Entanglement Entropy for probe branes.

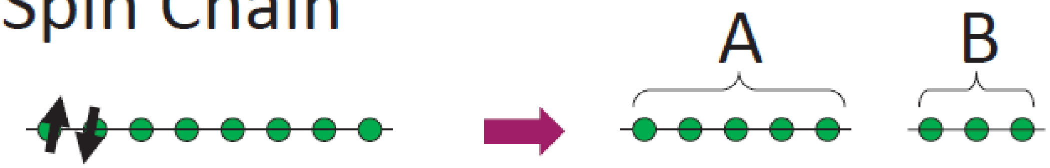
Andreas Karch (University of Washington)

work with Han-Chih Chang

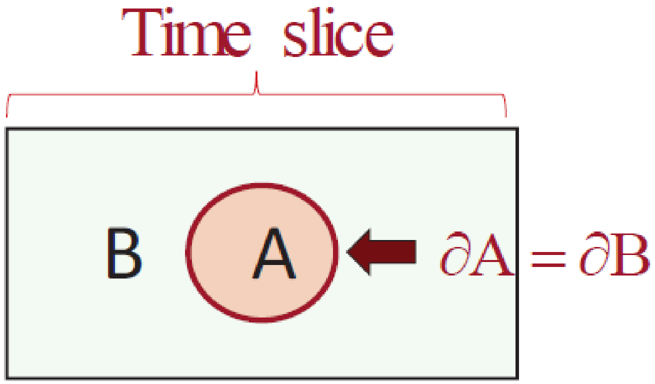
talk at Gauge/Gravity Duality Conference, Munich, Jul 31 2013

Entanglement Entropy

(1) Spin Chain



(2) QFT

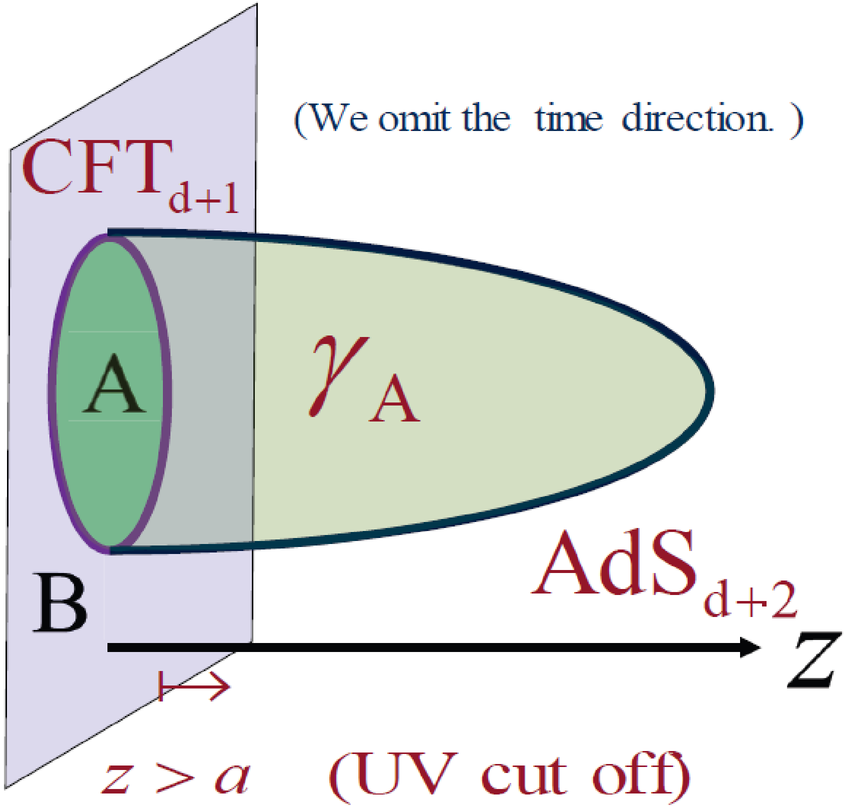


$$\rho_A = \mathrm{Tr}_B[\rho]$$

$$S_A = -\mathrm{Tr}[\rho_A \log \rho_A]$$

(picture from Takayanagi, 2012)

