Conformal anomalies in hydrodynamics (and entanglement entropy)

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Introduction

- Hydrodynamics- effective field theory description on scales $L \gg \ell_{mfp}$
  - Dynamics described in a derivative expansion by local conservation laws
  - Conserved charges are the degrees of freedom
- Anomalies in the underlying microscopic QFT are manifested in hydrodynamics
  - New non-dissipative transport coefficients, values fixed by entropy conservation
  - Example: chiral and mixed chiral gravitational (Son and Surowka 2009)
- How is the trace anomaly manifested in hydro? (work with S. Theisen, Y. Oz, S. Yankielowicz; arxiv:1301.3170

Hydro effective action

- Idea: *Equilibrium* hydro can be described by a single diffeo invariant action functional (Jensen, et.al 2012 and Banerjee, et. al 2012)

$$W = \int d^d x L(x); \quad T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta W}{\delta g_{\mu\nu}}. \tag{1}$$

Equilibrium implies Killing vector V^μ such that $\mathcal{L}_V g_{\mu\nu} = 0$. Identifying $u^\mu = V^\mu / \sqrt{-V^2}$, one can show the Killing equation implies the vanishing of the fluid shear and expansion and

$$\nabla_\mu u_\nu = -u_\mu a_\nu + \Omega_{\mu\nu}, \quad \nabla_\mu \ln T = -a_\mu. \tag{2}$$

- Build up the action using following variables: T, a_μ, $\Omega_{\mu\nu}$, invariants of background curvature $R_{\mu\nu\rho\sigma}$ and their derivatives

Conformal case

- Effective action is made up of conformally invariant terms, plus an anomalous contribution

$$W_{\rm total} = W_{\rm inv} + W_{\rm anom} \tag{3}$$

We require that

$$\delta_\sigma W_{\rm anom} = \int \sqrt{-g}\, \sigma\, T^\mu_\mu \tag{4}$$

where

$$\mathcal{A}(g,A) \equiv \langle T^\mu_\mu \rangle_{g,A} = -(-)^{d/2} a E_d + \sum_i c_i I_i \,. \tag{5}$$

Invariant action

Built out of conformal invariants

- Examples in $d = 2$ and $d = 4$

$$W_{\text{inv}}^{d=2} = \int d^2x \sqrt{-g} p_0 T^2 \tag{6}$$

$$W_{\text{inv}}^{d=4} = \int d^4x \sqrt{-g} \left(p_0 T^4 + \alpha_1 T^2 \mathcal{R} + \alpha_2 T^2 \Omega_{\mu\nu} \Omega^{\mu\nu} + \cdots \right) \tag{7}$$

p_0 is here a constant pressure and $\mathcal{R}$ is scalar curvature constructed from Weyl covariant connection.

Anomaly action

- In general this anomaly action is non-local, but in the hydro setting we find a local expression
- Action is the same as the anomalous dilaton action that arises from spontaneous breaking of conformal symmetry (Schwimmer and Theisen 2011)

Replace the scalar dilaton $\tau \to -\ln(T/T_0)$

$$W_{\rm anom} = -\int_0^1 dt \int \sqrt{-\det((T/T_0)^t g)}\, d^d x \; \ln(T/T_0)\mathcal{A}\left((T/T_0)^t g\right) \ , \qquad (8)$$

- Example: d=2

$$T^{\mu}_{\mu} = cR$$

$$W_{\rm anom} = c \int d^2x \sqrt{-g}\,(-\ln(T/T_0)R - a_\mu a^\mu)\,.$$

- Example: d=4

$$T^{\mu}_{\mu} = -aE_4 + cW^2 \tag{9}$$

$$W_{\rm Euler} = -a \int \sqrt{-g}\, d^4x \, \left(\ln(T/T_0)\, E_4 - 4G^{\mu\nu} a_\mu a_\nu + 4\, a_\lambda a^\lambda (\nabla_\mu a^\mu) - 2\,(a_\mu a^\mu)^2\right)$$

$$W_{\rm Weyl} = -c \int \sqrt{-g}\, d^4x \ln(T/T_0) W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma}\,.$$

Stress tensor

Evaluate stress tensor by taking the variation wrt to the metric:

