

# Holographic obstructions to symmetry-preserving regulators

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based on [arXiv:1306.3992](https://arxiv.org/abs/1306.3992) with:

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This talk is about (examples of) obstructions to symmetry-preserving regulators of QFT, esp. in 3+1 dimensions.

**Goal:** understand such obstructions by thinking about certain states of matter in one higher dimension with an energy gap (i.e.  $E_1 - E_{gs} > 0$  in thermodynamic limit).

More precisely: using their low-energy effective field theories (topological field theories (TFTs) in  $D = 4 + 1$ ).

These will be difficult phases to access in the lab!

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<sup>2</sup>they are 3+1 dimensional at least

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**Strategy:** use theories that obviously don't exist<sup>1</sup> to prove that certain slightly more reasonable-looking theories<sup>2</sup> don't exist even in principle<sup>3</sup>.

**One outcome:** Constraints on manifest electric-magnetic duality symmetry.

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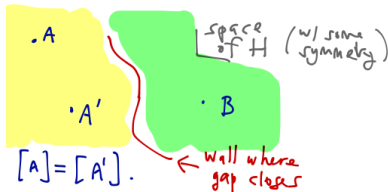
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# Realizations of symmetries in QFT and cond-mat

**Basic Q:** What are possible gapped phases of matter?

**Def:** Two gapped states are equivalent if they are adiabatically connected

(varying the parameters in the  $\mathbf{H}$  whose ground state they are to get from one to the other, without closing the energy gap).



One important distinguishing feature: how are the symmetries realized?

**Landau distinction:** characterize by *broken* symmetries

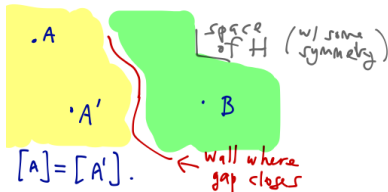
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**Mod out by Landau:** "What are possible (gapped) phases that don't break symmetries?" How do we distinguish them?

One (fancy) answer [Wen]: topological order.

# Topological order

Canonical examples: (abelian) *fractional* quantum Hall states in  $D = 2 + 1$

EFT is Chern-Simons-Witten gauge theory:  $S[a_I] = \sum_{IJ} \frac{K_{IJ}}{4\pi} \int a_I \wedge da_J$

electron current is  $j_\mu = \epsilon_{\mu\nu\rho} \partial_\nu a_\rho^I t_I$

3 intimately-connected features:

## 1. Fractionalization of symmetries

(i.e. emergent quasiparticle excitations carry quantum numbers (spin, charge) which are fractions of those of the constituents)

quasiparticles are anyons  
of charge  $e/k$

## 2. # of groundstates depends on the topology of space.

1  $\implies$  2: Pair-create qp-antiqp pair, move them around a spatial cycle, then re-annihilate.

This process  $\mathcal{F}_x$  maps one gs to another.

# of groundstates =  $|\det(K)|^{\text{genus}}$

Simplest case:  $K = k$ .  $\mathcal{F}_x = e^{i\oint_{C_x} a}$

$$\mathcal{F}_x \mathcal{F}_y = \mathcal{F}_y \mathcal{F}_x e^{2\pi i/k}$$



$\longrightarrow k^g$  groundstates.

## Topological order, cont'd

### 3. Requires *long-range entanglement*

[Kitaev-Preskill, Levin-Wen]:

2  $\implies$  3:  $S(A) \equiv -\text{tr } \rho_A \log \rho_A$ , the EE of the subregion  $A$  in the state in question.

$$S(A) = \Lambda \ell(\partial A) - \gamma \quad (\Lambda = \text{UV cutoff})$$

$\gamma \equiv$  “topological entanglement entropy”

$$\propto \log(\#\text{torus groundstates}) \geq 0.$$

(Deficit relative to area law.)

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c.f.: For a state w/o LRE

$$S(A) = \oint_{\partial A} s dl$$

(local at bdy)

$$= \oint (\Lambda + bK + cK^2 + \dots)$$

$$= \Lambda \ell(\partial A) + \tilde{b} + \frac{\tilde{c}}{\ell(\partial A)}$$

Pure state:

$$S(A) = S(\bar{A}) \implies b = 0.$$

[Grover-Turner-Vishwanath]



## Mod out by $W_n$ , too

“What are possible (gapped) phases that don't break symmetries and don't have topological order?”

# Mod out by Wen, too

“What are possible (gapped) phases that don't break symmetries and don't have topological order?”