Holographic Entanglement Entropy



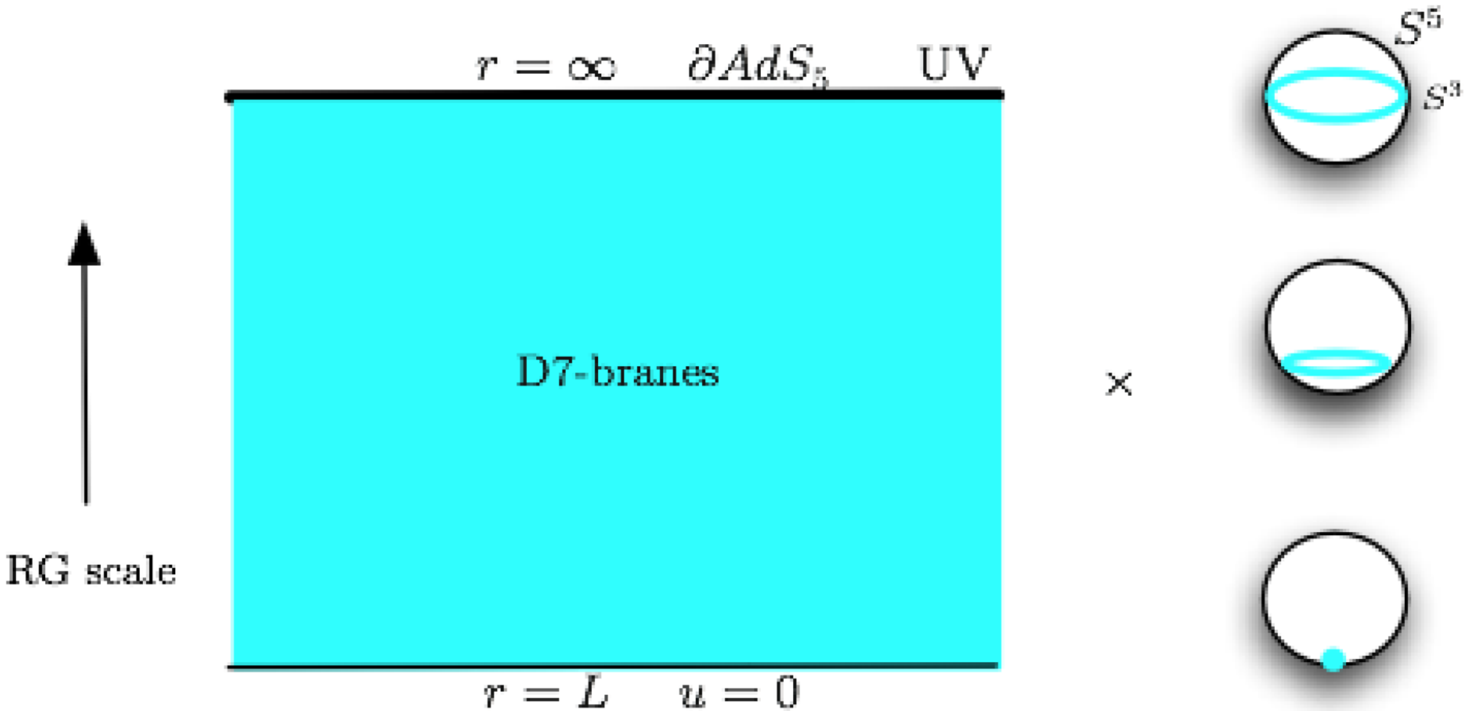
$$S_A = \mathrm{Min}_{\Sigma_A}\left[\frac{\mathrm{Area}(\Sigma_A)}{4G_N}\right]$$

(Ryu, Takayanagi)

(picture from Takayanagi, 2012)

Probe Branes

(AK, Katz)



(picture from CLMRW-review, 2011)

Probe Branes

(AK, Katz)

d+1 dimensional
spacetime

n+1 dimensional
worldvolume

$$\frac{L^{d-1}}{G_N} \gg T_0 L^{n+1} \gg 1$$

**Newton's
constant**

N^2

Probe brane tension **N**

L=curvature radius of background geometry

Probe Branes

(AK, Katz)

Gravity classical
dominant physics at O(N^2)
loops contribute at O(1)

“the glue”

$$\frac{L^{d-1}}{G_N} \gg \qquad\qquad 1$$

Newton’s constant

N^2

L=curvature radius of background geometry

Probe Branes

(AK, Katz)

Worldvolume equations classical
dominant flavor physics at O(N)
loops contribute at O(1)

“the quarks”

$$T_0 L^{n+1} \gg 1$$

Probe brane tension N

L=curvature radius of background geometry

Probe Branes

(AK, Katz)

No gravitational backreaction from probe brane
O(N^2) physics unaffected by order N physics
O(N) physics: wordvolume of flavor brane and leading order backreaction
at O(1) all hell breaks lose: 2nd order backreaction, glue loops, quark loops

$$\frac{L^{d-1}}{G_N} \gg T_0 L^{n+1} \gg 1$$

Newton's constant

N^2

Probe brane tension **N**

L=curvature radius of background geometry

The quark sector

Gravity bad -- DBI good.

