 \langle \chi_+ a_+ a_- \chi_- \rangle$$

$$\implies \langle 1\bar{3} \rangle \langle \chi_+ a_+ \chi_- a_- \rangle = -\langle 1\bar{4} \rangle \langle \chi_+ a_+ a_- \chi_- \rangle$$

- Including crossing relations, tree-level four-point amplitudes all related to one single amplitude.

- Can be packaged into two superamplitudes:

$$\mathcal{A}_{\Phi\Phi\Psi\Psi} = \frac{\langle 24 \rangle}{\langle \bar{3}2 \rangle} \delta^3(P)\delta^2(Q), \quad \mathcal{A}_{\Phi\Psi\Phi\Psi} = \frac{\langle 41 \rangle \langle 4\bar{1} \rangle - \langle 43 \rangle \langle 4\bar{3} \rangle}{\langle 1\bar{2} \rangle \langle 4\bar{1} \rangle} \delta^3(P)\delta^2(Q),$$

$$P^{\alpha\beta} = \sum_{i=1}^{4} \lambda_i^{(\alpha} \bar{\lambda}_i^{\beta)}, \;\; Q^\alpha = \sum_{i=1}^{4} \lambda_i^\alpha \bar{\eta}_i + \bar{\lambda}_i^\alpha \eta_i, \;\; \delta^2(Q) = Q^\alpha Q_\alpha,$$

$$\Phi = a_+ + \bar{\eta}\chi_+, \;\; \Psi = \chi_- + \eta a_-.$$

YMCS four-point amplitudes

- SUSY algebra less constraining: three-amplitude relations instead of two-amplitude relations:

$$
\begin{aligned}
&Q_2\langle\Psi_1\Psi_2 A\Psi_1\rangle = 0\\
&= -\lambda_1\langle\Phi\Psi_2 A\Psi_1\rangle - \bar{\lambda}_2\langle\Psi_1 AA\Psi_1\rangle + \lambda_3\langle\Psi_1\Psi_2\Psi_2\Psi_1\rangle - \lambda_4\langle\Psi_1\Psi_2 A\Phi\rangle
\end{aligned}
$$

- Can use the SUSY algebra to obtain all four-point amplitudes with external gauge fields from those without.

- Four-fermion amplitudes:

$$
\begin{aligned}
\langle\chi_+\chi_+\chi_-\chi_-\rangle = \langle\chi_-\chi_-\chi_+\chi_+\rangle &= -\frac{2\langle 34\rangle}{u+m^2}\left[\langle 12\rangle + im\frac{\langle 42\rangle}{\langle 4\bar{1}\rangle}\right],\\
\langle\chi_+\chi_-\chi_-\chi_+\rangle = \langle\chi_-\chi_+\chi_+\chi_-\rangle &= \frac{2\langle 41\rangle}{s+m^2}\left[\langle 23\rangle + im\frac{\langle 13\rangle}{\langle 1\bar{2}\rangle}\right].
\end{aligned}
$$

YMCS four-point amplitudes cont'd

Example of a nastier-looking amplitude:

$$\langle \chi_+ A A \chi_- \rangle = -\frac{\langle 41\rangle\langle \bar{4}\bar{1}\rangle}{\langle \bar{2}4\rangle\langle \bar{4}\bar{3}\rangle}\langle \Psi_2\Psi_2\Psi_1\Psi_1\rangle + \frac{\langle 43\rangle}{\langle \bar{2}4\rangle}\langle \Psi_1\Psi_2\Psi_2\Psi_1\rangle - \frac{\langle 41\rangle\langle 2\bar{4}\rangle}{\langle \bar{2}4\rangle\langle \bar{4}\bar{3}\rangle}\langle \Phi\Phi\chi_+\chi_-\rangle$$

$$= \frac{1}{\langle \bar{2}\bar{1}\rangle}\Bigg[-2\frac{\langle 41\rangle\langle 23\rangle}{\langle \bar{3}1\rangle}(s+2m^2) + 2\frac{\langle 12\rangle\langle 34\rangle}{\langle \bar{3}1\rangle}(s+4m^2)$$

$$- im\left(\langle 32\rangle\langle 34\rangle - 2\frac{\langle 42\rangle\langle 34\rangle}{\langle \bar{3}1\rangle\langle 4\bar{1}\rangle}(s+4m^2)\right)\Bigg]\frac{1}{u+m^2}$$