[nice review: Turner-Vishwanath, 1301.0330]

In the absence of topological order ('SRE', hence simpler),  
another answer: Put the model on the space with boundary.

A gapped state of matter in  $d + 1$  dimensions

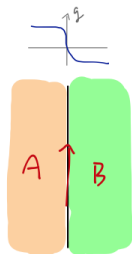
with short-range entanglement

can be (at least partially) characterized (within some symmetry class of hamiltonians) by (properties of) its edge states  
(i.e. what happens at an interface with the vacuum,  
or with another SRE state).

# SRE states are characterized by their edge states

**Rough idea:** just like varying the Hamiltonian in time to another phase requires closing the gap  $\mathbf{H} = \mathbf{H}_1 + g(t)\mathbf{H}_2$ , so does varying the Hamiltonian in space

$$\mathbf{H} = \mathbf{H}_1 + g(x)\mathbf{H}_2.$$



- ▶ Important role of SRE assumption: Here we are assuming that the bulk state has short-ranged correlations, so that changes we might make at the surface cannot have effects deep in the bulk.

# SPT states

**Def:** An *SPT state* (symmetry-protected topological state), protected by a symmetry group  $G$  is:

a SRE state, which is not adiabatically connected to a product state by local hamiltonians preserving  $G$ .

**e.g.:** free fermion topological insulators in  $3+1d$ , protected by  $U(1)$  and  $\mathcal{T}$ , have an odd number of Dirac cones on the surface.

# SPT states

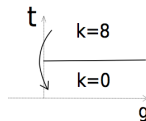
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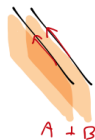
- ▶ Free fermion TIs classified [Kitaev: homotopy theory; Schneider et al: edge]

Interactions can affect the connectivity of the phase diagram in both directions:



- ▶ There are states which are adiabatically connected only via interacting hamiltonians [Fidkowski-Kitaev, 0904.2197].
- ▶ There are states whose existence requires interactions:  
e.g. Bosonic SPT states – w/o interactions, superfluid.

# Group structure of SPT states



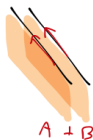
Simplifying feature:

SPT states (for given  $G$ ) form a group:

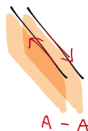
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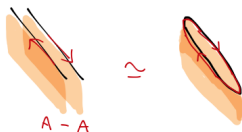
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Note: with topological order, even if we can gap out the edge states, there is still stuff going on (e.g. fractional charges) in the bulk. Not a group.

- [Chen-Gu-Wen, 1106.4772] conjecture: the group is  $H^{D+1}(BG, U(1))$ .
- $\exists$  'beyond-cohomology' states in  $D = 3 + 1$  [Senthil-Vishwanath]
- [Kitaev, unpublished] knows the correct construction of the group.



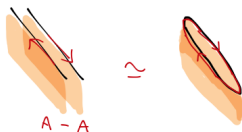
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This talk: an implication of this group structure

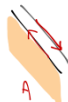
– which we can pursue by examples – is...

# Surface-only models

## Counterfactual:

Suppose the edge theory of an SPT state were realized *otherwise*  
– intrinsically in  $D$  dimensions, with a local hamiltonian.

Then we could paint that the conjugate local theory on the surface without changing anything about the bulk state.



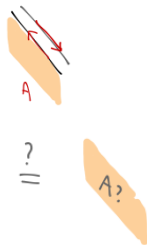
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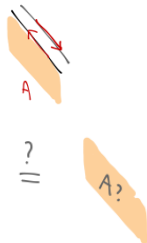
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But this contradicts the claim that we could characterize the  $D + 1$ -dimensional SPT state by its edge theory.

**Conclusion:** Edge theories of  $SPT_G$  states cannot be regularized intrinsically in  $D$  dims, preserving  $G$  – “surface-only models”.

[Wang-Senthil, 1302.6234 – general idea, concrete surprising examples of 2+1 surface-only states  
Wen, 1303.1803 – attempt to characterize the underlying mathematical structure, classify *all* such obstructions  
Wen, 1305.1045 – use this perspective to regulate the Standard Model on a 5d slab

## Summary of Nielsen-Ninomiya result on fermion doubling

The most famous example of such an obstruction was articulated by Nielsen and Ninomiya:

*It is not possible to regulate free fermions while preserving the chiral symmetry.*

(In odd spacetime dimensions, 'chiral symmetry' means 'parity'.)