(even I can solve it)

How far can I get without ever having to solve Einstein's equations?

There are some new quantities in the quark sector, the flavor sector that are manifestly only due to DBI fields (e.g. chiral condensate)

Gravity bad – DBI good

(see e.g. AK, O'Bannon, Thompson)

Generically, at order N backreaction matters.

$$S = N^2 S_{grav} + N\, S_{probe}$$

glue and quark sector

backreaction suppressed….

$$\delta h \propto G_N\, T_{\mu\nu} \propto {}^{1}/_{N}\;\; h$$

$$\boxed{\varepsilon = N^2 \varepsilon_{grav} + N\,(\varepsilon_{probe} + \frac{\delta\varepsilon_{grav}}{\delta g}\,\mathrm{h})}$$

$\varepsilon = {}^{\delta S}/_{\delta g_{00}}$ = energy density

… but enters at order N

Gravity bad – DBI good

(see e.g. AK, O'Bannon, Thompson)

Exception: the free energy = on-shell action.

$$S = N^2 S_{grav} + N\, S_{probe}$$

glue and quark sector

backreaction suppressed

$$\delta h \propto G_N\, T_{\mu\nu} \propto {}^{1}\!/_{N}\ \ h$$

$$\boxed{\omega = N^2 S_{grav} + N\,(S_{probe} + \overset{=0}{\cancel{\frac{\delta S_{grav}}{\delta g}}}\,\mathrm{h})}$$

contribution due to backreaction vanishes due to equations of motion!

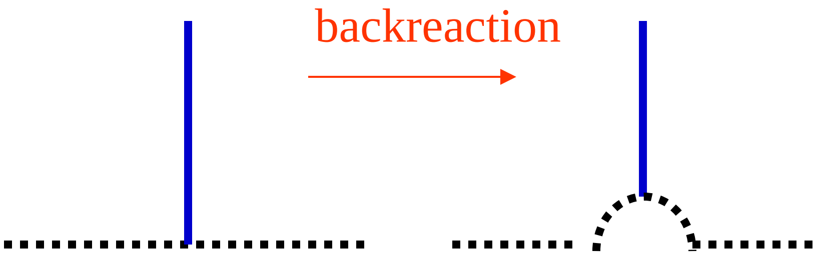
Gravity bad – DBI good

(see e.g. AK, O'Bannon, Thompson)

$$\boxed{\omega = N^2 S_{grav} + N\, S_{probe}}$$

good news! all equilibrium properties can be calculated without ever having to deal with backreaction!!!!

Including thermal entropy:



honest calculation:
calculate change in horizon area

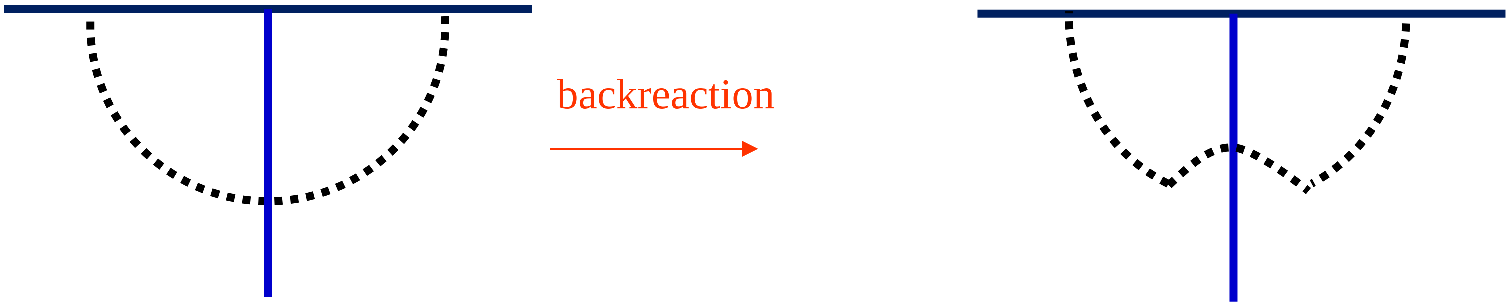
or

$$s = -\frac{\partial \omega}{\partial T}$$

but "cheat" works!

