Entanglement Entropy in Three-Dimensional Higher Spin Theories

Juan Jottar

Based on:

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with Jan de Boer
+
to appear
with G. Compère and W. Song

Outline

1. Motivation: higher spin theories in AdS_3
2. Entanglement entropy proposal
3. Testing the proposal
4. Outlook

Outline

Higher spin theories

- Broad definition: interacting theories of gravity coupled to a finite (or infinite) number of massless fields of spin $s > 2$.

- Motivation from holography: explore AdS/CFT in a regime where the bulk theory is not just classical (super-)gravity, and the dual theory is not necessarily strongly-coupled:
  - Critical $O(N)$ vector models in $3d$ in the large-N limit dual to higher spin theories of Fradkin-Vasiliev type in AdS_4. (Klebanov, Polyakov 2002; Giombi, Yin 2010-12; Maldacena, Zhiboedov 2011-12)
  - Two-dimensional minimal model CFTs with extended (W-)symmetries in the large-N limit dual to higher spin theories in AdS_3. (Gaberdiel, Gopakumar 2010)

- Motivation from GR: curvature, causal structure, etc are not invariant under the higher spin gauge symmetries ⇒ challenge traditional geometric notions.

AdS_3 gravity as a Chern-Simons theory

- Standard AdS_3 gravity can be written as an $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ Chern-Simons theory. (Achucarro, Townsend 1986; Witten 1988)

- Take $3d$ gravity with a negative cosmological constant $\Lambda = -1/\ell^2$. Combine dreibein e^a and (dual) spin connection $\omega^a = (1/2!)\epsilon^{abc}\omega_{bc}$ into $SL(2,\mathbb{R})$ connections

$$A = A^a J_a = \omega + \frac{e}{\ell}\,, \qquad \bar{A} = \bar{A}^a J_a = \omega - \frac{e}{\ell}$$

where the J_a satisfy the $so(2,1) \simeq sl(2,\mathbb{R})$ algebra $[J_a, J_b] = \epsilon_{ab}{}^c J_c$.

- Defining $CS(A) = A \wedge dA + \frac{2}{3} A \wedge A \wedge A$ one finds

$$I_{CS} \equiv \frac{k}{4\pi}\int_M \mathrm{Tr}\Big[CS(A) - CS(\bar{A})\Big]$$
$$= \frac{1}{16\pi G_3}\left[\int_M d^3x\sqrt{|g|}\left(\mathcal{R} + \frac{2}{\ell^2}\right) - \int_{\partial M}\omega^a \wedge e_a\right]$$

$$k = \frac{\ell}{4G_3}$$

Higher spin theories in AdS_3

- The higher spin theories in $3d$ are constructed by generalizing the gauge algebra.
- The pure higher spin sector of the $3d$ Vasiliev theory is $\text{hs}[\lambda] \oplus \text{hs}[\lambda]$ Chern-Simons theory.
- The dual CFT has an infinite tower of conserved currents and $\mathcal{W}_\infty[\lambda]$ symmetry. (Henneaux, Rey 2010; Gaberdiel, Hartman 2011)
- For $\lambda = N \in \mathbb{Z}$ it is possible to truncate the tower of higher spin fields to $s \leq N$. The bulk theory reduces to $sl(N, \mathbb{R}) \oplus sl(N, \mathbb{R})$ Chern-Simons. Different $sl(2)$ embeddings are possible.
- The asymptotic symmetry algebra is of W_N type. (Campoleoni, Fredenhagen, Pfenninger, Theisen 2010)

Extending the AdS_3/CFT_2 dictionary

- The BTZ black hole entropy (via Bekenstein-Hawking) and holographic entanglement entropy (via Ryu-Takayanagi) match universal CFT results:

Cardy entropy formula

$$S_{thermal} = 2\pi\sqrt{\frac{c}{6}L_0} + 2\pi\sqrt{\frac{c}{6}\bar{L}_0}$$

(Single interval) Entanglement entropy at finite temperature $T = \beta^{-1}$

$$S_A = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi\Delta x}{\beta}\right)\right)$$

where c is the central charge and a the UV cutoff.

- Question: How do we compute these entropies in higher spin theories?