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More precise (lattice) statement: A fermion action

$$S = \int_{\text{BZ}} \bar{d}^{2k} p \bar{\Psi}_p D(p) \Psi_p$$

cannot satisfy all three of these:

1.  $D(p)$  is smooth and periodic in the BZ (i.e. the FT of a local kinetic term on the lattice)
2. A single Dirac cone, i.e.  $D(p) \sim \gamma_\mu p^\mu$  for  $|p_\mu| \ll 1$ , and  $D$  invertible everywhere else.
3.  $\{\Gamma, D(p)\} = 0$ , where  $\Gamma$  is the chirality matrix ( $\gamma^5$ ).

# Illustration of fermion doubling

Simple illustration: attempt to regulate them on the lattice.

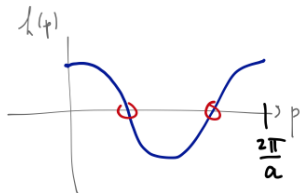
Then the momentum space is compact:

$$\text{for } n \in \mathbb{Z}, \quad e^{inap} = e^{ina\left(p + \frac{2\pi}{a}\right)} \quad \implies \quad \{p\} \simeq T^d \quad (\text{the Brillouin zone}).$$

The hamiltonian is of the form

$$\mathbf{H} = \int_{p \in \text{BZ}} h_{ab}(p) c_a^\dagger(p) c_b(p)$$

where  $h$  is a *periodic* map.



$$\text{e.g., in } 1+1\text{d: } \text{sign} \left( \frac{\partial h}{\partial p} \right) = \Gamma$$

- Friedan refinement: in each irreducible representation of the internal symmetry group there are no chiral fermions.
- Consistent with ABJ anomaly, since an exact symmetry of the lattice model is a symmetry.

# Recasting the NN result as a statement about SPT states

Consider free massive relativistic fermions in  
4+1 dimensions (with conserved  $U(1)$ ):

$$S = \int d^{4+1}x \bar{\Psi} (\not{\partial} + m) \Psi$$

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One proof of this:

Couple to external gauge field

$$\Delta S = \int d^5x A^\mu \bar{\Psi} \gamma_\mu \Psi.$$

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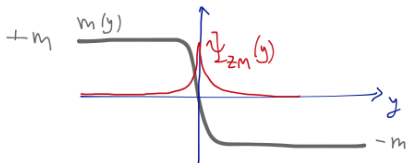
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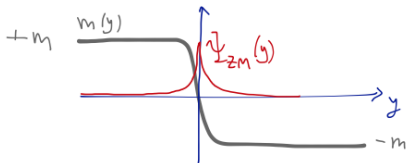
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More famous  $D = 3 + 1$  analog:

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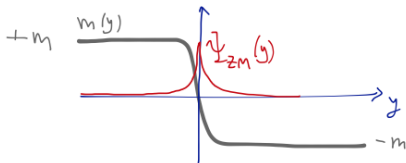
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$m = \pm m$  are distinct states ( $\theta = 0, \pi$ ):

$$\log \int [D\Psi] e^{iS_{3+1}[\Psi, A]} \propto \frac{m}{|m|} \int F \wedge F$$

Domain wall hosts a single Dirac cone in 2+1d. ⏪ ⏩ 🔍 🔄

# Strategy

Study a simple (unitary) gapped or topological field theory in  $4+1$  dimensions without topological order, with symmetry  $G$ .

Consider the model on the disk with some boundary conditions.

The resulting edge theory is

a “surface-only theory with respect to  $G$ ”

– it cannot be regulated by a local  $3 + 1$ -dim'l model while preserving  $G$ .

# What does it mean to be a surface-only state?

These theories are perfectly consistent and unitary – they *can* be realized as the edge theory of some gapped bulk. They just can't be regularized in a local way consistent with the symmetries without the bulk.

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2. Why '**probably**'? This perspective does not rule out emergent ("accidental") symmetries, not explicitly preserved in the UV.
3. It also does not rule out symmetric UV completions that include gravity, or decoupling limits of gravity/string theory.  
(**UV completions of gravity have their own complications!**)  
String theory strongly suggests the existence of Lorentz-invariant states of gravity with chiral fermions and lots of supersymmetry  
(the  $E_8 \times E_8$  heterotic string, chiral matter on D-brane intersections, self-dual tensor fields...)  
some of which can be decoupled from gravity.