DBI bad – Gravity good!!!

But for the EE we are stuck with needing leading order backreaction to even get order N contribution.



$$S^{EE} = \underbrace{S^{EE}_{grav}}_{O(N^2)} + \underbrace{\frac{\delta Area}{4G}}_{O(N)}$$

$\longleftarrow$ **O(1/N)**

$\longleftarrow$ **O(N^2)**

In this work we will give a general expression for δArea

Exceptions:

Casini, Huerta, Myers:

- spherical entangling surface
- conformal field theory

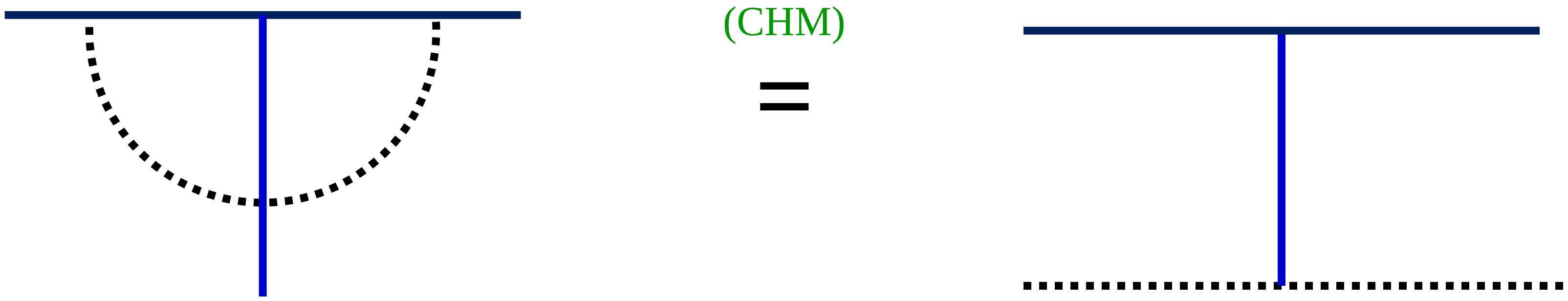
EE $\xrightarrow{\text{conformal map}}$ thermal entropy on hyperboloid

Jensen, O’Bannon:

still applies for conformal flavors (massless quarks!) on conformal defect

In this case cheat still applies.

Jensen-O’Bannon Calculation



$$ds^2 = \frac{-dt^2 + dx^2 + dz^2}{z^2}$$

$$ds^2 = -h(r)dt^2 + dH^2 + {dr^2}/{h(r)}$$

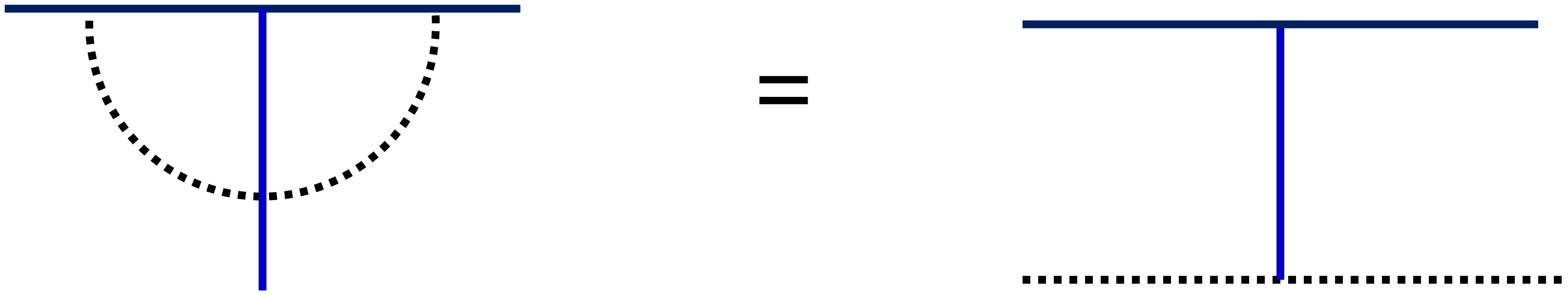
h(r)= $r^2 - 1$

T=0

T=1/2π

Casimir stress tensor on H appears thermal
Entangling surface = boundary of H

Jensen-O'Bannon Calculation



$O(N^2)$: (CHM)