- In general, it has a perfect fluid form

$$T^{\mu\nu} = \rho\, u^\mu u^\nu + P\, P^{\mu\nu} \ , \tag{10}$$

$$\rho = (d-1) p_0 T^d + corr. \tag{11}$$

$$P = p_0 T^2 + corr. \tag{12}$$

Traceless stress tensor up to anomaly corrections
$\rho = (d-1)P + anom.$

Rindler space

- Consider the full stress tensor $T^{\mu\nu}$ on Rinder space (Lorentzian analog of the Euclidean cone in Jensen, Loganayagam, Yarom 2012)

$$ds^2 = -x_1^2 dt^2 + dx_i dx^i \ , \tag{13}$$

$$T = \frac{T_0}{x_1} \ . \tag{14}$$

- Evaluate at $T = 1/2\pi$ since Minkowski vacuum is thermal

$$Z^{-1}\mathrm{Tr}(\mathrm{e}^{-2\pi \mathrm{H_R}}\mathrm{T}^{\mu\nu}) = \langle 0|\mathrm{T}^{\mu\nu}|0\rangle \ = 0, \tag{15}$$

Pressure

- Find that contrary to naive expectations, the Euler anomaly term (dth order in derivatives in d dimensions) affects the pressure (zeroth order)

$$p_0 = (\xi(\lambda_*) + n(2\pi)^d a) T^d \ , \tag{16}$$

where $\xi(\lambda)$ represents a coupling dependent term that arises from local conformally invariant terms in the effective action and λ_* is the fixed point value of the coupling.
In $d = 2$, $\xi = 0$ (action is exact) and $n = 1$, reproduces the “Cardy formula".

Entropy

Thermal entropy S

$$S = \frac{\partial}{\partial T_0}(T_0 W) = \int \sqrt{-g} d^{d-1}x \frac{\partial L}{\partial T_0} \tag{17}$$

Gives zero for anomalous part of effective lagrangian. Anomalies contribute to the entropy via the coefficients of the invariant parts of the action, which we have seen depend on the anomaly...

- Thermal entropy of CFT on $R \times H^{d-1}$ or on static patch of dS equiv to the entanglement entropy in flat spacetime with spherical ent. surface (Casini, Huerta, Myers 2011)
- Logarithmic divergent term in entanglement entropy is universal and is proportional to the Euler anomaly a

Free Field calculation

Metric: $ds^2 = -d\tau^2 + \mathcal{R}^2(du^2 + \sinh^2 u \, d\Omega^2_{d-2})$

- $R \times H^{d-1}$, static dS, and Rindler are related by conformal transformations

Note: Stress tensors are related up to conformal factors, plus a constant shift, coming from integration of the trace anomaly
Stress tensor on hyperbolic (Candelas and Dowker 1979, Emparan 1999)

$$\langle T^{\nu}_{\mu} \rangle = \frac{\pi^2}{90\beta^4} \left(n_0 + \frac{7}{4} n_{1/2} + 2n_1 + \frac{5\beta^2}{8\pi^2\mathcal{R}^2}(n_{1/2} + 8n_1) \right) diag(-3, 1, 1, 1) \tag{18}$$

Compute thermal entropy, extract Euler anomaly

$$a = \frac{1}{360} \left(n_0 + 11/2 n_{1/2} + 32 n_1 \right) \tag{19}$$

versus textbook answer

$$a = \frac{1}{360} \left(n_0 + (11/2) n_{1/2} + 62 n_1 \right). \tag{20}$$

Gauge field entanglement

Mismatch for free spin-1 fields

- Consider a different calculation of entropy. Log term in entropy via heat kernel on Euclidean static dS, e.g.

$$W = -\frac{1}{2}\int_{\epsilon^2}^{\infty} \frac{ds}{s} K(s) = -\frac{1}{32\pi^2}\left(\frac{a_0}{2\epsilon^4} + \frac{a_1}{\epsilon^2} - a_2 \ln(\epsilon^2) + \cdots\right) \tag{21}$$

This gives the correct value for the Euler anomaly in the spin-1 case!

- Implies old results for gauge field stress tensor are incorrect...

Only way to fix is if the coefficient in the T^2 part of the spin-1 energy density is doubled

Kabat term

In the case of (euclidean) Rindler space, this doubling is the consequence of the spin-1 “contact term" found by Kabat 1995

- Vector heat kernel on this space has the general form:

$$g_{\mu\nu} K^{\mu\nu}_{\text{vector}}(s, x) = 2K_{\text{scalar}}(s, x) + \int_s^\infty ds' \nabla^2 K_{\text{scalar}}(s', x)) \tag{22}$$

Not just two minimally coupled scalars

- Forthcoming work with Y. Oz and S. Theisen