## Some known examples of surface-only states

All in  $D = 2 + 1$ , realized on the surface of  $D = 3 + 1$  SPTs protected by time-reversal:

- ▶ Non-linear sigma model on  $S^2$  with Hopf term at  $\theta = \pi$

$$Z = \sum_{\text{instanton number, } n} (-1)^n Z_n$$

[model of high- $T_C$ : Dzyaloshinskii-Polyakov-Wiegmann 88, surface only: Vishwanath-Senthil 12]

- ▶ “Algebraic vortex liquid”: an insulating state of bosons (or a paramagnet) with massless fermionic vortices

[proposed by Fisher et al 06, surface only: Wang-Senthil 13]

- ▶ “All-fermion toric code”: a version of  $\mathbb{Z}_2$  gauge theory where  $e, m, \varepsilon = em$  are all fermions.

[surface only: Burnell-Chen-Fidkowski-Vishwanath, Wang-Senthil 13]

# A simple example

**Plan:** Implement the 4+1d analog of the “K-matrix approach” to 2+1d SPTs of [Lu-Vishwanath 12].

1. Solve the model – when is it an EFT for an SPT state?
2. Identify the edge states, and the symmetry  $G$  protecting them. (Whatever we get is surface-only with respect to  $G$ .)

# A simple topological field theory in 4+1 dimensions

Consider 2-forms  $B_{MN}$  in 4 + 1 dimensions, with action

$$S[B] = \frac{K_{IJ}}{2\pi} \int_{\mathbb{R} \times \Sigma} B^I \wedge dB^J$$

In  $4\ell + 1$  dims,  $K$  is a skew-symmetric integer  $2N_B \times 2N_B$  matrix.

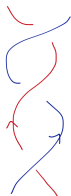
Note:  $B \wedge dB = \frac{1}{2}d(B \wedge B)$ .

Independent of choice of metric on  $\mathbb{R} \times \Sigma_{2p}$ .

Related models studied in: [Horowitz 1989, Blau et al 1989, Witten 1998,

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Simplest case ( $N_B = 1$ ) is realized in IIB strings on  $AdS_5 \times S^5$ ,

$B \equiv B_{NSNS}$ ,  $C \equiv C_{RR}$ :

$$S_{IIB} \ni \frac{1}{2\pi} \int_{AdS_5 \times S^5} F_{RR}^{(5)} \wedge B \wedge dC = \frac{N}{2\pi} \int_{\mathbb{R} \times \Sigma} B \wedge dC$$

Foreshadowing: Type IIB S-duality acts by  $B \leftrightarrow C$ .

## 'Trivial but difficult'

$$S[B] = \frac{K_{IJ}}{2\pi} \int_{\mathbb{R} \times \Sigma} B^I \wedge dB^J$$

gauge redundancies:  $B^I \simeq B^I + d\lambda^I$ ,  $\lambda^I$  are 1-forms

large gauge equivalences:  $B^I \simeq B^I + n^\alpha \omega_\alpha$ ,  $[\omega^\alpha] \in H^2(\Sigma, \mathbb{Z})$ ,  $n^\alpha \in \mathbb{Z}^{b^2(\Sigma)}$ .

# 'Trivial but difficult'

$$S[B] = \frac{K_{IJ}}{2\pi} \int_{\mathbb{R} \times \Sigma} B^I \wedge dB^J$$

gauge redundancies:  $B^I \simeq B^I + d\lambda^I$ ,  $\lambda^I$  are 1-forms

large gauge equivalences:  $B^I \simeq B^I + n^\alpha \omega_\alpha$ ,  $[\omega^\alpha] \in H^2(\Sigma, \mathbb{Z})$ ,  $n^\alpha \in \mathbb{Z}^{b^2(\Sigma)}$ .

- heavy machinery: [Freed-Moore-Segal 2006] I believe this machinery is not necessary if we consider only  $\Sigma_4$  without torsion homology.

- This sort of model has been used

[Witten, 90s; Shatashvili, unpublished; Maldacena-Moore-Seiberg 01; Belov-Moore 03-06]

to 'holographically' *define* the partition function of the edge.

Mainly  $D = 4\ell + 3$ : 1+1d chiral CFTs, conformal blocks of 5+1d (2,0) theory.

- The simplest case ( $N_B = 1$ ) is equivalent to a  $\mathbb{Z}_k$  2-form gauge theory. [Maldacena-Moore-Seiberg hep-th/0108152, Hansson-Oganesyan-Sondhi cond-mat/0404327]

(More below.) ( $k > 2$  is a finite-temperature quantum memory.)