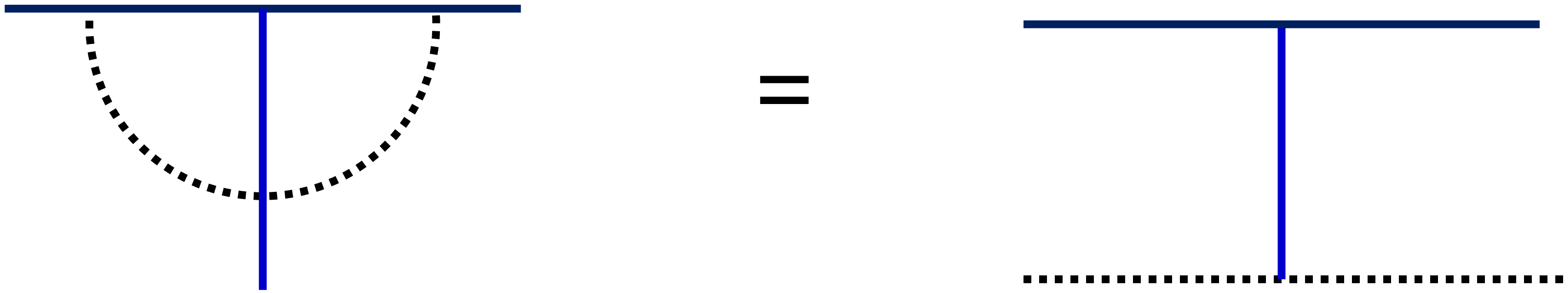
$$S = \frac{V^H_{d-1}}{4G_N} \xleftarrow[r_H \,=\, 2\pi T/(d-1)]{d/dT}$$

$$S = \frac{1}{16\pi G_N}\int d^{d+1}x\sqrt{g}\left(\overset{\text{constant}}{\overset{\swarrow}{R}} + \frac{d(d-1)}{L^2}\right)$$

Free energy = C_0 volume of spacetime
= $-(C_0/d)\, r_h^{\,d}$ vol(H)

$$C_0 = d/(8\pi G_N)$$

Jensen-O’Bannon Calculation



O(N^2):

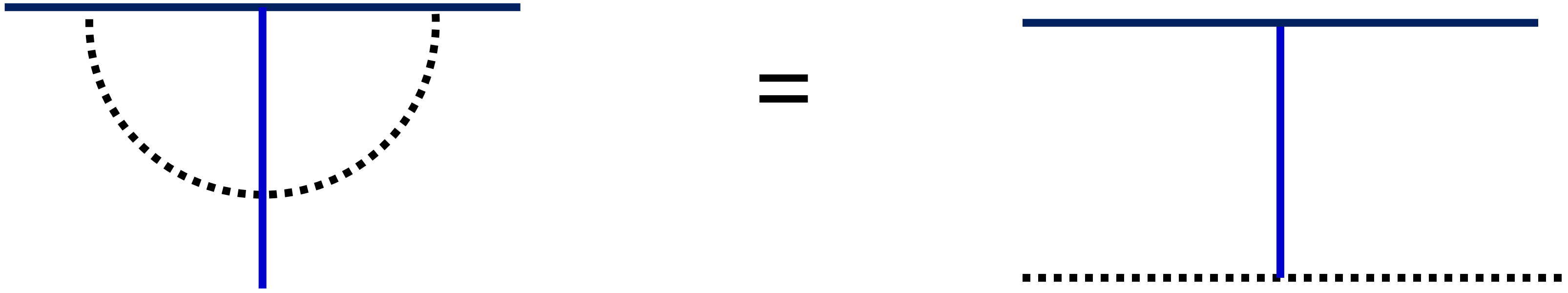
(CHM)

$$S = \frac{V^H_{d-1}}{4G_N}$$

$$\begin{aligned} V^H_m &= V^S_{m-1} \int_\epsilon^1 dy \, \frac{(1-y^2)^{(m-2)/2}}{y^m} \\ &= p_1 \left(\frac{1}{\epsilon}\right)^{m-1} + p_3 \left(\frac{1}{\epsilon}\right)^{m-3} + \dots \\ &\quad \dots + \begin{cases} p_{m-1}\left(\frac{1}{\epsilon}\right) + p_m + \mathcal{O}(\epsilon) & m \text{ even} \\ p_{m-2}\left(\frac{1}{\epsilon}\right) + q\log(\epsilon) + \mathcal{O}(1) & m \text{ odd.} \end{cases} \end{aligned}$$