$$+ \frac{1}{\langle \bar{4}\bar{3}\rangle}\Bigg[\frac{\langle 12\rangle\langle 34\rangle}{\langle \bar{2}4\rangle}(s-u) - 2\frac{\langle 23\rangle\langle 41\rangle}{\langle \bar{2}4\rangle}(t+s) - 2\frac{\langle 23\rangle\langle 4\bar{1}\rangle\langle 1\bar{3}\rangle}{\langle \bar{1}\bar{2}\rangle\langle \bar{3}4\rangle}(s+2m^2)$$

$$+ 2im\frac{\langle 13\rangle\langle 14\rangle}{\langle \bar{2}4\rangle\langle 1\bar{2}\rangle}(t+s) + im\frac{\langle 23\rangle}{\langle \bar{1}\bar{2}\rangle}(u-t)\Bigg]\frac{1}{s+m^2}.$$

BCFW for massless lines in $d = 3$

Recall: we had a linear shift in $d = 4$

$$p_i \to p_i + zq, \qquad p_j \to p_j - zq$$

this will **not** work in $d = 3$

$$q = \alpha\, p_i + \beta\, p_j + \gamma\, p_i \wedge p_j$$

requiring $p_i^2 = p_j^2 = 0$ means requiring $q \cdot p_i = q \cdot p_j = q^2 = 0$ but then $\alpha = \beta = \gamma = 0$.

Resolution: Use a non-linear shift [Gang, Huang, Koh, Lee, Lipstein (2011)]

$$p_{\substack{i\\j}} \to \frac{1}{2}(p_i + p_j) \pm z^2 q \pm z^{-2} \tilde{q}, \qquad q + \tilde{q} = \frac{1}{2}(p_i - p_j)$$

then $q^2 = \tilde{q}^2 = q \cdot (p_i + p_j) = \tilde{q} \cdot (p_i + p_j) = 0$ and $2\, q \cdot \tilde{q} = -p_i \cdot p_j$ can be solved!

N.B. undeformed case is now $z = 1$.

BCFW for massive lines in $d = 3$

In terms of spinor variables the BCFW shift is expressed as

$$\begin{pmatrix} \lambda_i \\ \lambda_j \end{pmatrix} \to \begin{pmatrix} \frac{1}{2}\left(z+z^{-1}\right) & \frac{i}{2}\left(z-z^{-1}\right) \\ -\frac{i}{2}\left(z-z^{-1}\right) & \frac{1}{2}\left(z+z^{-1}\right) \end{pmatrix} \begin{pmatrix} \lambda_i \\ \lambda_j \end{pmatrix}.$$

This can be extended to the massive case just by doing the same to the $\bar{\lambda}$'s:

$$\begin{pmatrix} \bar{\lambda}_i \\ \bar{\lambda}_j \end{pmatrix} \to \begin{pmatrix} \frac{1}{2}\left(z+z^{-1}\right) & \frac{i}{2}\left(z-z^{-1}\right) \\ -\frac{i}{2}\left(z-z^{-1}\right) & \frac{1}{2}\left(z+z^{-1}\right) \end{pmatrix} \begin{pmatrix} \bar{\lambda}_i \\ \bar{\lambda}_j \end{pmatrix}.$$

We then can express the recursion relation as

$$A(z=1) = -\frac{1}{2\pi i} \sum_{f,j} \oint_{z_{f,j}} \frac{A_L(z) A_R(z)}{\hat{p}_f(z)^2 + m^2} \frac{1}{z-1},$$

where f labels splittings, $\hat{p}_f(z)^2 + m^2 = a_f z^{-2} + b_f + c_f z^2$, and j labels its four roots.

Applying BCFW to CSM and YMCS

- The question of applicability has to do with the large-z behaviour of the amplitudes.

- We need $A(z) \to 0$ when $z \to \infty$.

- The YMCS component amplitudes do not have this property.

- The CSM component amplitudes don't either, but the superamplitude does.

- Thus the CSM theory seems amenable to BCFW recursion.

Future directions

- Compute 6-pt. amplitudes in CSM and see if BCFW gives the same result.
- Explore the theories at loop-level.
- Does there exist a superamplitude expression for YMCS?
- Understand how to compute amplitudes with external gauge fields in YMCS.

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Thanks!