# Bulk physics

(When is this the EFT for an SPT state?)

# When does the 4+1d CS theory have topological order?

Consider  $p$  forms in  $2p + 1$  dimensions:

$$S[B] = \frac{K_{IJ}}{2\pi} \int_{\mathbb{R} \times \Sigma_{2p}} B^I \wedge dB^J$$

For now, suppose that  $\partial \Sigma_{2p} = \emptyset$ .

Gauge-inequivalent operators labelled by  $[\omega_\alpha] \in H^p(\Sigma, \mathbb{Z})$ :

$$\mathcal{F}_{\omega_\alpha}(m) \equiv e^{2\pi i m_I^\alpha \int_{\omega_\alpha} B^I}$$

large gauge eq  $\implies m_I^\alpha \in \mathbb{Z}$ . ETCRs  $\implies$  Heisenberg algebra:

$$\mathcal{F}_{\omega_\alpha}(m) \mathcal{F}_{\omega_\beta}(m') = \mathcal{F}_{\omega_\beta}(m') \mathcal{F}_{\omega_\alpha}(m) e^{2\pi i m_I^\alpha m_J'^\beta (K^{-1})^{IJ} \mathcal{I}_{\alpha\beta}}.$$

$\int_\Sigma \omega_\alpha \wedge \omega_\beta = \mathcal{I}_{\alpha\beta}$ , intersection form (symmetric for  $\Sigma_4$ , AS for  $\Sigma_2$ ).

In 2 + 1:  $\mathcal{I} \approx \mathbb{1}_{g \times g} \otimes i\sigma^2$

the irrep of this algebra has dimension  $|\det(K)|^g$ .

In 4 + 1: the irrep of this algebra has dimension  $|\text{Pfaff}(K \otimes \mathcal{I})|$ .

Fact about 4-manifolds:  $\mathcal{I}$  is unimodular  $\implies$

$$\boxed{\text{SRE} \iff |\text{Pfaff}(K)| = 1}.$$



# Zeromode quantum mechanics

A more direct construction of the groundstates.

Expand in zeromodes  $b^{I\alpha} \simeq b^{I\alpha} + 2\pi$ :

$$B^I = \sum_{\alpha=1}^{b^2(\Sigma_4, \mathbb{Z})} \omega_\alpha b^{I\alpha}(t), \quad \text{span}\{[\omega_\alpha]\} = H^2(\Sigma_4, \mathbb{Z}),$$

$$S = \frac{K_{IJ}}{2\pi} \int dt \int_{\Sigma_4} \omega_\alpha \wedge \omega_\beta b^{I\alpha} \dot{b}^{J\beta} = \frac{K_{IJ}}{2\pi} \int dt \mathcal{I}_{\alpha\beta} b^{I\alpha} \dot{b}^{J\beta}$$

which describes a particle in  $b^2(\Sigma)$  dimensions with a magnetic field of strength  $k$  (for each pair of dimensions), in the LLL.

As in 2+1d, Maxwell-like terms

$$\Delta S = \int_{\Sigma \times \mathbb{R}} \frac{1}{m} (dB \wedge \star dB + dC \wedge \star dC) \propto \int dt \frac{1}{m} \dot{b}^2$$

$m < \infty$  brings down higher landau levels.

# This is a model of bosons

**Low-energy evidence:** I did not have to choose a spin structure to put this on an arbitrary 4-manifold.

(unlike  $U(1)_{k=1}$  CS theory in  $d = 2 + 1$ .)

Comment about spin structure:

On a manifold that admits spinors, the intersection form is even  
( $\mathcal{I}(v, v) \in 2\mathbb{Z}$ )

$\implies$  to describe an EFT for a *fermionic* SPT state,  
we should consider  $k \in \mathbb{Z}/2$ .

[Belov-Moore] 'spin Chern-Simons theory'.

**High-energy (i.e. cond-mat) evidence:**

Conjecture for a local lattice model of bosons which produces this EFT [Kraevc, JM, Swingle, in progress]:

# Which model of bosons?

[Kravec, JM, Swingle, in progress]

- Put rotors  $e^{ib_p}$  on the *plaquettes*  $p$  of a 4d spatial lattice.

$$e^{ib_p} |n_p\rangle = |n_p + 1\rangle.$$

- Put charge- $k$  bosons  $\Phi_\ell = \Phi_{-\ell}^\dagger$  on the *links*  $\ell$ .

$$[\Phi_\ell, \Phi_\ell^\dagger] = 1$$

[Wegner, ..., Motrunich-Senthil,

Levin-Wen, Walker-Wang, Burnell et al]

$n_p \equiv \#$  of 'sheets' covering the plaquette.