area law

Jensen-O'Bannon Calculation



O(N): (JO)

$$S = \frac{2\pi T_0}{d-1} V^H_{n-1}$$

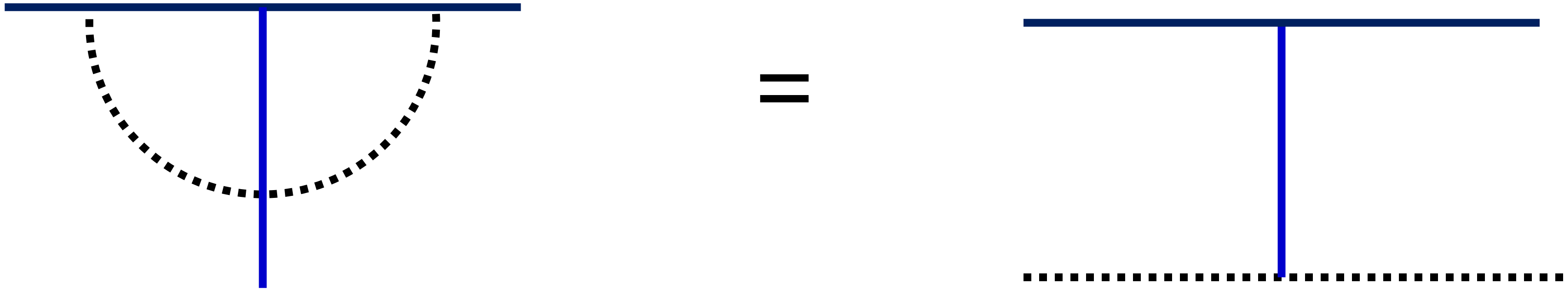
d/dT ←

$$S_{probe} = -T_0 \int d^{n+1}z \sqrt{g_I}$$

Free energy ~ volume of spacetime

(for n<d, 1/(d-1) replaced by 1/d for n=d)

Jensen-O'Bannon Calculation



O(N): (JO)

"flavor central charge"
number of DOFs on defect

$$S = \boxed{\frac{2\pi T_0}{d-1}} \underline{V^H_{n-1}}$$

Has exactly the functional form of a spherical entangling surface in an n+1 dimensional CFT.

Limitations of Jensen – O'Bannon

- Massless flavors only

  no topological phases

  not applicable to probe branes realizing quantum Hall effect, topological insulators, ….

- No worldvolume gauge fields

  no novel compressible quantum liquids

  no holographic quantum liquid

Limitations of Jensen – O'Bannon

- Massless flavors only
- No worldvolume gauge fields

not applicable to any of the systems we really want to be able to calculate the EE in!

Want a method that works for any probe brane

use Jensen-O'Bannon to validate our new method.

EE for probe branes.

$$S_A = (\pi T_0) \int \left(d^{d-1} w \sqrt{\gamma}\right) \left(d^{n+1} z \sqrt{g_I}\right) \left(T^{\mu\nu}_{min} G_{\mu\nu\rho\sigma} T^{\rho\sigma}_{probe} \right)$$

tension
O(N) constant

minimal
area

probe
worldvolume

double integral

gravitational
Green’s function

EE for probe branes.

= gravitational interaction between two energy distributions

$$S_A = (\pi T_0) \int \left(d^{d-1} w \sqrt{\gamma}\right) \left(d^{n+1} z \sqrt{g_I}\right) \left(T^{\mu\nu}_{min} G_{\mu\nu\rho\sigma} T^{\rho\sigma}_{probe}\right)$$

$$S_{min} = \frac{1}{4G_N} \int d^{d-1} w \, \sqrt{\gamma} \equiv \frac{1}{4G_N} \int d^{d-1} w \, \mathcal{L}_{min}$$

$$T^{\mu\nu}_{min} = \frac{2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}_{min}\right)}{\delta g_{\mu\nu}}$$