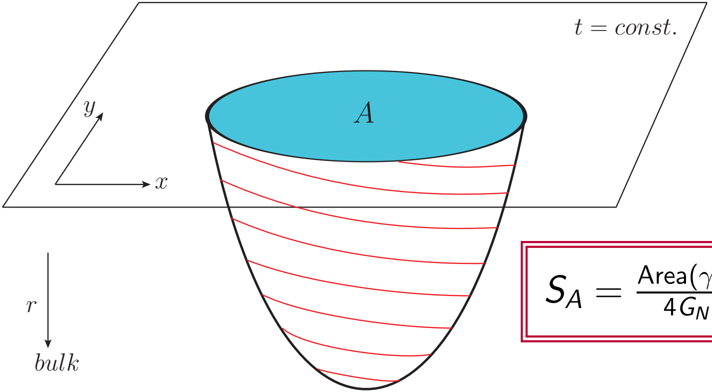
Outline

Entanglement entropy

- Consider a quantum system in a pure (or mixed) state, with density operator $\rho = |\Psi\rangle\langle\Psi|$ (or $\rho = e^{-\beta H}$).
- Partition the Hilbert space as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ ($B = A^c$), the reduced density matrix for subsystem A is defined as $\rho_A = \mathrm{Tr}_B \rho$.
- The entanglement entropy S_A associated with A is then given by the Von Neumann entropy of ρ_A: $S_A = -\mathrm{Tr}_A \, \rho_A \log \rho_A$.
- Universal results in 2d CFTs. (Holzhey, Larsen, Wilczek 1994; Calabrese, Cardy, 2004) If A is a single interval of length Δx in an 1d system:
  - System on the ∞ line at zero temperature: $S_A \simeq (c/3) \log (\Delta x / a)$
  - Periodic system of total length L: $S_A \simeq (c/3) \log \left(\frac{L}{\pi a} \sin (\pi \Delta x / L) \right)$
  - System on the ∞ line in a thermal (mixed) state: $S_A \simeq (c/3) \log \left(\frac{\beta}{\pi a} \sinh (\pi \Delta x / \beta) \right)$

Holographic entanglement entropy

- Generically, entanglement entropy is notoriously difficult to compute in field theory, even for free theories.
- However, in theories with a gravity dual it can be computed using the Ryu-Takayanagi prescription:



$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N}$$

Entanglement entropy for higher spin theories

- What replaces the R-T prescription in a $3d$ higher spin theory?. Write something that:
  - Involves the natural objects in the h.s. theory: Wilson lines.
  - Is invariant under the diagonal gauge group (rotations of the local frame), and under gauge transformations that do not change the state in the dual theory.
  - Encodes the geodesic length in the pure gravity case.
- One is lead to consider

$$W(P,Q) \equiv \mathrm{Tr}_{\mathcal{R}}\left[\mathcal{P}\exp\left(\int_Q^P \bar{A}\right)\mathcal{P}\exp\left(\int_P^Q A\right)\right]$$

Entanglement entropy for higher spin theories

- For $AAdS_3$ solutions of pure gravity, placing P and Q on the boundary, and at equal times, we find

$$W_{2d}(P,Q)\big|_{\rho_P=\rho_Q=\rho_0} \xrightarrow[\rho_0\to\infty]{} \exp d(P,Q)$$

  where $d(P,Q)$ is the geodesic distance.

- Since ent ent in AdS_3 is related to the geodesic distance via the Ryu-Takayanagi prescription, we propose

$$S_{ent} = k_{cs} \log \mathrm{Tr}_{\mathcal{R}} \left[\mathcal{P} \exp\left(\int_Q^P \bar{A} \right) \mathcal{P} \exp\left(\int_P^Q A \right) \right] \Bigg|_{\rho_P=\rho_Q=\rho_0\to\infty}$$

- This is a well-defined object in the higher spin theory as well. In the absence of field-theoretical results to compare against, we will perform some plausibility tests.

Outline

Test 1: Recover AdS_3 results

- For solutions of pure gravity we obtain

$$\text{Poincaré-patch:} \qquad S_{PAdS_3} = \frac{c}{3}\log\left[\frac{\Delta x}{a}\right]$$

$$\text{global:} \qquad S_{AdS_3} = \frac{c}{3}\log\left[\frac{\ell}{a}\sin\left(\frac{\Delta\varphi}{2}\right)\right]$$

$$\text{black hole:} \qquad S_{BTZ} = \frac{c}{6}\log\left[\frac{\beta_+\beta_-}{\pi^2 a^2}\sinh\left(\pi\frac{\Delta x}{\beta_+}\right)\sinh\left(\pi\frac{\Delta x}{\beta_-}\right)\right]$$

  in agreement with CFT results and holographic calculations. (Ryu,Takayanagi, 2006; Hubeny, Rangamani, Takayanagi, 2007)

- The higher spin theory with diagonally-embedded $sl(2)$ contains a truncation to Einstein gravity plus Abelian Chern-Simons fields. Our prescription also gives the right result for black holes carrying $U(1)$ charges, as recently confirmed by an independent CFT calculation. (Caputa, Mandal, Sinha 2013)

Test 2: Thermal entropy

- The entanglement entropy should approach the thermal entropy in the limit $\Delta x \gg \beta$ (i.e. as subsystem A approaches the full system). E.g.

$$S_A = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi\Delta x}{\beta}\right)\right) \xrightarrow[\Delta x \gg \beta]{} \frac{\pi c}{3\beta}\Delta x = s_{thermal}\,\Delta x$$

- Since the causal structure is not invariant under the higher spin gauge symmetries, we cannot calculate the entropy with the usual methods.