$\Phi_\ell^\dagger$  creates a string segment.

$\Phi_\ell^\dagger \Phi_\ell \equiv \#$  of strings covering the link.

$$\mathbf{H} = - \underbrace{\sum_{\text{links}, \ell \in \Delta_1} \left( \sum_{p \in s(\ell)} n_p - k \Phi_\ell^\dagger \Phi_\ell \right)^2}_{\mathbf{H}_1, \text{ gauss law. happy when sheets close, or end on strings}} - \underbrace{\sum_{\text{volumes}, v \in \Delta_3} \prod_{p \in \partial v} e^{ib_p}}_{\mathbf{H}_3 \sim B^2, \text{ makes sheets hop.}} + h.c.$$

$$- \underbrace{\Gamma \sum_{p \in \Delta_2} n_p^2}_{\mathbf{H}_2 \sim E^2. \text{ discourages sheets.}} - t \underbrace{\sum_{p \in \Delta_2} e^{ikb_p} \prod_{\ell \in \partial p} \Phi_\ell^\dagger}_{\mathbf{H}_{\text{strings}}, \text{ hopping term for matter strings}} + h.c. + V (|\Phi|^2)$$

When  $\Gamma = 0$ , the model is solvable:

Soup of oriented closed 2d sheets, groups of  $k$  can end on strings.

## Which model of bosons, cont'd

Condense  $\Phi_\ell = v e^{i\varphi_\ell}$ :  $\mathbf{H}_{\text{strings}} = - \sum_p t v^4 \cos \left( k b_p - \sum_{\ell \in \partial p} \varphi_\ell \right)$

$$\implies (e^{i b_p})^k = \mathbb{1}, |n_p\rangle \simeq |n_p + k\rangle.$$

Leaves behind  $k$  species of (unoriented) sheets.

**Groundstates:** equal-superposition sheet soup.  $k^{b_2}$  sectors for  $\mathcal{I} = \mathbb{1}$ .

## Continuum limit.

$U(1) \xrightarrow{\text{Higgs}} \mathbb{Z}_k$  2-form gauge theory: (subscripts indicate form degree)

$$L = \frac{tv^4}{2} (d\varphi_1 + kB_2) \wedge \star (d\varphi_1 + kB_2) + \frac{1}{g^2} dB_2 \wedge \star dB_2$$

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path integral  
manipulations

$$\simeq \frac{k}{2\pi} B \wedge dC + \underbrace{\frac{1}{8\pi tv^4} dC \wedge \star dC + \frac{1}{g^2} dB \wedge \star dB}_{\text{irrelevant perturbation, ignore when } E < tv^4, g}$$

with  $dC \simeq 2\pi tv^4 \star (d\varphi + kB)$ .

[Maldacena-Moore-Seiberg hep-th/0108152, Hansson-Oganesyan-Sondhi cond-mat/0404327]

# Edge states

(Whatever we find there is a surface-only state.)

# Review of edge states of 2+1 CS theory.

Consider abelian CS theory on the LHP:

$$S = \frac{k}{4\pi} \int_{\mathbb{R} \times \text{LHP}} A \wedge dA$$

EoM for  $A_0$ :  $0 = F$

$$\implies A = \mathbf{i}g^{-1}dg = d\phi, \quad \phi \simeq \phi + 2\pi.$$

Only gauge transfs which approach  $\mathbb{1}$  at the bdy preserve  $S_{CS}$

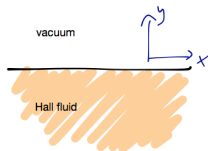
$\implies \phi$  is dynamical.

Boundary condition:  $0 = A - v(\star_2 A)$

i.e.  $A_t = vA_x$ .  $v$  is UV data.

[Witten, Elitzur et al, Wen, ...

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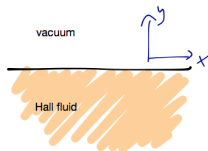
$$S_{CS}[A = d\phi] = \frac{k}{2\pi} \int dt dx \left( \partial_t \phi \partial_x \phi + v (\partial_x \phi)^2 \right).$$

Conclusion:  $\phi$  is a chiral boson.

$kv > 0$  required for stability.