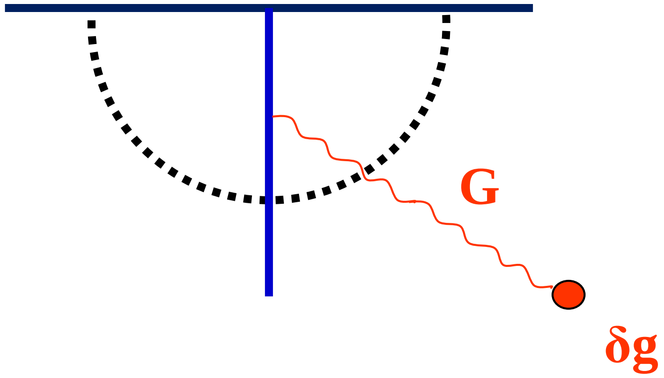
“stress tensor” of probe brane

“stress tensor” of minimal area

EE for probe branes - derivation

$$S_A = (\pi T_0)\int \left(d^{d-1}w\sqrt{\gamma}\right)\left(d^{n+1}z\sqrt{g_I}\right)\left(T^{\mu\nu}_{min}\underbrace{G_{\mu\nu\rho\sigma}T^{\rho\sigma}_{probe}}\right)$$

$\delta g_{\mu\nu}$
linearized backreaction



EE for probe branes - derivation

$$S_A = (\pi T_0) \int (d^{d-1}w\sqrt{\gamma})\,(d^{n+1}z\sqrt{g_I})\left(T^{\mu\nu}_{min}\underbrace{G_{\mu\nu\rho\sigma}T^{\rho\sigma}_{probe}}_{\delta g_{\mu\nu}}\right)$$

corresponding change in minimal area:

$$\delta S_{min} = \frac{1}{4G_N}\int d^{d-1}w\,\sqrt{\gamma}\left(\frac{T^{\mu\nu}_{min}}{2}(\delta g)_{\mu\nu} + \cancel{\frac{\delta\mathcal{L}_{min}}{\delta x^{\mu}_{M}}}\,\delta x^{\mu}_{M}\right)$$

=0 by eom.

change of embedding

Comments

- Works in time-dependent backgrounds

Hubeny, Rangamani, Takayanagi:

Minimal area → Extremal area

same action!

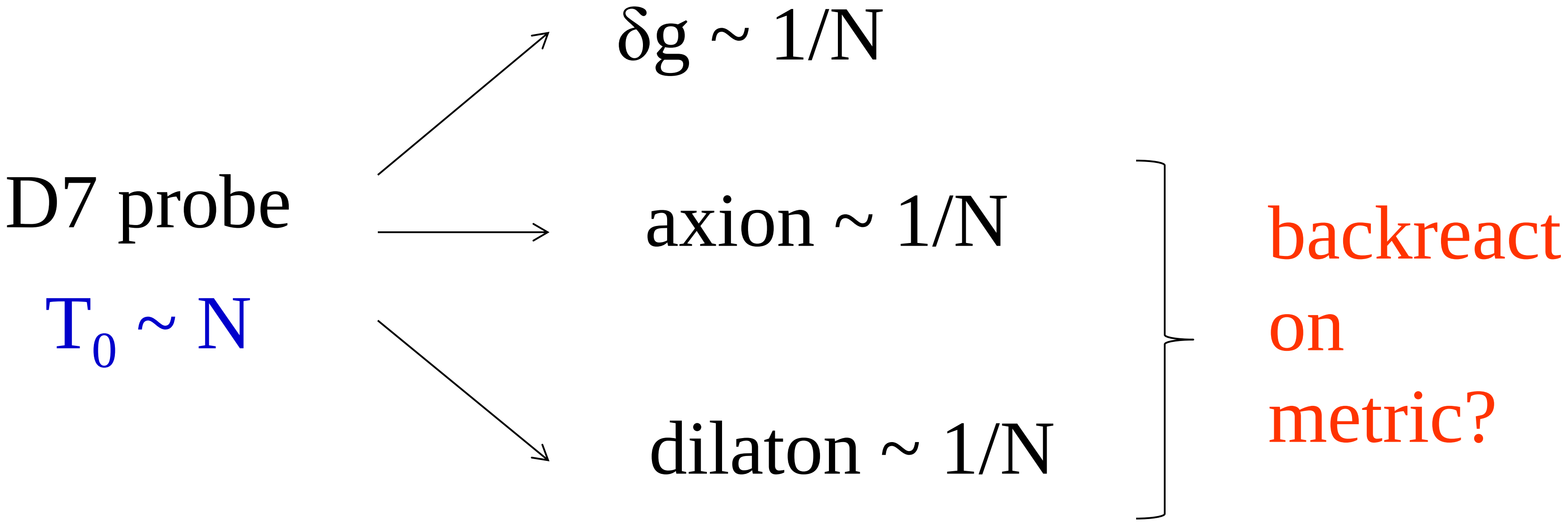
use retarded Green's function for G

Comments

- Works with higher curvature corrections
    - **minimal area requirement gets replaced with more general surface**
    - **still follows from an action principle, action defining the surface is area + curvature corrections**
    - **still allows definition of $T^{min}_{\mu\nu}$**