- Proceed indirectly by demanding existence of a well-defined partition function. E.g. $N = 3$:

$$Z(\tau,\alpha) = \mathrm{Tr}\left[e^{2\pi i\left(\tau\hat{L}+\alpha\hat{W}\right)}\right] = e^{S+2\pi i(\tau\mathcal{L}+\alpha\mathcal{W})}$$

with the integrability condition

$$\frac{\partial\mathcal{L}}{\partial\alpha} = \frac{\partial\mathcal{W}}{\partial\tau} \qquad \text{and} \qquad \tau = \frac{i}{2\pi}\frac{\partial S}{\partial\mathcal{L}}\,, \quad \alpha = \frac{i}{2\pi}\frac{\partial S}{\partial\mathcal{W}}$$

- Gutperle and Kraus showed that integrability follows from demanding smoothness (connection has trivial holonomy around contractible cycle of the Euclidean torus) and obtained:

$$S = 4\pi\sqrt{\frac{c}{6}\mathcal{L}}\sqrt{1-\frac{3}{4C}}$$

with $C = C(\mathcal{W}/\mathcal{L}^{3/2})$.

- Using the canonical formalism for charges in GR it was found instead (Perez, Tempo, Troncoso 2013)

$$S_{can} = 4\pi\sqrt{\frac{c}{6}\mathcal{L}}\left(1-\frac{3}{2C}\right)^{-1}\sqrt{1-\frac{3}{4C}}$$

which agrees with a perturbative result obtained using the metric formulation of the higher spin theory and Wald's entropy formula. (Campoleoni et. al. 2012)

Thermodynamics from the on-shell action

- From the point of view of the action, the two formulations follow by adding different boundary terms, and in general one finds (de Boer, J.I.J. 2013)

$$S_{holo} = -2\pi i k_{cs} \operatorname{Tr}\Big[A_z\left(\tau A_z + \bar{\tau} A_{\bar{z}}\right) - \text{barred}\Big]$$

$$S_{can} = -2\pi i k_{cs} \operatorname{Tr}\Big[\left(A_z + A_{\bar{z}}\right)\left(\tau A_z + \bar{\tau} A_{\bar{z}}\right) - \text{barred}\Big]$$

- The holonomy conditions imply $(\tau A_z + \bar{\tau} A_{\bar{z}}) = u^{-1}(iL_0)u$. Since for constant connections A_z and $A_{\bar{z}}$ commute by the e.o.m.:

$$S_{holo} = 2\pi k_{cs} \operatorname{Tr}_N\Big[\lambda_z L_0 - \text{barred}\Big]$$

$$S_{can} = 2\pi k_{cs} \operatorname{Tr}_N\Big[\lambda_\varphi L_0 - \text{barred}\Big]$$

where λ_j is a diagonal matrix containing the eigenvalues of A_j.
- The entanglement functional involves the spectrum of A_φ, so it favors the canonical expression.

Holomorphic vs. canonical

- The canonical expression for the entropy is also obtained by considering a generalized version of the conical singularity method. (Ammon, Castro, Iqbal 2013)
- The holomorphic formalism has been well understood from the point of view of the dual CFT: the black hole partition function corresponds to the theory deformed perturbatively by $\int d^2x\, \mu\mathcal{W}$. (Ammon et. al. 2011, Kraus, Perlmutter 2011; Gaberdiel, Hartman, Jin 2012)
- On the other hand, the CFT interpretation of the canonical formalism has proven elusive.
- The key to remedy this is to analyze the different notions of conserved charges at play.

Conserved charges at finite chemical potential

- In the holomorphic formulation the charges (expectation values) are obtained from traces of A_z, while the chemical potentials (sources) are incorporated by turning on the $A_{\bar{z}}$ component. The energy (mass) of the higher spin black hole is still $\mathcal{L} \sim \mathrm{Tr}\left[A_z^2\right]$, as for BTZ.
- On the other hand, the canonical charges in the presence of the deformation are (Bañados, Canto, Theisen 2012; Compère, Song 2013)

$$\tilde{\mathcal{L}} = \mathcal{L} + 3\mu\mathcal{W} + \frac{16}{3k}\mu^2\mathcal{L}^2 \sim \mathrm{Tr}\left[A_\varphi^2\right]$$
$$\tilde{\mathcal{W}} = \mathcal{W} + \frac{32\mathcal{L}^2\mu}{3k} + \frac{16\mathcal{L}\mathcal{W}\mu^2}{k} + \ldots \sim \mathrm{Tr}\left[A_\varphi^3\right]$$