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$$S_{CS}[A = d\phi] = \frac{k}{2\pi} \int dt dx \left( \partial_t \phi \partial_x \phi + v (\partial_x \phi)^2 \right) v(x,y)$$

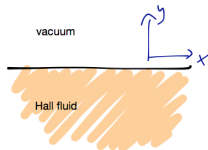
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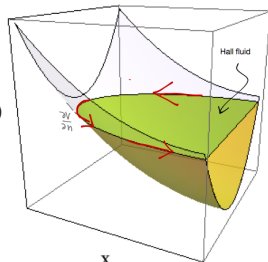
Note: The Hamiltonian depends on the boundary conditions; the  $\mathcal{H}$  does not.

[Witten, Elitzur et al, Wen, ...

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microscopic picture:  
y



## Edge states of 4+1d CS theory.

Focus on the simplest case where  $K = k\mathbf{i}\sigma^2$ ,  $S = \frac{k}{2\pi} \int B \wedge dC$ .

$$S[B, C] = \frac{k}{2\pi} \int_{\mathbb{R} \times \Sigma_4} B \wedge dC + \frac{1}{4g^2} \int_{\mathbb{R} \times \partial\Sigma_4} (C \wedge \star_4 C + B \wedge \star_4 C)$$

bc is:  $\left(\frac{k}{2\pi} B - \frac{1}{2g^2} \star_4 C\right) |_{\partial\Sigma_4} = 0$ .

$$\int DB e^{iS} = \delta[dC] \quad \Longrightarrow \quad C = da$$

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This is ordinary Maxwell theory! We know how to regularize this! e.g.

$$H = - \sum_{\text{vertices}, v \in \Delta_0} \left( \sum_{\ell \in s(v)} n_\ell - q_v \right)^2 - \sum_{p \in \Delta_2} \prod_{\ell \in \partial(p)} e^{ib_\ell} + h.c. - \sum_{\ell \in \Delta_1} \Gamma n_\ell^2.$$

$\Delta_p \equiv \{p\text{-simplices}\}$ .  $s(v) \equiv \{\text{edges incident on } v \text{ (oriented ingoing)}\}$  and

$[b_p, n_p] = \mathbf{i}$  is a number-phase representation.  $b_p \equiv b_p + 2\pi$ ,  $n_p \in \mathbb{Z}$ .

# Symmetries

(The conclusion must be that the lattice model above breaks a crucial symmetry of the bulk.)

# Continuous symmetries

In 2+1d CS theory, as it arises from QH states, we have a conserved current (electron number):

$$0 = \partial^\mu J_\mu \quad \Longrightarrow \quad J_\mu \equiv \frac{1}{2\pi} \epsilon_{\mu\nu\rho} \partial_\nu A_\rho$$

$J$  is conserved iff  $A$  is single-valued.

In 4+1, the analog is pairs of string currents

$$J_{\mu\nu}^I \equiv \frac{1}{2\pi} \epsilon_{\mu\nu\rho\sigma\lambda} \partial_\rho B_{\sigma\lambda}^I$$

## Demonstration that different $K$ are different states

**In**  $D = 2 + 1$ : couple the particle currents  $J^I = \star dA^I$  to *external* 1-form potentials,  $\mathcal{A}_I$ :

$$\log \int [DA^I] e^{i \int k A dA + i \int J_I \mathcal{A}^I} = \int_{2+1} (k^{-1})_{IJ} \mathcal{A}^I d\mathcal{A}^J$$

– quantized Hall response (integer if no topological order,  $\det k = 1$ ).

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**In**  $D = 4 + 1$ : Couple the string currents  $J^I = \star dB^I$  to *external* 2-form potentials,  $\mathcal{B}_I$ :

$$\log \int [DB^I] e^{i \int K B dB + i \int J_I \mathcal{B}^I} = \int (K^{-1})_{IJ} \mathcal{B}^I d\mathcal{B}^J .$$

quantized 'string Hall' response (integer if no topological order,  $\text{Pfaff} K = 1$ ).

# Symmetries

- ▶ Translation invariance is a red herring (I think!).

The lattice model should have the same edge states.

- ▶ Stringy symmetries:  $J_{\ell 0}^B|_{bdy} = E_\ell$ ,  $J_{\ell 0}^C|_{bdy} = -B_\ell$ .