Properties

No secondary backreaction!



D7 probe

$T_0 \sim N$

$\delta g \sim 1/N$

axion ~ 1/N

dilaton ~ 1/N

backreact on metric?

$G \sim 1/N^2$

Secondary Backreaction

axion ~ 1/N $G \sim 1/N^2$

IIB-action = 1/G $(\text{axion})^2$

stress tensor = $N^2 (\text{axion})^2 \sim O(1)$

$\delta g \sim G$ (stress tensor) $\sim 1/N^2$

secondary backreaction is subleading

turned on in background

Exception:

axion $\sim 1/N$ Axion ~ 1 $G \sim 1/N^2$

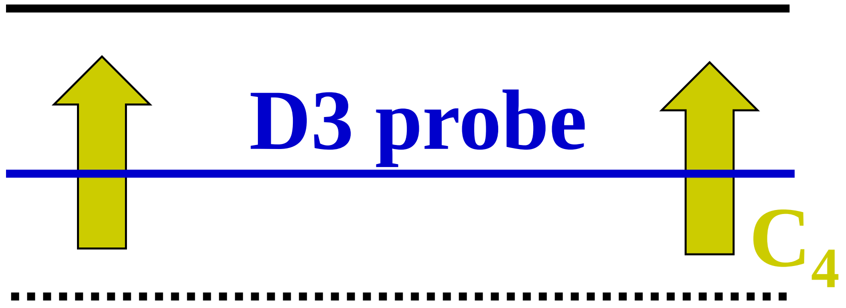
IIB-action $= 1/G\,(\text{Axion} + \text{axion})^2$

stress tensor $= \ldots + N^2\,(\text{axion Axion}) \sim O(N)$

$\delta g \sim G\,(\text{stress tensor}) \sim 1/N$

secondary backreaction is important if brane sources field that has non-trivial background

Exception: Examples

- 

- D6 flavor brane in ABJM

(ABJM has RR 2-form background)

But: **D5 with F_{rt} is fine. Sources C_4, but orthogonal to background.**

Properties

Backreaction in internal space irrelevant.

trace reversed Einstein:

$$R_{\mu\nu} = \tilde{T}_{\mu\nu} \qquad \tilde{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{D-2} g_{\mu\nu} T^{\rho}_{\rho}$$

Entangling surface is:

- **codimension 2 minimal surface**
- **wrapping internal manifold**

makes co-dimension 2 special

and hence does not source internal tensor modes

Properties

Backreaction in internal space irrelevant.

Instead of integrating over D-dim product space, can do integral in (d+1) dimensional spacetime with "effective" probe stress tensor.

$$T_{\mu\nu}^{probe,eff} = \int_{x_I} \sqrt{g_I}\, \underset{\uparrow}{T_{\mu\nu}^{probe}} \, .$$

internal space

No internal O(N) backreaction:

EE for D3/D7 (AdS$_5$ x S^3) =
EE for spacetime filling probe in AdS$_5$

EE for D3/D5 (AdS$_4$ x S^2) =
EE for co-dim 1 probe in AdS$_5$=
Randall-Sundrum brane

very simple to get fully backreacted metric!

Discussion

Gave universal formula for probe EE. Two important properties:

- no secondary backreaction
- no internal backreaction

Discussion

We checked our formula in the simplest examples:

- spherical entangling surface
- strip entangling surface

for

- D3/D7
- D3/D5
- toy models (spacetime filling and RS)

and found perfect agreement with

- fully backreacted solution (toy-models)
- Jensen-O’Bannon calculation (sphere)

To-Do list:

A whole new world to explore, including:

topological phases

probe branes realizing
quantum Hall effect, topological insulators, ….

novel compressible quantum liquids

holographic quantum liquid,
θ=d-1 non-relativistic critical points