- One can show that the canonical entropy, as a function of the tilded charges, has the same form as the holomorphic entropy. The same is true about the partition function as a function of the tilded sources $\tilde{\tau}$, $\tilde{\alpha}$ which are conjugate to the canonical charges. (Compère, J.I.J, Song, to appear)

Selecting the representation

- In the principal embedding we can rewrite our thermal entropy formula as
$$s_{thermal} = k_{cs} \mathrm{Tr}_N \left[\left(\lambda_x - \bar{\lambda}_x \right) L_0 \right] = k_{cs} \langle \vec{\lambda}_x - \vec{\bar{\lambda}}_x, \vec{\rho} \rangle$$
where $\vec{\rho}$ is the Weyl vector of $sl(N)$.

- In the principal embedding, the representation $\mathcal{R}$ that gives the right thermal limit $\Delta x \gg \beta$ of the entanglement functional is then the one with highest weight $\Lambda = \vec{\rho}$. The dimension of this rep is $\dim(\mathcal{R}) = 2^{N(N-1)/2}$.

- With this criterion one can determine the representation in other embeddings as well.

Test 3: Strong subadditivity

- An important property of entanglement entropy is that it is strongly subadditive

$$S(A) + S(B) \geq S(A \cup B) + S(A \cap B)$$
$$S(A) + S(B) \geq S(A \cap B^c) + S(B \cap A^c)$$

- In one spatial dimension, these inequalities imply that the single-interval entanglement entropy is a concave and non-decreasing function of the interval length.

- We have verified these properties numerically in different examples, including the spin-3 black hole. Caveat: Short distance singularities at the scale $\Delta x \simeq \mu$ suggest a redefinition of the cutoff is needed due to the effects of the irrelevant perturbation.

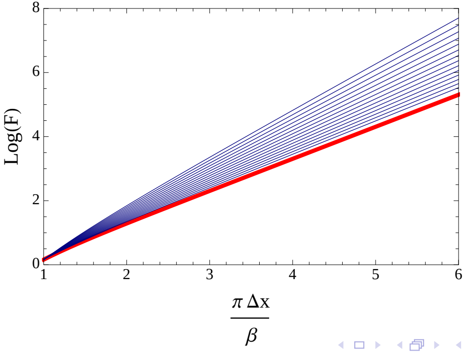
Example: entanglement at finite β and finite spin-3 charge

- Result depends on the dimensionless ratios $(\Delta x/\beta)$ and (μ/β):

$$S_A = \frac{c}{3} \log \left[\frac{\beta}{\pi\, a} F\left(\frac{\Delta x}{\beta}, \frac{\mu}{\beta} \right) \right]$$

- When the higher spin deformation is switched off $(\mu = \mathcal{W} = 0)$

$$S_A = \frac{c}{3} \log \left[\frac{\beta}{\pi\, a} \sinh\left(\frac{\pi \Delta x}{\beta} \right) \right]$$



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Summary

- We provided a holographic "guess" for the entanglement entropy in $2d$ CFTs with higher spin gravity duals, in particular in the presence of deformations by higher spin currents (irrelevant operators).
- The proposal passes several checks, but seems to breakdown for distances on the scale of the higher spin chemical potentials.
- We expect to clarify these issues via a constructive proof based on the replica trick and Rényi entropies:

$$S_A = \lim_{n\to 1} S_A^{(n)} = -\lim_{n\to 1} \frac{1}{n-1} \ln \mathrm{Tr}\left[\rho_A^n\right]$$

- The Ryu-Takayanagi prescription in AdS_3/CFT_2 has been recently proven in this way (Faulkner 2013; Hartman 2013), and extended beyond the semiclassical limit. (Barrella et. al. 2013)

Proving the higher spin entanglement proposal (in progress)

- One way to attack the problem is by computing

$$\mathrm{Tr}\left[\rho_A^n\right] = \frac{Z_n}{Z_1}$$

  where Z_n is the partition function on an n-sheeted cover of the original manifold obtained by cutting along subsystem A and cyclically gluing n copies (replicas). In the present case this amounts to evaluating the Chern-Simons action for bulk solutions with the topology of the branched cover at the boundary.

- Alternatively, perform a direct CFT calculation using twist fields:

$$\mathrm{Tr}\left[\rho_A^n\right] = \left\langle \sigma_n(u_1)\tilde{\sigma}_n(v_1)\ldots\sigma_n(u_N)\tilde{\sigma}_n(v_N)e^{\int d^2x\,\mu W}\right\rangle_{\mathbb{C}\text{ or }\mathcal{T}^2}$$