$E_\ell \equiv \partial_t a_\ell - \partial_\ell a_t$   $B_\ell \equiv \epsilon_{\ell ij}(\partial_i a_j - \partial_j a_i)$  are ordinary E&M fields

$$J_{y0}^C = \epsilon_{ijk} \partial_i C_{jk} = \epsilon_{ijk} \partial_i \partial_j a_k = \vec{\nabla} \cdot \vec{B}$$

$$J_{y0}^B = \epsilon_{ijk} \partial_i B_{jk} = \epsilon_{ijk} \partial_i \epsilon_{jkl} E_\ell = \vec{\nabla} \cdot \vec{E}.$$

This is ordinary charge, of course it has to be conserved.

- ▶  $\mathcal{C}$ :  $(B, C) \rightarrow -(B, C)$  is  $(\vec{E}, \vec{B}) \rightarrow -(\vec{E}, \vec{B})$ . Preserved in pure U(1) lattice gauge theory.
- ▶  $\mathcal{TP}$ :  $t \rightarrow -t$ ,  $x^M \rightarrow -x^M$ ,  $\mathbf{i} \rightarrow -\mathbf{i}$ ,  $B \rightarrow -B$ ,  $C \rightarrow C$  as two-forms. Acts in the usual way on the EM field as  $(E, B) \rightarrow (E, -B)$ .

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- ▶

**EM DUALITY!:**  $(B, C) \rightarrow (C, -B)$

is a manifest symmetry of the bulk theory.

Unbreakable in the IR.

# EM duality is the culprit

**Conclusion:** electric-magnetic duality is the symmetry protecting the edge states.

(i.e.  $G = \text{SL}(2, \mathbb{Z})$ , though the  $S$  operation is enough.)

To kill the photon, we must *add* degrees of freedom: matter with charge  $\vec{q} = (q_e, q_m)$ , which condenses (to Higgs or confine it).

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**Corollary:** it is not possible to gauge electric-magnetic duality symmetry.

∃ recent literature with continuum arguments for this impossibility:

[Deser 1012.5109, Bunster 1101.3927, Saa 1101.6064]

# Concluding remarks

## 7d CS theory and the (2,0) superconformal theory

While we're at it, consider the following 6+1d TFT: [Witten, Belov-Moore]

$$S_7[C^{(3)}] = \frac{k}{4\pi} \int_{\mathbb{R} \times \Sigma_6} C^{(3)} \wedge dC^{(3)}$$

For  $k = 1$ , no topological order.

gauss law:  $C^{(3)} = dc^{(2)}$

$$S_7[C^{(3)} = dc^{(2)}] = \frac{k}{4\pi} \int_{\mathbb{R} \times \partial\Sigma_6} \epsilon \partial c^{(2)} \cdot \left( \partial_t c^{(2)} + v \epsilon \partial c^{(2)} \right).$$

(boundary condition:  $C_{0ij} = v(\star_6 C)_{ij}$ .)

Conclusion:  $c$  is a self-dual 2-form potential in 5+1d.

$\implies$  The 'topological sector' of the  $A_0$  (2,0) superconformal theory in 6d – the worldvolume theory of M5-branes – is an SPT state with respect to  $G = \{\mathbb{1}\}$ .

The conjecture that it can be consistently decoupled from gravity underlies much recent progress in the field formerly known as strings [Witten, Gaiotto...].

e.g. it makes various deep 4d QFT dualities manifest.



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Is there an associated anomaly?

3. And what about supersymmetry?

Gauging this leads to supergravity!

$\exists$  holographic obstructions to SUSY-preserving regulators?

## Final remarks

- ▶ The relationship between these CS gauge theories and their edge states is *not* a holographic duality: the bulk and edge are not equivalent.

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- ▶ The relationship between these CS gauge theories and their edge states is *not* a holographic duality: the bulk and edge are not equivalent.

Rather, we're using the edge states to *label* the bulk phase.

- ▶ If we demand a local regulator without gravity, the fact that the Standard Model is a surface-only state is evidence for extra dimensions.

The end

Thanks for listening.

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Some entry points to the literature:

- ▶ Wen [1201.1281](#): 'gentle' review of topological order.
- ▶ Levin-Gu [1202.3120](#): Simple  $D = 2 + 1$  SPTs.
- ▶ Lu-Vishwanath [1205.3156](#): K-matrix description of  $D = 2 + 1$  SPTs.
- ▶ Senthil-Levin [1206.1604](#): Boson integer QHE.
- ▶ Senthil-Vishwanath [1209.3058](#):  $D = 3 + 1$  boson SPTs.
- ▶ Turner-Vishwanath [1301.0330](#): brief, nice partial review.